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# THE SINGLE PARTICLE SCHRÖDINGER FLUID AND MOMENTS OF INERTIA OF DEFORMED NUCLEI

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**Abstract.** We have applied the theory of the single-particle Schroedinger fluid to the nuclear collective motion of axially deformed nuclei. A counter example of an arbitrary number of independent nucleons in the anisotropic harmonic oscillator potential at the equilibrium deformation has been also given. Moreover, the ground states of the doubly even nuclei in the s-d shell $^{20}$Ne, $^{24}$Mg, $^{28}$Si, $^{32}$S and $^{36}$Ar are constructed by filling the single-particle states corresponding to the possible values of the number of quanta of excitations $n_x$, $n_y$, and $n_z$. Accordingly, the cranking-model, the rigid-body model and the equilibrium-model moments of inertia of these nuclei are calculated as functions of the oscillator parameters $\sim \omega_x$, $\sim \omega_y$ and $\sim \omega_z$ which are given in terms of the non deformed value $\sim \omega_0^0$ depending on the mass number $A$ the number of neutrons $N$, the number of protons $Z$, and the deformation parameter $\beta$. The calculated values of the cranking-model moments of inertia of these nuclei are in good agreement with the corresponding experimental values and show that the considered axially deformed nuclei may have oblate as well as prolate shapes and that the nucleus $^{24}$Mg is the only one which is highly deformed. The rigid-body model and the equilibrium-model moments of inertia of the two nuclei $^{20}$Ne and $^{24}$Mg are also in good agreement with the corresponding experimental values.

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## 1 Introduction

It is well known that the shell model explains many nuclear properties, but fails to account the large nuclear quadrupole moments and spheroidal shapes which many nuclei posses. It is also clear that such effects cannot be obtained from any model which considers the pairwise filling of the individual orbits of spherical potential to be a good approximation to nuclear structure. Such large effects can only arise from coordinates motion of many nucleons. We may characterize such motion by assuming that the particle motion and the surface motion are couples.

Because the surface is distorted at some moment, the potential felt by a particle is not spherically symmetric, the particles will move in orbits appropriate to an aspherical shell-model potential. To express the particle-surface coupling mathematically, it is necessary to introduce some collective variables to describe the cooperative modes of motion. The simpler model has sometimes been called the collective model, and the distorted shell model the unified model.

The nuclear collective rotation [1] is a topic of the nuclear structure theory some fifty years old which has grown steadily both in the sophistication of its theory and in the range of data to which it relates. The most central parameter of collective rotation is the moment of inertia of deformed nuclei [2–5]. Consequently, the investigation of the nuclear moments of inertia is a sensitive check for the validity of the nuclear structure theories.

The quantum fluid [6] is considered to be completely transparent internally with respect to motion of the constituent particles, and to receive disturbances solely by way of surface deformations. Its near incompressibility comes about, not by particle to particle push, as in an ordinary liquid, but by more subtle means. It is capable of collective oscillations, but it is the wall which organizes these disturbances, not nucleon to nucleon interactions. Oscillations experience a damping, but the mechanism of the damping is unlike that encountered in ordinary liquids. The rotational properties of the quantum fluid are quite different from those of ordinary fluids.

Moreover, the study of the velocity fields for the rotational motion led to the formulation of the so-called the Schroedinger fluid [7]. Since the Schroedinger fluid theory is at present an independent particle model, the cranking model approximation for the velocity fields and the moments of inertia play the dominant role in this theory.

In the present paper we have applied the theory of the single particle Schroedinger fluid to the nuclear collective motion and to the calculations of the nuclear moments of inertia. Also, an example was given for an arbitrary number of independent particles in the anisotropic harmonic oscillator potential at the equilibrium deformation. Moreover, the moments of inertia of the doubly even axially deformed nuclei in the s-d shell: $^{20}$N, $^{24}$Mg, $^{28}$Si, $^{32}$S and $^{36}$Ar are calculated according to the concepts of the single particle Schroedinger fluid for both of the cranking model and the rigid body model. The equilibrium moments of inertia of these nuclei are also calculated.

## 2 The Schroedinger Fluid

The polar form of the time-dependent $K^{\text{th}}$-single particle wave function is given by [8]

$$\Psi(\mathbf{r},\alpha(t),t) = \Phi(\mathbf{r},\alpha(t)) \exp\left\{ -i\frac{M}{\hbar} S_K(\mathbf{r},\alpha(t)) - \frac{i}{\hbar}\int_0^t \varepsilon_K(\alpha(t'))dt' \right\}, \quad (1)$$

where $\alpha$ represents some time-dependent collective parameters, $S$ is a real function and $\Phi$ is a positive real function. In the case of rotation, the parameter $\alpha$ becomes the angle of rotation, $\theta$. The single-particle Hamiltonian $H$ is $\alpha$ dependent through its potential and the time-dependent Schroedinger equation

$$H(\mathbf{r}, \mathbf{p}, \alpha(t))\Psi(\mathbf{r}, \alpha(t), t) = i\hbar\frac{\partial}{\partial t}\Psi_K(\mathbf{r}, \alpha(t), t), \tag{2}$$

can be separated into real and imaginary parts, by using Eq. (1), and as a result two equations are obtained. The first is the continuity equation

$$\rho\nabla.\mathbf{v} + \mathbf{v}.\nabla\rho = -\frac{\partial\rho}{\partial t}, \tag{3}$$

where the density $\rho = \Phi^2$ and the irrotational velocity field $\mathbf{v}$ is defined by

$$\mathbf{v} = -\nabla S, \tag{4}$$

$$S = \frac{i\hbar}{2M}\ln(\Psi/\Psi^*), \tag{5}$$

The second equation is

$$(H + V_{\text{dyn}})\phi_i = \varepsilon_i\phi_i, \tag{6}$$

which is a modified Schroedinger equation through the modified dynamical potential

$$V_{\text{dyn}} = -M\left(\frac{\partial S}{\partial t} - \frac{1}{2}\mathbf{v}^2\right). \tag{7}$$

In addition to the irrotational velocity field $\mathbf{v}$, which has been result from the fluid dynamical equation, other velocity fields which satisfy the continuity equation of the Schroedinger equation occur. Among these velocity fields are the incompressible velocity field, the regular velocity field, the geometric velocity field and the rigid body velocity field. For rotations, the rigid body velocity field $\mathbf{v}_{\text{rig}}$ is defined by

$$\mathbf{v}_{\text{rig}} = \Omega \times \mathbf{r}. \tag{8}$$

It is seen that this velocity field is incompressible, regular and also of a geometric type.

In the adiabatic approximation where, $\frac{\partial\alpha}{\partial t} \to 0$, the collective kinetic energy of a nucleon in the nucleus is given by [8]

$$T_K = \frac{1}{2}M\int \rho_K\mathbf{v}_T.(\Omega \times \mathbf{r})d\tau, \tag{9}$$

and the collective kinetic energy $T$ of the nucleus is given by

$$T = \frac{1}{2}M\int \rho_T\mathbf{v}_T.(\Omega \times \mathbf{r})d\tau, \tag{10}$$

where $\rho_T$ is the total density distribution of the nucleus and $\mathbf{v}_T$ is the total velocity field

$$v_T = \frac{\sum\limits_{K=occ} \rho_k \mathbf{v}_K}{\sum\limits_{K=occ} \rho_k} \tag{11}$$

## 3 Single Nucleon in the Harmonic Oscillator Potential

The single particle wave functions for a nucleon in the average harmonic oscillator potential of the nucleus are given in the form of products of the three one-dimensional oscillator functions given, in the usual notations, by

$$U_{n_x n_y n_z}(x, y, z) = U_{n_x}(\xi) U_{n_y}(\eta) U_{n_z}(\zeta), \tag{12}$$

where

$$U_{n_x}(\xi) = \left\{ 2^{n_x} n_x! \sqrt{\frac{\pi\hbar}{M\omega_x}} \right\}^{-1/2} \exp\left(\frac{1}{2}\xi^2\right) H_{n_x}(\xi), \tag{13}$$

and $n_x = 0, 1, 2, \ldots$ .

Similar equations hold for $U_{n_y}(\eta)$ and $U_{n_z}(\zeta)$. In Eq. (13) $H_{n_x}(\xi)$ is the Hermite polynomial of degree $n_x$ and the dimensionless variables $\xi$, $\eta$ and $\zeta$ are defined by

$$\xi = \sqrt{\frac{M\omega_x}{\hbar}} x, \ etc. \tag{14}$$

If the $z$-axis is an axis of symmetry, so that $\omega_x = \omega_y$, the intrinsic energy of the single particle state is given by

$$\varepsilon_{n_x, n_y, n_z} = \hbar\omega_x(n_x + n_y + 1) + \hbar\omega_z(n_z + \frac{1}{2}) \tag{15}$$

In the adiabatic approximation the $K^{\text{th}}$ single particle wave function is approximated by a sum of two functions one of which is real and the other is imaginary. The first function, the quasi-static wave function, which is the real part of the wave function satisfies the quasi-static Schroedinger wave equation and the second function, the imaginary part, is the first-order time-dependent perturbation correction to the wave function and is given for rotation about the $z$-axis by [8]

$$\mu_K = \Omega \sum_{j=K} \frac{\langle j|L_z|K\rangle}{\varepsilon_j - \varepsilon_K} U_j, \tag{16}$$

where $L_z$ is the $z$-component of the single-particle orbital angular momentum. We can calculate the cranking correction to the wave function explicitly, obtaining

$$\mu_{n_x,n_y,n_z}(x,y,z) = U_{n_x}(\xi)\mu_{n_y n_z}(y,z) =$$
$$= -\frac{\Omega}{2\sqrt{\omega_y\omega_z}}U_{n_x}\left\{\sigma\sqrt{n_y n_z}U_{n_y-1}U_{n_z-1} + \frac{1}{\sigma}\sqrt{n_y(n_z+1)}U_{n_y-1}U_{n_z+1}\right.$$
$$\left. + \frac{1}{\sigma}\sqrt{(n_y+1)n_z}U_{n_y+1}U_{n_z-1} + \sigma\sqrt{(n_y+1)(n_z+1)}U_{n_y+1}U_{n_z+1}\right\}. \quad (17)$$

The functions with subscripts $n_x$, $n_y$, and $n_z$ are of arguments $\xi$,$\eta$ and $\zeta$, respectively, and

$$\sigma = \frac{\omega_y - \omega_z}{\omega_y + \omega_z}, \quad (18)$$

is a measure of the deformation of the potential.

We introduce one single parameter of deformation $\delta$ defined by [9]

$$\omega_x^2 = \omega_0^2\left(1 + \frac{2\delta}{3}\right) = \omega_y^2, \quad (19)$$

$$\omega_z^2 = \omega_0^2\left(1 - \frac{4\delta}{3}\right). \quad (20)$$

The condition of constant volume of the nucleus leads to

$$\omega_x\omega_y\omega_z = \text{const.} \quad (21)$$

Keeping this condition in the general case together with (19) and (20), $\omega_0$ has to depend on $\delta$ in the following way [9]

$$\omega_0 = \omega_0(\delta) = \omega_0^0\left(1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right)^{-1/6}, \quad (22)$$

where $\omega_0^0$ is the value of $\omega_0(\delta)$ for $\delta = 0$. The value of the oscillator parameter $\hbar\omega_0^0$ for nuclei with mass number $A$, number of neutrons $N$ and number of protons $Z$ is given by [10]

$$\hbar\omega_0^0 = 38.6A^{-1/3} - 127.0A^{-4/3} + 14.75A^{-4/3}(N-Z). \quad (23)$$

Another choice of the deformation parameter is defined as follows [9]

$$\delta = \frac{3}{2}\sqrt{\frac{5}{4\pi}}\beta \approx 0.95\beta. \quad (24)$$

The parameter $\beta$ is allowed to vary in the range $-0.50 \leq \beta \leq 0.50$.

## 4 Cranking Model And Rigid Body Moments of Inertia

It is well known that the cranking model moment of inertia is defined by [11]

$$\mathfrak{J}_{\text{cr}} = 2M\hbar^2 \sum_{j=i} \frac{|\langle j|L_x|i\rangle|^2}{\varepsilon_j - \varepsilon_i}. \tag{25}$$

We now examine the cranking model moment of inertia in terms of the velocity fields. Bohr and Mottelson [1] show that for the harmonic oscillator case at the equilibrium deformation, where

$$\frac{d}{d\sigma}\sum_{i=1}^{A}(E_{n_x n_y n_z})_i = 0 \tag{26}$$

and $A$ is the mass number, the cranking model moment of inertia is identically equal to the rigid body moment of inertia

$$\mathfrak{J}_{\text{cr}} = \mathfrak{J}_{\text{rig}} = \sum_{i=1}^{A} M\langle y_i^2 + x_i^2\rangle. \tag{27}$$

We note that the cranking model moment of inertia $\mathfrak{J}_{\text{cr}}$ and the rigid body moment of inertia $\mathfrak{J}_{\text{rig}}$ are equal only when the harmonic oscillator is at the equilibrium deformation. At other deformations, they can, and do, deviate substantially from one another [8]

The following expressions for the cranking model moment of inertia $\mathfrak{J}_{\text{cr}}$ and the rigid body moment of inertia $\mathfrak{J}_{\text{rig}}$ can be easily obtained [8]

$$\mathfrak{J}_{\text{cr}} = \frac{E}{\omega_0^2}\frac{1}{6+2\sigma}\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2}\left[\sigma^2(1+q) + \frac{1}{\sigma}(1-q)\right], \tag{28}$$

$$\mathfrak{J}_{\text{rig}} = \frac{E}{\omega_0^2}\frac{1}{6+2\sigma}\left(\frac{1+\sigma}{1-\sigma}\right)^{1/2}[(1+q) + \sigma(1-q)] \tag{29}$$

where $q$ is the anisotropy of the configuration which is defined by

$$q = \frac{\sum\limits_{occ}\left(n_y + \frac{1}{2}\right)}{\sum\limits_{occ}\left(n_z + \frac{1}{2}\right)}, \tag{30}$$

and $E$ is the total single particle energy,

$$E = \sum_{occ}\left[\hbar\omega_y(n_x + n_y + 1) + \hbar\omega_z\left(n_z + \frac{1}{2}\right)\right]. \tag{31}$$

Analyzing the experimental data concerning the ground states of the nuclei $^{20}$Ne, $^{24}$Mg, $^{28}$Si, $^{32}$S and $^{36}$Ar one can easily fill the occupied orbits by neutrons and protons and as a consequence, formulas (30) and (31) can be easily calculated for these nuclei.

## 5 Results and Conclusions

In Table 1 we present the calculated values of the moments of inertia of some doubly even deformed nuclei in the s-d shell: $^{20}$Ne, $^{24}$Mg, $^{28}$Si, $^{32}$S and $^{36}$Ar according to the cranking model, $\mathfrak{J}_{\text{cr}}$, and the rigid body model, $\mathfrak{J}_{\text{rig}}$, formulas (28) and (29), together with the values of the deformation parameter $\beta$ and the oscillator parameter $\hbar\omega_0^0$. In Table 1, also, we present the experimental values of the moments of inertia $\mathfrak{J}_{\text{exp}}$ of these nuclei, obtained from the low-lying rotational spectra of these nuclei [12].

**Table 1**. Moments of inertia of the nuclei $^{20}$Ne, $^{24}$Mg, $^{28}$Si, $^{32}$S and $^{36}$Ar.

| Nucleus | $\beta$ | $\hbar\omega_0^0$ (MeV) | $\frac{\hbar^2}{2\mathfrak{J}_{\text{cr}}}$ (KeV) | $\frac{\hbar^2}{2\mathfrak{J}_{\text{rig}}}$ (KeV) | $\frac{\hbar^2}{2\mathfrak{J}_{\text{exp}}}$ (KeV) |
|---|---|---|---|---|---|
| $^{20}$Ne | 0.22 | 11.88 | 276.04 | 305.40 | 279.90 |
| $^{20}$Ne | -0.24 | 11.88 | 281.30 | 328.21 | |
| $^{24}$Mg | 0.39 | 11.55 | 237.58 | 213.22 | 237.90 |
| $^{24}$Mg | -0.44 | 11.55 | 232.29 | 244.43 | |
| $^{28}$Si | 0.26 | 11.22 | 321.36 | 192.41 | 324.60 |
| $^{28}$Si | -0.29 | 11.22 | 320.83 | 212.26 | |
| $^{32}$S | 0.27 | 10.91 | 358.47 | 162.02 | 371.72 |
| $^{32}$S | -0.32 | 10.91 | 365.38 | 179.23 | |
| $^{36}$Ar | 0.27 | 10.62 | 372.91 | 138.96 | 374.55 |
| $^{36}$Ar | -0.32 | 10.62 | 370.22 | 152.78 | |

In Table 2 we present the calculated values of the equilibrium moments of inertia, $\mathfrak{J}_{\text{equ}}$, for the deformed doubly even nuclei in the s-d shell: $^{20}$Ne, $^{24}$Mg, $^{2S}$Si, $^{32}$S and $^{36}$Ar together with the values of the deformation parameter, $\beta$, at which the cranking model and the rigid body model moments of inertia are equal, and the values of the oscillator parameter $\hbar\omega_0^0$.

It is seen from Table 1 that the calculated values of the moments of inertia of the considered nuclei according to the cranking model by using the concepts of the single-particle Schroedinger fluid are in good agreement with the corresponding experimental values, a result which shows that the concept of this fluid is reliable and can be applied successfully to deformed nuclei in the s-d shell. It is seen, also, from Table 1 that the nuclei $^{20}$Ne, $^{28}$Si, $^{32}$S and $^{36}$Ar have nearly equal values of the deformation parameter $0.22 \leq \beta \leq 0.27$ (or $-0.32 \leq 0 \leq -0.24$). Table 1 shows, also, that the calculated values of the moments of inertia according to the rigid body model for the two nuclei $^{20}$Ne and $^{24}$Mg are in good agreement with the corresponding experimental values while the calculated values for the three nuclei $^{28}$Si, $^{32}$S and $^{36}$Ar are not in agreement with the corresponding experimental values. Moreover, it is seen from Table 1 that, according to the calculations, the considered deformed nuclei may have oblate as well as prolate deformations. The analysis of the quadrupole moments

**Table 2**. Equilibrium moments of inertia of $^{20}$Ne, $^{24}$Mg, $^{28}$Si, $^{32}$S and $^{36}$Ar.

| Nucleus | $\beta$ | $\hbar\omega_0^0$ (MeV) | $\frac{\hbar^2}{2\Im_{\text{equ}}}$ (KeV) | $\frac{\hbar^2}{2\Im_{\text{exp}}}$ (KeV) [12] |
|---|---|---|---|---|
| $^{20}$Ne | 0.24 | 11.88 | 305.16 | 279.90 |
| $^{24}$ Mg | 0.36 | 11.55 | 212.80 | 237.90 |
| $^{28}$Si | 0.17 | 11.22 | 193.60 | 324.60 |
| $^{32}$S | 0.14 | 10.91 | 163.30 | 371.72 |
| $^{36}$Ar | 0.11 | 10.62 | 140.10 | 374.55 |

of the considered nuclei show that, among all the considered nuclei, the nucleus $^{28}$Si may only have an oblate shape while the others have prolate deformations. Furthermore, according to the calculations the nucleus $^{24}$Mg is the only one which is highly deformed, It may be more reasonable to assume that this nucleus has a triaxial shape and not an axial shape.

It is seen from Table 2 that the values of the equilibrium moments of inertia of the two nuclei $^{20}$Ne and $^{24}$Mg are in good agreement with the corresponding experimental values while the equilibrium moments of inertia of the other three nuclei are not in good agreement with the corresponding experimental values.

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