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EXACT SELF-CONSISTENT PLANE-SYMMETRIC SOLUTIONS TO THE SPINOR AND SCALAR FIELD EQUATIONS

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Abstract. We consider a system of nonlinear spinor and scalar fields with minimal coupling in general relativity. The nonlinearity in the spinor field Lagrangian is given by an arbitrary function of the invariants generated from the bilinear spinor forms $S = \bar{\psi}\psi$ and $P = i\bar{\psi}\gamma^5\psi$; the scalar Lagrangian is chosen as an arbitrary function of the scalar invariant $\Omega = \varphi_{,\alpha}\varphi^{,\alpha}$, that becomes linear at $\Omega \to 0$. The spinor and the scalar fields in question interact with each other by means of a gravitational field which is given by a plane-symmetric metric. Exact plane-symmetric solutions to the gravitational, spinor and scalar field equations are obtained. Spinor field equations with the nonlinear term being a power function of I, i.e., $L_N = \lambda I^n$, where $I = S^2, P^2$, or $S^2 \pm P^2$, λ is the self-coupling constant and n is the power of nonlinearity, are thoroughly investigated. Role of gravitational field in the formation of the field configurations with limited total energy, spin and charge has been investigated. Influence of the change of the sign of energy density of the spinor and scalar fields on the properties of the configurations obtained has been examined. It has been established that under the change of the sign of the scalar field energy density the system in question can be realized physically iff the scalar charge does not exceed some critical value. In case of spinor field no such restriction on its parameter occurs. In general it has been shown that the choice of spinor field nonlinearity can lead to the elimination of scalar field contribution to the metric functions, but leaving its contribution to the total energy unaltered.

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1 Introduction

At present the nonlinear generalization of classical field theory remains one of the possible ways to overcome the difficulties of the theory which considers elementary particles as mathematical points. In this approach elementary particles are modeled by soliton-like solutions of corresponding nonlinear equations. The gravitational field equation is nonlinear by nature and the field itself is universal and unscreenable. These properties lead to definite physical interest for the proper gravitational field to be considered. Nevertheless, papers, dealing with soliton-like solutions of nonlinear field equations, ignore the proper gravitational field in the initial field system more often than not.

The existence of stable particle-like classical elementary excitations in a model 5-D universe was obtained in [1]. The authors showed that if a torsion invariant is included in the free Lagrangian, the particle-like stable solutions exist having definite positive rest energy, spin, and corresponding anti-particles.

In recent years cosmological models exhibiting plane symmetry have been studied by various authors [2-11]. Though at the present state of evolution, the universe is spherically symmetric and the matter distribution in it is isotropic and homogeneous, in the early stage of evolution (say close to big bang singularity), neither the assumption of spherical symmetry nor of isotropy can be strictly valid. From this point of view many authors consider plane symmetry, which is less restrictive than spherical symmetry and provides an avenue to study inhomogeneities, since such models play an important role in understanding some essential features of the universe such as formation of galaxies during the early stage of evolution and process of homogenization. Inhomogeneous plane-symmetric model was first considered by Taub [12,13]. As was defined by Taub [12], a space-time will be said to have plane symmetry if it admits the three parameter group generated by the transformation

$$y^* = y + a, \tag{1.1a}$$

$$z^* = z + b, \tag{1.1b}$$

and

$$\begin{aligned} y^* &= y\cos\theta + z\sin\theta, \\ z* &= -y\sin\theta + z\cos\theta. \end{aligned} \tag{1.1c}$$

The metric of space-time admitting plane symmetry may be written as [13]

$$ds^2 = e^{2\chi}dt^2 - e^{2\alpha}dx^2 - e^{2\beta}(dy^2 + dz^2), \tag{1.2}$$

where the speed of light c is taken to be unity and χ, α, β are functions of x and t alone. Note that though (1.2) is one of the most general form of a universe admitting plane symmetry, in this paper we consider the metric functions χ, α, β to be time independent. The purpose of the paper is to study the role of nonlinear spinor and scalar field in the formation of configurations with localized energy density and limited total energy, spin and charge of the spinor field. Within the scope of this report we show that the spinor field is the more sensitive to the proper gravitational one than the scalar or electromagnetic fields.

2 Fundamental Equations and General Solutions

The Lagrangian of the nonlinear spinor, scalar and gravitational fields can be written in the form

$$K = \frac{R}{2\kappa} + L_{\rm sp} + L_{\rm sc} \tag{2.1}$$

with

$$L_{\rm sp} = \frac{i}{2}[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi] - m\bar\psi\psi + L_N, \tag{2.2}$$

and

$$L_{\rm sc} = \Psi(\Omega), \quad \Omega = \varphi_{,\alpha}\varphi^{,\alpha} \tag{2.3}$$

Here R is the scalar curvature and κ is the Einstein's gravitational constant. The nonlinear term L_N in spinor Lagrangian describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field having the form

$$S = \bar\psi\psi, \quad P = i\bar\psi\gamma^5\psi, \quad \nu^\mu = (\bar\psi\gamma^\mu\psi), \quad A^\mu = (\bar\psi\gamma^5\psi), \quad T^{\mu\nu} = (\bar\psi\sigma^{\mu\nu}\psi),$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$. Invariants, corresponding to the bilinear forms, look like

$$I = S^2, \quad J = P^2, \quad I_\nu = \nu_\mu\nu^\mu = (\bar\psi\gamma^\mu\psi)g_{\mu\nu}(\bar\psi\gamma^\nu\psi)$$
$$I_A = A_\mu A^\mu = (\bar\psi\gamma^5\gamma^\mu\psi)g_{\mu\nu}(\bar\psi\gamma^5\gamma^\nu\psi)$$
$$I_T = T_{\mu\nu}T^{\mu\nu} = (\bar\psi\sigma^{\mu\nu}\psi)g_{\mu\alpha}g_{\nu\beta}(\bar\psi\sigma^{\alpha\beta}\psi)$$

According to the Pauli–Fierz theorem, [14] among the five invariants only I and J are independent as all other can be expressed by them: $I_\nu = -I_A = I + J$ and $I_T = I - J$. Therefore, we choose the nonlinear term $L_N = F(I, J)$, thus claiming that it describes the nonlinearity in the most general of its form.

The scalar Lagrangian L_{sc} is an arbitrary function of invariant $\Omega = \varphi_{,\alpha}\varphi^{,\alpha}$, satisfying the condition

$$\lim_{\Omega\to 0}\Psi(\Omega) = \frac{1}{2}\Omega + \cdots \tag{2.4}$$

The static plane-symmetric metric we choose in the form

$$ds^2 = e^{2\chi}dt^2 - e^{2\alpha}dx^2 - e^{2\beta}(dy^2 + dz^2), \tag{2.5}$$

where the metric functions χ, α, β depend on the spatial variable x only and obey the coordinate condition

$$\alpha = 2\beta + \chi. \tag{2.6}$$

Variation of (2.1) with respect to spinor field $\psi(\bar\psi)$ gives nonlinear spinor field equations

$$i\gamma^\mu\nabla_\mu\psi - \Phi\psi + i\mathscr{G}\gamma^5\psi = 0, \tag{2.7a}$$
$$i\nabla_\mu\bar\psi\gamma^\mu + \Phi\bar\psi + i\mathscr{G}\bar\psi\gamma^5 = 0, \tag{2.7b}$$

with

$$\Phi = m - \mathscr{D} = m - 2D\frac{\partial F}{\partial I}, \quad \mathscr{G} = 2P\frac{\partial F}{\partial J}$$

whereas, variation of (2.1) with respect to scalar field yields the following scalar field equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\Big(\sqrt{-g}\,g^{\nu\mu}\frac{d\Psi}{d\Omega}\varphi_{,\mu}\Big) = 0. \tag{2.8}$$

Varying (2.1) with respect to metric tensor $g_{\mu\nu}$, we obtain the Einstein's field equation

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = -\kappa T^\mu_\nu, \tag{2.9}$$

which in view of (2.5) and (2.6) is written as follows

$$G^0_0 = e^{-2\alpha}(2\beta'' - 2\chi'\beta' - \beta'^2) = -\kappa T^0_0 \tag{2.10a}$$
$$G^1_1 = e^{-2\alpha}(2\chi'\beta' + \beta'^2) = -\kappa T^1_1 \tag{2.10b}$$
$$G^2_2 = e^{-2\alpha}(b'' + \chi'' - 2\chi'\beta' - \beta'^2 = -\kappa T^2_2 \tag{2.10c}$$
$$G^3_3 = G^2_2, \quad T^3_3 = T^2_2. \tag{2.10d}$$

Here prime denotes differentiation with respect to x and T^μ_ν is the energy-momentum tensor of the spinor and scalar fields

$$T^\nu_\mu = T^\nu_{\mathrm{sp}\mu} + T^\nu_{\mathrm{sc}\mu} \tag{2.11}$$

The energy-momentum tensor of the spinor field is

$$T^\rho_{\mathrm{sp}\mu} = \frac{i}{4}g^{\rho\nu}\big(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi\big) - \delta^\rho_\mu L_{\mathrm{sp}} \tag{2.12}$$

where L_{sp} with respect to (2.7) takes the form

$$L_{\mathrm{sp}} = -\frac{1}{2}\Big(\bar\psi\frac{\partial F}{\partial\bar\psi} + \frac{\partial F}{\partial\psi}\psi\Big) - F, \tag{2.13}$$

and the energy-momentum tensor of the scalar one is

$$T^\nu_{\mathrm{sc}\mu} = 2\frac{s\Psi}{d\Omega}\varphi_{,\mu}\varphi^{,\mu} - \delta^\nu_\mu\Psi, \quad \Omega = -(\varphi')^2e^{-2\alpha}, \quad \varphi' = \frac{d\varphi}{dx}. \tag{2.14}$$

In (2.7) and (2.12) ∇_μ denotes the covariant derivative of spinor, having the form [15,16]

$$\nabla_\mu\psi = \frac{\partial\psi}{\partial x^\mu} - \Gamma_\mu\psi, \tag{2.15}$$

where $\Gamma_\mu(x)$ are spinor affine connection matrices. γ matrices in the above equations are connected with the flat space-time Dirac matrices $\bar\gamma$in the following way

$$g_{\mu\nu}(x) = e_\mu^a(x)e_\nu^b(x)\eta_{ab}, \quad \gamma_\mu(x) = e_\mu^a(x)\bar\gamma_a, \tag{2.16}$$

where $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ and e_μ^a is a set of tetrad 4-vectors. Using (2.16) we obtain

$$\gamma^0(x) = e^{-\chi}\bar\gamma^0, \quad \gamma^1(x) = e^{-\alpha}\bar\gamma^1, \quad \gamma^2(x) = e^{-\beta}\bar\gamma^2, \quad \gamma^3(x) = e^{-\beta}\bar\gamma^3. \tag{2.17}$$

From

$$\gamma_\mu(x) = \frac{1}{4}g_{\rho\sigma}(x)\big(\partial_\mu e_\delta^n e_b^\rho - \Gamma_{\mu\delta}^\rho\big)\gamma^\sigma\gamma^\delta, \tag{2.18}$$

one finds

$$\begin{aligned}\Gamma_0 &= -\frac{1}{2}\bar\gamma^0\bar\gamma^1 e^{-2\beta}\chi', \qquad \Gamma_1 = 0,\\ \Gamma_2 &= \frac{1}{2}\bar\gamma^2\bar\gamma^1 e^{-2(\chi+\beta)}\beta', \quad \Gamma_3 = \frac{1}{2}\bar\gamma^3\bar\gamma^1 e^{-(\chi+\beta)}\beta'.\end{aligned} \tag{2.19}$$

Flat space-time matrices $\bar\gamma$ we will choose in the form, given in [17]:

$$\bar\gamma^0 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}, \qquad \bar\gamma^1 = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},$$
$$\bar\gamma^2 = \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}, \qquad \bar\gamma^3 = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}.$$

Defining γ^5 as follows,

$$\begin{aligned}\gamma^5 &= -\frac{i}{4}E_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho, \quad E_{\mu\nu\sigma\rho} = \sqrt{-g}\,\varepsilon_{\mu\nu\sigma\rho}, \quad \varepsilon_{0123} = 1,\\ \gamma^5 &= -i\sqrt{-g}\,\gamma^0\gamma^1\gamma^2\gamma^3 = -i\bar\gamma^0\bar\gamma^1\bar\gamma^2\bar\gamma^3 = \bar\gamma^5,\end{aligned}$$

we obtain

$$\bar\gamma^5 = \begin{pmatrix}0&0&-1&0\\0&0&0&-1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}$$

The scalar field equation (2.8) has the solution

$$\frac{d\Psi}{d\Omega}\varphi' = \varphi_0, \quad \varphi_0 = \text{const.} \tag{2.20}$$

The equality (2.20) for a given $\Psi(\Omega)$ is an algebraic equation for φ' that is to be defined through metric function $e^{\alpha(x)}$.

We will consider the spinor field to be the function of the spatial coordinate x only $[\psi = \psi(x)]$. Using (2.15), (2.17) and (2.19), we find

$$\gamma^\mu \Gamma_\mu = -\frac{1}{2} e^{-\alpha} \alpha' \bar{\gamma}^1. \tag{2.21}$$

Then taking into account (2.21), we rewrite the spinor field equation (2.7a) as

$$i\bar{\gamma}^1 \Bigl(\frac{\partial}{\partial x} + \frac{\alpha'}{2}\Bigr)\psi + ie^{\alpha}\Phi\psi + e^{\alpha}\mathcal{G}\gamma^5\psi = 0. \tag{2.22}$$

Further setting $V(x) = e^{\alpha/2}\psi(x)$ with

$$V(x) = \begin{pmatrix} V_1(x) \\ V_2(x) \\ V_3(x) \\ V_4(x) \end{pmatrix}$$

for the components of spinor field from (2.22) one deduces the following system of equations:

$$V_4' + ie^{\alpha}\Phi V_1 - e^{\alpha}\mathcal{G} V_3 = 0, \tag{2.23a}$$
$$V_3' + ie^{\alpha}\Phi V_2 - e^{\alpha}\mathcal{G} V_4 = 0, \tag{2.23b}$$
$$V_2' - ie^{\alpha}\Phi V_3 - e^{\alpha}\mathcal{G} V_1 = 0, \tag{2.23c}$$
$$V_1' - ie^{\alpha}\Phi V_4 - e^{\alpha}\mathcal{G} V_2 = 0. \tag{2.23d}$$

As one sees, the equation (2.23) gives following relations

$$V_1^2 - V_2^2 - V_3^2 + V_4^2 = \text{const.} \tag{2.24}$$

Using the solutions obtained one can write the components of spinor current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{2.25}$$

Taking into account that $\bar{\psi} = \psi^\dagger \bar{\gamma}^0$, where $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ and $\psi_j = e^{-\alpha/2}V_j$, $j = 1, 2, 3, 4$ for the components of spinor current we write

$$j^0 = [V_1^*V_1 + V_2^*V_2 + V_3^*V_3 + V_4^*V_4]e^{-(\alpha+\chi)}, \tag{2.26a}$$
$$j^1 = [V_1^*V_4 + V_2^*V_3 + V_3^*V_2 + V_4^*V_1]e^{-2\alpha}, \tag{2.26b}$$
$$j^2 = -i[V_1^*V_4 - V_2^*V_3 + V_3^*V_2 - V_4^*V_1]e^{-(\alpha+\beta)}, \tag{2.26c}$$
$$j^3 = [V_1^*V_3 - V_2^*V_4 + V_3^*V_1 - V_4^*V_2]e^{-(\alpha+\beta)}. \tag{2.26d}$$

Since we consider the field configuration to be static one, the spatial components of spinor current vanishes, i.e.,

$$j^1 = 0, \quad j^2 = 0, \quad j^3 = 0. \tag{2.27}$$

This supposition gives additional relation between the constant of integration. The component j^0 defines the charge density of spinor field that has the following chronometric-invariant form

$$\rho = (j_0 \cdot j^0)^{1/2}. \tag{2.28}$$

The total charge of spinor field is defined as

$$Q = \int \rho\sqrt{^{-3}g}\, dV, \quad dV = dxdydz. \tag{2.29a}$$

Since $\rho = \rho(x)$, i.e., matter distribution takes place along x axis only, for Q to make any sense we should integrate it for any finite range by y and z and then normalize it to unity. Sometimes Q is defined as

$$Q = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \rho\sqrt{^{-3}g}\, dxdydz \Bigg/ \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} dydz. \tag{2.29b}$$

In what follows we perform integration by y and z in the limit (0,1) and define the total charge (normalized) as

$$Q = \int\limits_{-\infty}^{\infty} \rho\sqrt{^{-3}g}\, dx. \tag{2.29}$$

Let us consider the spin tensor [17]

$$S^{\mu\nu,\varepsilon} = \frac{1}{4}\bar\psi\{\gamma^\varepsilon\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\varepsilon\}\psi. \tag{2.30}$$

We write the components $S^{ik,0}$ $(i,k = 1,2,3)$, defining the spatial density of spin vector explicitly. From (2.30) we have

$$S^{ij,0} = \frac{1}{4}\bar\psi\{\gamma^0\sigma^{ij} + \sigma^{ij}\gamma^0\}\psi = \frac{1}{2}\bar\psi\gamma^0\sigma^{ij}\psi \tag{2.31}$$

that defines the projection of spin vector on k axis. Here i, j, k take the values 1, 2, 3 and $i \neq j \neq k$. Thus, for the projection of spin vectors on the X, Y and Z axis we find

$$S^{23,0} = [V_1^* V_2 + V_2^* V_1 + V_3^* V_4 + V_4^* V_3]e^{-\alpha-2\beta-\chi}, \tag{2.32a}$$
$$S^{31,0} = [V_1^* V_2 - V_2^* V_1 + V_3^* V_4 - V_4^* V_3]e^{-2\alpha-\beta-\chi}, \tag{2.32b}$$
$$S^{12,0} = [V_1^* V_1 - V_2^* V_2 + V_3^* V_3 + V_4^* V_4]e^{-2\alpha-\beta-\chi}. \tag{2.32c}$$

The chronometric invariant spin tensor takes the form

$$S^{ij,0}_{\rm ch} = (S_{ij,0}S^{ij,0})^{1/2}, \tag{2.33}$$

and the projection of the spin vector on k axis is defined by

$$S_k = \int\limits_{\infty}^{\infty} S_{\rm ch}^{ij,0} \sqrt{-^3 g}\, dx. \qquad (2.34)$$

(In (2.34), as well as in (2.29) integrations by y and z are performed in the limit (0,1)).

From (2.7) one can write the equations for $S = \bar\psi\psi$, $P = i\bar\psi\gamma^5\psi$ and $A = \bar\psi\bar\gamma^5\bar\gamma^1\psi$

$$S' + \alpha' S + 2e^{\alpha}\mathscr{G} A = 0, \qquad (2.35a)$$

$$P' + \alpha' P + 2e^{\alpha}\Phi A = 0, \qquad (2.35b)$$

$$A' + \alpha' A + 2e^{\alpha}\Phi P + 2e^{\alpha}\mathscr{G} S = 0. \qquad (2.35c)$$

Note that, A in (2.35) is indeed the pseudo-vector A^1. Here for simplicity, we use the notation A. From (2.35) immediately follows

$$S^2 + P^2 - A^2 = C_0 e^{2\alpha}, \quad C_0 = \text{const}. \qquad (2.36)$$

Let us now solve the Einstein equations. To do it we first write the expression for the components of the energy-momentum tensor explicitly. Using the property of flat space-time Dirac matrices and the explicit form of covariant derivative ∇_μ, for the spinor field one finds

$$T_{\rm sp1}^1 = mS - F(I,J), \quad T_{\rm sp0}^0 = T_{\rm sp2}^2 = T_{\rm sp3}^3 = \mathscr{D} S - \mathscr{G} P - F(I,J) \qquad (2.37)$$

On the other hand, taking into account that the scalar field φ is also a function of x only $[\varphi = \varphi(x)]$ for the scalar field one obtains

$$T_{\rm sc1}^1 = 2\Omega\frac{d\Psi}{d\Omega} - \Psi(\Omega), \quad T_{\rm sc0}^0 = T_{\rm sc2}^2 = T_{\rm sc3}^3 = -\Psi(\Omega). \qquad (2.38)$$

In view of $T_0^0 = T_2^2$, subtraction of Einstein equations (2.10a) and (2.10c) leads to the equation

$$\beta'' - \chi'' = 0, \qquad (2.39)$$

with the solution

$$\beta(x) = \chi(x) + Bx, \qquad (2.40)$$

where B is the integration constant. The second constant has been chosen to be trivial, since it acts on the scale of Y and Z axes only. In account of (2.39) from (2.6) one obtains

$$\beta'' = \frac{1}{3}\alpha'', \quad \chi'' = \frac{1}{3}\alpha''. \qquad (2.41)$$

Solutions to the equation (2.41) together with (2.6) and (2.40) lead to the following expression for $\beta(x)$ and $\gamma(x)$:

$$\beta(x) = \frac{1}{3}\big(\alpha(x) + Bx\big), \quad \chi(x) = \frac{1}{3}\Big(\alpha(x) - 2Bx\Big). \qquad (2.42)$$

Equation (2.10b), being the first integral of (2.10a) and (2.10c), is a first order differential equation. Inserting β and γ from (2.42) and T_1^1 in account of (2.11), (2.37) and (2.38) into (2.10b) for α one gets

$$\alpha'^2 - B^2 = -3\kappa e^{2\alpha}\Big[mS - F(I,J) + 2\Omega\frac{d\Psi}{d\Omega} - \Psi(\Omega)\Big]. \tag{2.43}$$

As one sees from (2.35) and (2.36), the invariants are the functions of α, so is the right hand side of (2.43), hence can be solved in quadrature. In the sections to follow, we analyze the equation (2.43) in details given the concrete form of nonlinear term in spinor Lagrangian.

3 Analysis of the Results

In this section we shall analyze the general results obtained in the previous section for concrete nonlinear term.

3.1 Case with Linear Spinor and scalar fields

Let us consider the self-consistent system of linear spinor and massless scalar field equations. By doing so we can compare the results obtained with those of the self-consistent system of nonlinear spinor and scalar field equations, hence clarify the role of nonlinearity of the fields in question in the formation of regular localized solutions such as static solitary wave or solitons [18,19].

In this case for the scalar field we have $\Psi(\Omega) = \Omega/2$. Inserting this into (2.20), we obtain

$$\varphi'(x) = \varphi_0. \tag{3.1}$$

From (2.38) in account of (3.1) we get

$$-T_{\rm sc1}^1 = T_{\rm sc0}^0 = T_{\rm sc2}^2 T_{\rm sc3}^3 = -\frac{1}{2}\Omega = \frac{1}{2}\varphi_0^2 e^{-2\alpha}. \tag{3.2}$$

On the other hand for the linear spinor field we have

$$T_{\rm sp1}^1 = mS, \quad T_{\rm sp0}^0 = T_{\rm sp2}^2 T_{\rm sp3}^3 = 0. \tag{3.3}$$

As one can easily verify, for the linear spinor field the equation (2.35a) results

$$S = C_0 \exp[-\alpha(x_0)]. \tag{3.4}$$

Taking this relation into account and the fact that

$$\alpha'(x) = -\frac{1}{S}\frac{dS}{dx}$$

from (2.43) we write

$$\int \frac{dS}{\sqrt{(1+\bar\kappa/2)B^2S^2 - 3\kappa C_0^2 S}} = x, \quad \bar\kappa = 2\kappa\varphi_0^2/B^2, \tag{3.5}$$

with the solution

$$S(x)=\frac{M^2}{H^2}\cosh^2(\tilde{H}x),\ M^2=3\kappa C_0^2,\ H^2=B^2(1+\bar{\kappa}/2),\ \tilde{H}=H/2. \quad (3.6)$$

Further we define the functions ψ_j. Taking into account that in this case

$$\mathscr{F}(S) = mC_0/S\sqrt{H^2S^2 - M^2S},$$

for $N_{1,2}$ in view of (3.6) we find

$$N_{1,2}(x) = \pm(2H/3\kappa C_0)\tanh(\tilde{H}x) + R_{1,2}.$$

We can then finally write

$$\psi_{1,2} = ia_{1,2}E(x)\cosh[f(x) + R_{1,2}], \quad \psi_{3,4} = ia_{2,1}E(x)\sinh[f(x) + R_{2,1}], \quad (3.7)$$

where $E(x) = \sqrt{3\kappa mC_0/H^2}\cosh(Hx)$ and $f(x) = (2H/3\kappa C_0)\tanh(Hx)$. For the scalar field energy density we find

$$T^0_{\rm sc0}(x) = \frac{1}{2}\varphi_0^2e^{-2\alpha} = \frac{M^4\varphi_0^2}{2C_0^2H^4}\cosh^4(\tilde{H}x). \quad (3.8)$$

It is clear from (3.8) that the scalar field energy density is not localized.

Let us consider the case when the scalar field possesses negative energy density. Then we have $\Psi(\Omega) = -(1/2)\Omega$ and

$$-T^1_{\rm sc1} = T^0_{\rm sc0} = T^2_{\rm sc2} = T^3_{\rm sc3} = \frac{1}{2}\Omega = -\frac{1}{2}\varphi_0e^{-2\alpha}. \quad (3.9)$$

Then for S we get

$$\int\frac{dS}{\sqrt{(1-\bar{\kappa}/2)B^2S^2 - 3\kappa C_0^2S}} = x. \quad (3.10)$$

As one sees, the field system considered here is physically realizable iff 1 $1-\bar{\kappa}/2 > 0$, i.e., the scalar charge $|\varphi_0| < \sqrt{2/3\kappa B}$. Moreover, in the specific case with $B = 0$, independent to the quantity of scalar charge φ_0, the existence of scalar field with negative energy density in general relativity is impossible (even in absence of linear spinor field).

For the total charge Q of the system in this case we have

$$Q = 2a^2\int\limits_{-\infty}^{\infty}\cosh\Big[\frac{4H}{3\kappa C_0}\tanh(\tilde{H}x) + 2R\Big]\Big(\frac{C_0H^2}{M^2\cosh^2(\tilde{H}x)}\Big)^{3/2}e^{2Bx/3}dx < \infty. \quad (3.11)$$

It can be shown that, in case of linear spinor and scalar fields with minimal coupling both charge and spin of spinor field are limited. The energy density of the system, in view of (3.3) is defined by the contribution of scalar field only:

$$T^0_0(x) = T^0_{\rm sc0}(x) = \frac{1}{2}\frac{\varphi_0^2M^4}{C_0^2H^4}\cosh^4(\tilde{H}x). \quad (3.12)$$

From (3.12) follows that, the energy density of the system is not localized and the total energy of the system $E = \int_\infty^\infty T^0_0\sqrt{-^3g}\,dx$ is not finite.

3.2 Nonlinear Spinor and Linear Scalar Fields

Case I: $F = F(I)$. Let us consider the case when the nonlinear term in spinor field Lagrangian is a function of $I(S)$ only, that leads to $\mathscr{G} = 0$. From (2.35) as in case of linear spinor field we find $S = C_0 e^{-\alpha(x)}$. Proceeding as in foregoing subsection, for S from (2.43) we write

$$\frac{dS}{dx} = \mathscr{L}(s), \quad \mathscr{L}(S) = \sqrt{B^2 S^2 - 3\kappa C_0^2 \Big[mS - F(S) + 2\Omega\frac{d\Psi}{d\Omega} - \Psi(\Omega)\Big]} \tag{3.13}$$

with the solution

$$\int \frac{dS}{\mathscr{L}(s)} = \pm(x + x_0). \tag{3.14}$$

Given the concrete form of the functions $F(S)$ and $\Psi(S)$, from (3.14) yields S, hence α, β, χ.

Let us now go back to spinor field equations (2.23). Setting $V_j(x) = U_j(S)$, $j = 1, 2, 3, 4$ and taking into account that in this case $\mathscr{G} = 0$, for $U_j(S)$ we obtain

$$\frac{dU_4}{dS} + i\mathscr{F}(S)U_1 = 0, \tag{3.15a}$$

$$\frac{dU_3}{dS} + i\mathscr{F}(S)U_2 = 0, \tag{3.15b}$$

$$\frac{dU_2}{dS} - i\mathscr{F}(S)U_3 = 0, \tag{3.15c}$$

$$\frac{dU_1}{dS} - i\mathscr{F}(S)U_4 = 0, \tag{3.15d}$$

with $\mathscr{F}(S) = \Phi\mathscr{L}(S)C_0/S$ Differentiating (3.15a) with respect to S and inserting (3.15d) into it for U_4 we find

$$\frac{d^2U_4}{dS^2} - \frac{1}{\mathscr{F}}\frac{d\mathscr{F}}{dS}\frac{dU_4}{dS} - \mathscr{F}^2 U_4 = 0 \tag{3.16}$$

that transforms to

$$\frac{1}{\mathscr{F}}\frac{d}{dS}\Big(\frac{1}{\mathscr{F}}\frac{dU_4}{dS}\Big) - U_4 = 0, \tag{3.17}$$

with the first integral

$$\frac{dU_4}{dS} = \pm\sqrt{U_4^2 + C_1}\cdot\mathscr{F}(S), \quad C_1 = \text{consta.} \tag{3.18}$$

For $C_1 = a_1^2 > 0$ from (3.18) we obtain

$$U_4(S) = a_1 \sinh N_1(S), \quad N_1 = \pm\int \mathscr{F}(S)dS + R_1, \quad R_1 = \text{const.} \tag{3.19}$$

whereas, for $C_1 = -b_1^2 < 0$ from (3.18) we obtain

$$U_4(S) = a_1 \cosh N_1(S) \tag{3.20}$$

Inserting (3.19) and (3.20) into (3.15d) one finds

$$U_1(S) = ia_1 \cosh N_1(S), \quad U_1(S) = ib_1 \sinh N_1(S). \tag{3.21}$$

Analogically, for U_2 and U_3 we obtain

$$U_2(S) = a_2 \sinh N_2(S), \quad U_3(S) = b_2 \cosh N_2(S). \tag{3.22}$$

and

$$U_2(S) = ia_2 \cosh N_2(S), \quad U_2(S) = ib_2 \sinh N_2(S). \tag{3.23}$$

where $N_2 = \pm \int \mathscr{F}(S)dS + R_2$ and a_2, b_2 and R_2 are the integration constants. Thus we find the general solutions to the spinor field equations (3.15) containing four arbitrary constants.

Using the solutions obtained, from (2.26) we find the components of spinor current

$$j^0 = [a_1^2 \cosh(2N_1(S)) + a_2^2 \cosh(2N_2(S))]e^{-(\alpha+\chi)}, \tag{3.24a}$$
$$j^1 = 0, \tag{3.24b}$$
$$j^2 = -[a_1^2 \sinh(2N_1(S)) - a_2^2 \sinh(2N_2(S))]e^{-(\alpha+\beta)}, \tag{3.24c}$$
$$j^3 = 0. \tag{3.24d}$$

The supposition (2.27) leads to the following relations between the constants: $a_1 = a_2 = a$ and $R_1 = R_2 = R$, since $N_1(S) = N_2(S) = N(S)$. The chronometric-invariant form of the charge density and the total charge of spinor field are

$$\rho = 2a^2 \cosh(3N(S))e^{-\alpha}, \tag{3.25}$$

$$Q = 2a^2 \int\limits_{-\infty}^{\infty} \cosh(2N(S))e^{\alpha-\chi}dx. \tag{3.26}$$

From (2.31) we find

$$S^{12,0} = 0, \quad S^{13,0} = 0, \quad S^{23,0} = a^2 \cosh(2N(S))e^{-2a}. \tag{3.27}$$

Thus, the only nontrivial component of the spin tensor is $S^{23,0}$ that defines the projection of spin vector on X axis. From (2.33) we write the chronometric invariant spin tensor

$$S^{23,0}_{\rm ch} = a^2 \cosh(2N(S))e^{-\alpha}, \tag{3.28}$$

and the projection of the spin vector on X axis

$$S_1 = a^2 \int\limits_{-\infty}^{\infty} \cosh(2N(S))e^{\alpha-\chi}dx. \tag{3.29}$$

(in (2.34), as well as in (2.29) integrations by y and z are performed in the limit (0,1)). Note that the integrands both in (3.26) and (3.29) coincide.

Let us now analyze the result obtained choosing the nonlinear term in the form $F(I) = \lambda S^n = \lambda I^{n/2}$ with $n \geq 2$ and λ is the parameter of nonlinearity. For $n = 2$ we have Heisenberg-Ivanenko type nonlinear spinor field equation

$$ie^{-\alpha}\bar{\gamma}^1\Big(\partial_x + \frac{1}{2}\alpha')\psi - m\psi + 2\lambda(\bar{\psi}\psi)\psi = 0. \tag{3.30}$$

Setting $F = S^2$ into (3.14) we come to the expression for S that is similar to that for linear case with

$$H^2 \to H_1^2 = B^2 + 3\kappa\lambda C_0 + 3\kappa\varphi_0^2. \tag{3.31}$$

Let us write the functions ψ_j explicitly. In this case we have

$$\mathscr{F}(S) = m(C_0 - 2\lambda S)/S\sqrt{H_1^2 S^2 - M^2 S},$$

and

$$N_{1,2}(x) = (2H_1/3\kappa C_0)\tanh(\bar{H}_1 x) - 2\lambda C_0 x + R_{1,2}, \quad \bar{H}_1 = H_1/2.$$

We can then finally write

$$\begin{aligned}\psi_{1,2}(x) &= ia_{1,2}\frac{\sqrt{3\kappa m C_0}}{H_1}\cosh(\bar{H}_1 x)\cosh N_{1,2}(x)\\ \psi_{3,4}(x) &= ia_{2,1}\frac{\sqrt{3\kappa m C_0}}{H_1}\cosh(\bar{H}_1 x)\cosh N_{2,1}(x)\end{aligned} \tag{3.32}$$

Let us consider the energy-density distribution of the field system:

$$T_0^0 = \Big(\lambda + \frac{1}{2}\frac{\varphi_0^2}{C_0^2}\Big)\frac{M^4}{H_1^4}\cosh^4(\bar{H}_1 x). \tag{3.33}$$

From (3.33) follows that, the energy density of the system is not localized and the total energy of the system

$$E = \int\limits_{-\infty}^{\infty} T_0^0\sqrt{-^3 g}\,dx$$

is not finite. Note that, the energy density of the system can be trivial, if

$$\lambda + \frac{1}{2}\frac{\varphi_0^2}{C_0^2} = 0. \tag{3.34}$$

It is possible, iff the sign of energy density of spinor and scalar fields are different.

Let us write the total charge of the system.

$$Q = 2a^2\int\limits_{-\infty}^{\infty}\cosh\Big[\frac{4H_1}{3\kappa C_0}\tanh(\bar{H}_1 x) - 4\lambda C_0 x + 2R\Big]\Big(\frac{C_0 H_1^2}{M^2\cosh^2(\bar{H}_1 x)}\Big)^{3/2} e^{2Bx/3}dx. \tag{3.35}$$

If $12\lambda^2 C_0^2 + \lambda C_0(4B - \kappa C_0) - \kappa\varphi_0^2/2 < 0$, the integral (3.35) converges, that means the possibility of existence of finite charge and spin of the system.

In case of $n > 2$, the energy density of the system in question is

$$T_0^0 = \lambda(n-1)S^n + \frac{1}{2}\frac{\varphi_0^2}{C_0^2}S^2, \tag{3.36}$$

which shows that the regular solutions with localized energy density exists iff $S = \bar{\psi}\psi$ is a continuous and limited function and $\lim\limits_{x\to\pm\infty} S(x) \to 0$. The condition, when S possesses the properties mentioned above is

$$\int \frac{dS}{\sqrt{(1+\bar{\kappa}/2)B^2S^2 - 3\kappa C_0^2(mS - \lambda S^n)}} = x. \tag{3.37}$$

As one sees from (3.37), for $m \neq 0$ at no value of x, S becomes trivial, since as $S \to 0$, the denominator of the integrand beginning from some finite value of S becomes imaginary. It means that for $S(x)$ to be trivial at spatial infinity ($x \to \infty$), it is necessary to choose massless spinor field setting $m = 0$ in (3.37). Note that, in the unified nonlinear spinor theory of Heisenberg, the massive term is absent, and according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [21]. It should be emphasized that in the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system [22]. Thus without losing the generality we can consider massless spinor field putting $m = 0$. Note that in the sections to follow where we consider the nonlinear spinor term as $F = P^n$, or $F = (K_\pm)^n$ with $K_\pm = (I \pm J)$, we will study the massless spinor field only.

From (3.37) for $m = 0$, $\lambda > 0$ and $n > 2$ for $S(x)$ we obtain

$$S(x) = \Bigl[-H_1/\sqrt{3\kappa\lambda C_0^2(\zeta^2-1)}\Bigr]^{2/(n-2)}, \quad \zeta = \cosh[(n-2)\bar{H}_1 x], \tag{3.38}$$

from which follows that $\lim\limits_{x\to 0}|S(x)| \to \infty$. It means that $T_0^0(x)$ is not bounded at $x = 0$ and the initial system of equations does not possess solutions with localized energy density.

If we set in (3.37) $m = 0$, $\lambda = -\Lambda^2 < 0$ and $n > 2$, then for S we obtain

$$S(x) = \Bigl[H_1/\sqrt{3\kappa\lambda C_0^2\zeta}\Bigr]^{2/(n-2)}. \tag{3.39}$$

It is seen from (3.39) that $S(x)$ has maximum at $x = 0$ and $\lim\limits_{x\to\pm\infty} S(x) \to 0$. For energy density we have

$$T_0^0 = -\Lambda^2(n-1)S^n + \frac{1}{2}\frac{\varphi_0^2}{C_0^2}S^2, \tag{3.40}$$

where S is defined by (3.39). In view of S it follows that $T_0^0(x)$ is an alternating function. Let us find the condition when the total energy of the system is bound

$$E = \int_{-\infty}^{\infty} T_0^0 \sqrt{-^3 g}\, dx < \infty. \tag{3.41}$$

For this we write the integrand of (3.41)

$$\varepsilon(x) = T_0^0 \sqrt{-^3 g} = C_0^{5/3} \Big[\frac{\varphi_0^2}{2C_0^2} - \frac{(n-1)H_1^2 \zeta^2}{3\kappa\lambda C_0^2}\Big]\Big[\frac{H_1^2 \zeta}{3\kappa \Lambda^2 C_0^2}\Big]^{1/3(n-2)} e^{2Bx/3}. \tag{3.42}$$

From (3.42) follows that $\lim\limits_{x\to-\infty} \varepsilon(x) \to 0$ for any value of the parameters, while $\lim\limits_{x\to+\infty} \varepsilon(x) \to 0$ iff $H > 2B$ or $\kappa\varphi_0^2 > 2B^2$. Note that in this case the contribution of scalar field to the total energy in positive and finite:

$$T_{\rm sc0}^0 = \frac{\varphi_0^2}{2C_0^2} S^2, \quad E_{\rm sc} = \int_{-\infty}^{\infty} T_{\rm sc0}^0 \sqrt{-^3 g}\, dx < \infty. \tag{3.43}$$

Note that in the case considered the scalar field is linear and massless. As far as in absence of spinor field energy density of the linear scalar field is not localized and the total energy in not finite, in the case considered the properties of the field configurations are defined by those of nonlinear spinor field. The contribution of nonlinear spinor field to the total energy is negative. Moreover, it remains finite even in absence of scalar field for $n > 2$ [23].

The components of spinor field in this case have the form

$$\begin{aligned}\psi_{1,2}(x) &= i a_{1,2} E(x) \cosh N_{1,2}(x),\\ \psi_{3,4}(x) &= a_{2,1} E(x) \sinh N_{2,1}(x),\end{aligned} \tag{3.44}$$

where

$$E(x) = \frac{1}{\sqrt{C_0}} \Big[H_1 / \sqrt{3\kappa\Lambda^2 C_0^2 \zeta}\Big]^{1/(n-2)}$$

and

$$N_{1,2}(x) = -\frac{2nH_1\sqrt{\zeta^2 - 1}}{3\kappa C_0 (n-2)\zeta} + R_{1,2}.$$

For the solutions obtained we write the chronometric-invariant charge density of the spinor field ρ:

$$\rho(x) = \frac{2a^2}{C_0} \cosh\Big(-\frac{4nH_1\sqrt{\zeta^2-1}}{3\kappa C_0(n-2)\zeta} + 2R\Big)\Big(\frac{H_1}{3\kappa\Lambda^2 C_0^2 \zeta^2}\Big)^{1/(n-2)}. \tag{3.45}$$

As one sees from (3.45), the charge density is localized, since $\lim\limits_{x\to\pm\infty} \rho(x) \to 0$. Nevertheless, the charge density of the spinor field, coming to unit invariant volume

$\rho\sqrt{^{-3}g}$, is not localized:

$$\rho\sqrt{^{-3}g} = 2a^2 \cosh[2N(x)]e^{\alpha-\gamma} = 2a^2 \cosh[2N(x)](C_0/S)^{2/3}e^{2Bx/3}. \quad (3.46)$$

It leads to the fact that the total charge of the spinor field is not bounded as well. As far as the expression for chronometric-invariant tensor of spin (3.28) coincides with that of p(jc)/2, the conclusions made for $\rho(x)$ and Q will be valid for the spin tensor $S_{\rm ch}^{23,0}$ and projection of spin vector on X axis S_1, i.e., $S_{\rm ch}^{23,0}$ is localized and S_1 is unlimited.

The solution obtained describes the configuration of nonlinear spinor and linear scalar fields with localized energy density but with the metric that is singular at spatial infinity, as in this case

$$e^{2\alpha} = (C_0/S)^2 = C_0^2\Bigl(\frac{3\kappa\Lambda C_0^2\zeta}{H_1^2}\Bigr)^{2/(n-2)}\Bigr|_{x\to\pm\infty} \to \infty. \quad (3.47)$$

Let us consider the massless spinor field with

$$F = -\Lambda^2 S^{-\nu}, \quad \nu = \text{const.} > 0. \quad (3.48)$$

In this case the energy density of the system of nonlinear spinor and linear scalar fields with minimal coupling takes the form

$$T_0^0 = \Lambda^2(\nu+1)S^{-\nu} + \frac{\varphi_0^2}{2C_0^2}S^2. \quad (3.49)$$

For S in this case we get

$$\int \frac{dS}{\sqrt{(1+\bar{\kappa}/2)B^2S^2 - 3\kappa C_0^2\Lambda^2 S^{-\nu}}} = x \quad (3.50)$$

with the solution

$$S(x) = \Bigl[\frac{3\kappa\Lambda^2 C_0^2}{H_1^2}\zeta_1^2\Bigr]^{\nu/(\nu+2)}, \quad \zeta_1 = \cosh[(\nu+2)\bar{H}_1 x]. \quad (3.51)$$

For energy density in this case we have

$$T_0^0(x) = \Lambda^2(\nu+1)\Bigl[\frac{H_1^2}{3\kappa C_0^2\Lambda^2\zeta_1^2}\Bigr]^{\nu/(\nu+2)} + \frac{\varphi_0^2}{2C_0^2}\Bigl[\frac{3\kappa C_0^2\Lambda^2\zeta_1^2}{H_1^2}\Bigr]^{2/(\nu+2)}. \quad (3.52)$$

It follows from (3.52) that the contribution of the spinor field in the energy density is localized while for the scalar field it is not the case.

The energy density distribution of the field system, coming to unit invariant volume is

$$\begin{aligned}\varepsilon(x) = T_0^0\sqrt{^{-3}g} &= \Bigl[\Lambda^2(\nu+1)S^{-\nu} + \frac{\varphi_0^2}{2C_0^2}S^2\Bigr]e^{2\alpha-\gamma}\\ &= \Bigl(\frac{H_1^2(\nu+1)}{3\kappa\zeta_1^2} + \frac{\varphi_0^2}{2}\Bigr)\Bigl(\frac{H_1^2}{3\kappa C_0^2\Lambda^2\zeta_1^2}\Bigr)^{1/3(\nu+2)}e^{2Bx/3}. \quad (3.53)\end{aligned}$$

As one sees from (3.53) $\varepsilon(x)$ is a localized function, i.e., $\lim_{x\to\pm\infty} \varepsilon(x) \to 0$, if $H > 2B$ or $\kappa\varphi_0^2 > 2B^2$. In this case the total energy is also finite.

The components of spinor field in this case have the form

$$\begin{aligned} \psi_{1,2} &= ia_{1,2}E(x)\cosh N_{1,2}(x), \\ \psi_{3,4} &= a_{2,1}E(x)\sinh N_{2,1}(x), \end{aligned} \tag{3.54}$$

where

$$E(x) = \frac{1}{\sqrt{C_0}}\Big[\frac{\sqrt{3\kappa\Lambda^2 C_0^2}}{H_1^2}\zeta_1\Big]^{1/(\nu+2)}$$

and

$$N_{1,2}(x) = -\frac{2H\nu\sqrt{\zeta_1^2-1}}{3\kappa C_0(\nu+2)\zeta_1} + R_{1,2}.$$

The chronometric-invariant charge density of the spinor field coming to unit invariant volume with $a_1 = a_2 = a$ and $N_1 = N_2$ reads

$$\begin{aligned} &\rho\sqrt{-^3g} = 2a^2\cosh[2N(x)]e^{\alpha-\gamma} \\ &= 2a^2(C_0)^{2/3}\cosh\Big(2R - \frac{4H_1\nu\sqrt{\zeta_1^2-1}}{3\kappa C_0(\nu+2)\zeta_1}\Big)\Big(\frac{H_1^2}{3\kappa C_0^2\Lambda^2\zeta_1^2}\Big)^{2/3(\nu+2)}e^{2Bx/3}. \end{aligned} \tag{3.55}$$

It follows from (3.55) that $\rho\sqrt{-^3g}$ is a localized function and the total charge Q is finite. The spin of spinor field is limited as well.

Case II: $F = F(J)$. Here we consider the massless spinor field with the nonlinearity $F = F(J)$. In this case from (2.35b) immediately follows

$$P = D_0\exp[-\alpha(x)], \quad D_0 = \text{const.} \tag{3.56}$$

From (2.23) we now have

$$V_4' - \mathscr{E}V_3 = 0, \tag{3.57a}$$
$$V_3' - \mathscr{E}V_4 = 0, \tag{3.57b}$$
$$V_2' + \mathscr{E}V_1 = 0, \tag{3.57c}$$
$$V_1' + \mathscr{E}V_2 = 0, \tag{3.57d}$$

where we denote $\mathscr{E} = e^{\alpha}\mathscr{G}$. From (3.57d) and (3.57c) we obtain

$$V''_1 - \frac{\mathscr{E}'}{\mathscr{E}}V'_1 - \mathscr{E}^2V_1 = 0, \tag{3.58}$$

with the solution

$$V_1(x) = D_1\exp\Big(\int\mathscr{E}dx\Big) + iD_2\exp\Big(-\int\mathscr{E}dx\Big). \tag{3.59}$$

Inserting (3.59) into (3.57d) we obtain

$$V_2(x) = -D_1 \exp\Big(\int \mathscr{E} dx\Big) + iD_2 \exp\Big(-\int \mathscr{E} dx\Big). \tag{3.60}$$

Analogically, from (3.57b) and (3.57a) we get

$$V_3(x) = D_3 \exp\Big(\int \mathscr{E} dx\Big) + iD_4 \exp\Big(-\int \mathscr{E} dx\Big), \tag{3.61}$$

$$V_4(x) = -D_3 \exp\Big(\int \mathscr{E} dx\Big) + iD_4 \exp\Big(-\int \mathscr{E} dx\Big). \tag{3.62}$$

Here D_1, D_2, D_3 and D_4 are the integration constants obeying

$$4(D_1 D_2 - D_3 D_4) = D_0.$$

Using the solutions obtained and noticing that

$$V_1 = -V_2^*, \quad V_2 = -V_1^*, \quad V_2 = -V_4^*, \quad V_4 = -V_3^*,$$

from (2.26) we now find the components of spinor current

$$j^0 = 2\Big[(D_1^2 + D_3^2)\exp\Big(2{\textstyle\int}\mathscr{E} dx\Big) + (D_2^2 + D_4^2)\exp\Big(-2{\textstyle\int}\mathscr{E} dx\Big)\Big]e^{-(\alpha+\chi)}, \tag{3.63a}$$

$$j^1 = -4\Big[D_1 D_3 \exp\Big(2{\textstyle\int}\mathscr{E} dx\Big) - D_2 D_4 \exp\Big(-2{\textstyle\int}\mathscr{E} dx\Big)\Big]e^{-2\alpha}, \tag{3.63b}$$

$$j^2 = 4[D_1 D_4 + D_2 D_3]\exp[-(\alpha+\beta)], \tag{3.63c}$$

$$j^3 = 0. \tag{3.63d}$$

The supposition (2.27) that the spatial components of the spinor current are trivial leads to the fact that one of the pairs (D_1, D_2), (D_3, D_4) is trivial, i.e., in this case we have two of the four components of the spinor field have trivial solutions. Setting $D_3 = D_4 = 0$, we obtain the chronometric-invariant form of the charge density and the total charge of spinor field

$$\rho = 2\Big[D_1^2 \exp\Big(2\int \mathscr{E} dx\Big) + D_2^2 \exp\Big(-2\int \mathscr{E} dx\Big)\Big]e^{-\alpha} \tag{3.64}$$

$$Q = 2\int_{-\infty}^{\infty}\Big[D_1^2 \exp\Big(2\int \mathscr{E} dx\Big) + D_2^2 \exp\Big(-2\int \mathscr{E} dx\Big)\Big]e^{\alpha-\chi} dx. \tag{3.65}$$

From (2.31) we find

$$S^{12,0} = 0, \tag{3.66a}$$

$$S^{23,0} = -2\Big[D_1^2 \exp\Big(2\int \mathscr{E} dx\Big) - D_2^2 \exp\Big(-2\int \mathscr{E} dx\Big)\Big]e^{-(\alpha+2\beta+\chi)}, \tag{3.66b}$$

$$S^{31,0} = 4iD_1 D_2 e^{-(2\alpha+\beta+\chi)}. \tag{3.66c}$$

Thus, in this case we have two nontrivial components of the spin tensor $S^{23,0}$ and $S^{31,0}$. Those define the projections of spin vector on X and Y axis, respectively. From (2.33) we write the chronometric invariant spin tensor

$$S^{23,0}_{\rm ch} = -2\Big[D_1^2 \exp\Big(2\int \mathscr{E} dx\Big) - D_2^2 \exp\Big(-2\int \mathscr{E} dx\Big)\Big] e^{-(\alpha+\chi)}, \quad (3.67a)$$

$$S^{31,0}_{\rm ch} = 4iD_1D_2 e^{-(\alpha+\chi)}. \quad (3.67b)$$

and the projections of the spin vector on X and Z axes are

$$S_1 = -2\int_{-\infty}^{\infty}\Big[D_1^2 \exp\Big(2\int \mathscr{E} dx\Big) - D_2^2 \exp\Big(-2\int \mathscr{E} dx\Big)\Big] e^{\alpha-2\chi)} dx, \quad (3.68a)$$

$$S_2 = 4iD_1D_2 \int_{\infty}^{\infty} \exp(\alpha - 2\chi) dx. \quad (3.68b)$$

Let us choose $F = \lambda P^n = \lambda J^{n/2}$. In this case for energy density we get

$$T_0^0 = \lambda(n-1)P^n + \frac{1}{2}\frac{\varphi_0^2}{2D_0^2}P^2. \quad (3.69)$$

From (3.69) it follows that the regular solution with localized energy density exists iff P is continuous and finite and $\lim\limits_{x\to\pm\infty} P(x) \to 0$. For P from (2.43) in this case we have

$$\int \frac{dP}{\sqrt{B^2P^2 + 3\kappa D_0^2(\lambda P^n + \varphi_0^2 P^2/2D_0^2)}} = x. \quad (3.70)$$

As one sees from (3.70), at spatial infinity $P(x)$ becomes trivial.

Case III: $F = F(K_\pm)$. Here we consider the massless spinor field with the nonlinearity $F = F(K_\pm)$, where $K_\pm = I \pm J = S^2 \pm P^2$. In this case from (2.35) we obtain

$$K_\pm = K_0 \exp[-2\alpha(x)], \quad K_0 = \text{const}. \quad (3.71)$$

Setting $F = \lambda K_\pm^n$ for $K_\pm$ from (2.43) in this case we obtain

$$\int \frac{dK_\pm}{\sqrt{B^2K_\pm^2 + 3\kappa K_0(\lambda K_\pm^{(n-1)} + \varphi_0^2 K_\pm^2/2K_0)}} = 2x. \quad (3.72)$$

Energy density in the case considered has the form

$$T_0^0 = (2n-1)\lambda K_\pm^n + \frac{1}{2}\frac{\varphi_0^2}{K_0}K_\pm. \quad (3.73)$$

From (3.73) follows that the initial system has regular solutions with localized energy density iff $K_\pm$ is continuous and finite and $K_\pm(x)|_{x\to\infty} \to 0$. As in previous case, from (3.72) we conclude that $K_{\pm}(x)|_{x\to\infty} \to 0$. Contrary to cases, when $F = F(I)$ or $F = F(J)$, the equations for spinor field in this case are rather complicated. In this case we have $\Phi = -2n\lambda K_{\pm}^{n-1} S$ and $\mathscr{G} = \pm 2n\lambda K_{\pm}^{n-1} P$. Noticing that $K_{\pm} = S^2 \pm P^2 = K_0$ one can redefine S and P as $\cos(\zeta)[\cosh(\zeta)]$ and $\sin(\zeta)[\sinh(\zeta)]$. Inserting them into (2.23) one finds the corresponding equations for spinor field. A detailed analysis of an analogous system can be found in [24].

3.3 Nonlinear Scalar Field in Absence of Spinor One

Let us consider the system of gravitational and nonlinear scalar fields. As a nonlinear scalar field equation we choose Born-Infeld one, given by the Lagrangian [19]

$$\Psi(\Omega) = -\frac{1}{\sigma}(1 - \sqrt{1 + \sigma\Omega}), \tag{3.74}$$

with $\Omega = \varphi_\alpha \varphi^\alpha$ and σ is the parameter of nonlinearity. From (3.74) we also have

$$\lim_{\sigma\to 0} \Psi(\Omega) = \frac{1}{2}\Omega \cdots \tag{3.75}$$

Inserting (3.74) into (2.20) for the scalar field we obtain the equation

$$\varphi'(x) = \frac{\varphi_0}{\sqrt{1 + \sigma\varphi_0^2 e^{-2\alpha(x)}}}, \tag{3.76}$$

that gives

$$\Omega = -(\varphi')^2 e^{-2\alpha} = \frac{\varphi_0^2 e^{-2\alpha(x)}}{1 + \sigma\varphi_0^2 e^{-2\alpha(x)}}. \tag{3.77}$$

From (3.76) follows that $\varphi'\Big|_{\sigma=0} = \varphi_0$.

For the case considered in this section we have

$$T_{\rm sc0}^0 = T_{\rm sc2}^2 = T_{\rm sc3}^3 = -\Psi(\Omega) = \frac{1}{\sigma}\Big(1 - \frac{1}{\sqrt{1 + \sigma\varphi_0^2 e^{-2\alpha(x)}}}\Big), \tag{3.78}$$

and

$$T_{\rm sc1}^1 = 2\Omega\frac{d\Psi}{d\Omega} - \Psi = \frac{1}{\sigma}\Big(1 - \sqrt{1 + \sigma\varphi_0^2 e^{-2\alpha(x)}}\Big). \tag{3.79}$$

Putting (3.79) into (2.43), in account of $m = 0$ and $F(I,J) = 0$ for α we find

$$\alpha' = \pm\sqrt{B^2 - \frac{3\kappa}{\sigma}e^{2\alpha}\Big(1 - \sqrt{1 + \sigma\varphi_0^2 e^{-2\alpha(x)}}\Big)}. \tag{3.80}$$

From (3.80) one finds

$$\int \frac{d\alpha}{\sqrt{B^2 - \frac{3\kappa}{\sigma}e^{2\alpha}\Big(1 - \sqrt{1 + \sigma\varphi_0^2 e^{-2\alpha(x)}}\Big)}} = -\frac{2}{B}\ln|\xi + \sqrt{\bar\kappa + \xi^2}|$$
$$+ \frac{1}{B\sqrt{1 + \bar\kappa/2}}\Big[\ln|\sqrt{2}B\sqrt{\bar\kappa + \xi^2} + \sqrt{2}B\sqrt{1 + \bar\kappa\xi/2}|$$
$$- \ln|\sqrt{3\kappa\varphi_0^2(\xi^2 - 2)}\,|\Big] = x \tag{3.81}$$

with

$$\xi^2 = 1 + \sqrt{1 + \sigma\varphi_0^2 e^{-2\alpha(x)}}.$$

As one sees from (3.81)

$$e^{2\alpha(x)}\Big|_{x\to+\infty} \approx \frac{\sigma\varphi_0^2}{2}\exp(2\sqrt{1+\bar{\kappa}/2Bx}) \to \infty, \tag{3.82}$$

$$e^{2\alpha(x)}\Big|_{x\to-\infty} \approx \frac{\sigma\varphi_0^2}{2}\exp(2Bx) \to 0. \tag{3.83}$$

Let us study the energy density distribution of nonlinear scalar field. From (3.78) we find

$$T_{\rm sc0}^0(x)\Big|_{x=-\infty} = \frac{1}{\sigma}, \quad T_{\rm sc0}^0(x)\Big|_{x=\infty} = 0, \tag{3.84}$$

which shows that the energy density of the scalar field is not localized. Nevertheless, the energy density on unit invariant volume is localized if $\kappa\varphi_0^2 > 2B^2$:

$$\varepsilon(x) = T_{\rm sc0}^0\sqrt{-^3g} = \frac{1}{\sigma}\Big(1 - \frac{1}{1+\sigma\varphi_0^2 e^{-2\alpha}}\Big)e^{5\alpha/3+2Bx/3}\Big|_{x\to\pm\infty} \to 0. \tag{3.85}$$

In this case the total energy of the scalar field is also bound. From (3.77) in account of (3.82) and (3.83) we also have

$$\Omega(x)\Big|_{x=-\infty} = -\frac{1}{\sigma}, \quad \Omega(x)\Big|_{x=+\infty} = 0, \tag{3.86}$$

showing that $\Omega(x)$ is kink-like.

3.4 Nonlinear Spinor and Nonlinear Scalar Field

Finally we consider the self-consistent system of nonlinear spinor and scalar fields. We choose the self-action of the spinor field as $F = \lambda S^n$, $n > 2$, where as the scalar field is taken in the form (3.74). Using the line of reasoning mentioned earlier, we conclude that the spinor field considered here should be massless. Taking into account that $\exp(-2\alpha) = S^2C_0^2$ for S we write

$$\int \frac{dS}{\sqrt{B^2 + 3\kappa C_0^2\Big[\lambda S^n + \Big(\sqrt{1+\sigma\varphi_0^2 S^2/C_0^2} - 1\Big)/\sigma\Big]}} = x. \tag{3.87}$$

From (3.87) one estimates

$$S(x)\Big|_{x\to 0} \sim \frac{1}{x^{2/(n-2)}} \to \infty. \tag{3.88}$$

On the other hand for the energy density we have

$$T_0^0 = \lambda(n-1)S^n + \frac{1}{\sigma}\Big(1 - \frac{1}{\sqrt{1+\sigma\varphi_0^2 S^2/C_0^2}}\Big) \tag{3.89}$$

that states that for T_0^0 to be localized 5 should be localized too and $\lim\limits_{x\to\pm\infty} S(x) \to 0$. Hence from (3.88) we conclude that $S(x)$ is singular and energy density in unlimited at $x = 0$.

For $\lambda = -\Lambda^2$ and $n > 2$ we have

$$\int \frac{dS}{\sqrt{B^2 + 3\kappa C_0^2\Big[-\Lambda^2 S^n + \Big(\sqrt{1+\sigma\varphi_0^2 S^2/C_0^2} - 1\Big)/\sigma\Big]}} = x. \tag{3.90}$$

In this case $S(x)$ is finite and its maximum value is defined from

$$S^n(x) = \frac{1}{3\kappa C_0^2\Lambda^2}\Big[B^2 S^2 + 3\kappa C_0^2 \frac{\sqrt{1+\sigma\varphi_0^2 S^2/C_0^2} - 1}{\sigma}\Big]. \tag{3.91}$$

Noticing that at spatial infinity effects of nonlinearity vanish, from (3.90) we find

$$S(x)\Big|_{x\to-\infty} \sim e^{Hx} \to 0, \quad S(x)\Big|_{x\to+\infty} \sim e^{-Hx} \to 0, \tag{3.92}$$

with $H = \sqrt{B^2 + 3\kappa\varphi_0^2/2} = B\sqrt{1+\bar{\kappa}/2}$. In this case the energy density T_0^0 defined by (3.89) is localized and the total energy of the system in bound. Nevertheless, spin and charge of the system unlimited.

Let us go back to the general case. For $F = F(S)$ we now have

$$T_1^1 = mS - F(S) + 2\Omega\frac{d\Psi}{d\Omega} - \Psi. \tag{3.93}$$

It follows that for the arbitrary choice of $\Psi(\Omega)$, obeying (2.4), we can always choose nonlinear spinor term that will eliminate the scalar field contribution in T_1^1, i.e., by virtue of total freedom we have here to choose $F(S)$, we can write

$$F(S) = F_1(S) + F_2(S), \quad F_2(S) = 2\Omega\frac{d\Psi}{d\Omega} - \Psi, \tag{3.94}$$

since $\Omega = \Omega(S^2)$. To prove this we go back to (2.20) that gives

$$\Omega\Big(\frac{d\Psi}{d\Omega}\Big)^2 = -\frac{\varphi_0^2 S^2}{C_0^2}. \tag{3.95}$$

Since Ψ is the function of Ω only, (3.95) comprises an algebraic equation for defining Ω as a function of S^2. For (3.94) takes place, we find

$$(\alpha')^2 - B^2 = -\frac{3\kappa C_0^2}{S^2}[mS - F_1(S)]. \tag{3.96}$$

As we see, the scalar field has no effect on space-time, but it contributes to energy density and total energy of the system as in this case

$$T_0^0 = SF'_1(S) - F_1(S) + S\frac{d}{d\Omega}\Big(-2\Omega\frac{d\Psi}{d\Omega} + \Psi\Big)\frac{d\Omega}{dS} + 2\Omega\frac{d\Psi}{d\Omega} - \Psi. \tag{3.97}$$

Note that in (3.93) with $F(S)$ arbitrary, we cannot choose $\Psi(\Omega)$ such that

$$2\Omega\frac{d\Psi}{d\Omega} - \Psi = F(S), \tag{3.98}$$

due to the fact that $\Psi(\Omega)$ is not totally arbitrary, since it has to obey

$$\lim_{\Omega\to 0}\Psi(\Omega) \to \frac{1}{2}\Omega, \quad \lim_{\Omega\to 0} 2\Omega\frac{d\Psi}{d\Omega} - \Psi = \frac{1}{2}\Omega = \frac{\varphi_0^2}{2C_0^2}S^2, \tag{3.99}$$

whereas at $S \to 0$, $F(S)$ behaves arbitrarily.

4 Conclusion

The system of nonlinear spinor and nonlinear scalar fields with minimal coupling has been thoroughly studied within the scope of general relativity given by a plane-symmetric space-time. Contrary to the scalar field, the spinor field nonlinearity has direct effect on space-time. Energy density and the total energy of the linear spinor and scalar field system are not bounded and the system does not possess real physical infinity, hence the configuration is not observable for an infinitely remote observer, since in this case

$$R = \int\limits_{-\infty}^{\infty} \sqrt{g_{11}}\,dx = \int\limits_{-\infty}^{\infty} e^{\alpha}dx = \frac{4C_0H}{M^2} < \infty. \tag{4.1}$$

But introduction of nonlinear spinor term into the system eliminates these shortcomings and we have the configuration with finite energy density and limited total energy which is also observable as in this case the system possesses real physical infinity. Thus we see, spinor field nonlinearity is crucial for the regular solutions with localized energy density. We also conclude that the properties of nonlinear spinor and scalar field system with minimal coupling are defined by that part of gravitational field which is generated by nonlinear spinor one.

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