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Cosmic Strings Interacting with Gravitational and Dilaton-Axion Waves

S. S. Yazadjiev

Department of Theoretical Physics, Faculty of Physics, Sofia University,
5 James Bourchier Blvd, BG-1164 Sofia, Bulgaria

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Abstract. We present a cylindrically symmetric solution to the dilaton-axion gravity equations. The found solution is (quasi)regular on the axis of symmetry, asymptotically flat and free from curvature singularities. From a physical point of view the solution describes a nonrotating cosmic string interacting with gravitational and dilaton-axion waves.

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1 Introduction

Grand unified theories predict the existence of cosmic strings formed as a consequence of phase transitions in the early Universe. Cosmic strings attracted much attention due to the possible physical consequences related to them: they act as gravitational lenses, may produce fluctuations in the microwave background, and may be responsible for galaxy formation in the Universe.

The straight cosmic strings are characterized by their radius, which is related to the Compton wavelength of Higgs scalar governing the symmetry breaking, therefore, the radius is extremely small, about 10^{-27} m. That is why, from macroscopic point of view, a cosmic string may be described as a line source and its gravitational field would have cylindrical symmetry. The space-time of an isolated cosmic string is described by flat space with a wedge cut out. In other words, the metric has a conical singularity on the symmetry axis. The deficit angle $\delta\phi$ is related to the string's energy density per unit length, ε, through the formula $\delta\phi = 8\pi\varepsilon$.

One would expect that the strings would interact with the matter and the radiation surrounding them. Exact solutions of Einstein's equations, describing rotating cosmic strings interacting with gravitational waves, were found by Xanthopoulos [1, 2]. These solutions are asymptotically flat, of Petrov type D, admitting quasiregular axis, and are free from curvature singularities. Later on, Xanthopoulos and Papadopoulos [3], using an analytic continuation of the Tomimatsu-Sato solutions, obtained exact solutions, which describe a beam-like shaped pulse of gravitational radiation scattered by a rotating cosmic string. Petrov type I solutions, which describe soliton-like waves interacting with a nonrotating cosmic string, were given and discussed in [4]. Rotating cosmic strings interacting with gravitational and electromagnetic waves were studied in [5].

The four-dimensional effective string actions attracted much interest in recent years as a possible generalization of the Einstein general relativity [6–17]. Solutions describing isolated static cosmic strings within the effective string actions were discussed in [19]. Rotating cosmic strings interacting with gravitational and scalar (dilaton) waves were studied in [20].

In the present paper we consider the so-called dilaton-axion gravity. We present a new solution, which can be interpreted as a nonrotating cosmic string interacting with gravitational and dilaton-axion waves.

2 General Equations and Exact Solutions

The dilaton-axion gravity in the Einstein frame is described by the action

$$S = \int d^4x\sqrt{-g}\left(R - 2g^{ab}\partial_a\varphi\partial_b\varphi - \frac{1}{2}e^{4\varphi}g^{ab}\partial_a\chi\partial_b\chi\right) \tag{1}$$

where g_{ab} is the space-time metric, R is the Ricci scalar curvature, and φ and χ are the dilaton and axion field, respectively.

We consider space-times with two commuting space-like Killing vectors, $K_1 = \partial/\partial z$ and $K_2 = \partial/\partial\phi$ and adopt the following Ansatz for the metric:

$$ds^2 = e^{2\Gamma(t,\rho)}(-dt^2 + d\rho^2) + \rho^2 d\phi^2 + dz^2. \tag{2}$$

It turns out that the solution takes much simpler form in the so-called prolate coordinates, η and μ, in which the metric can be written in the form

$$ds^2 = e^{2f(\eta,\mu)}\left(-\frac{d\eta^2}{1+\eta^2} + \frac{d\mu^2}{\mu^2-1}\right) + \rho^2(\eta,\mu)d\phi^2 + dz^2 \tag{3}$$

where

$$\rho^2(\eta,\mu) = (1+\eta^2)(\mu^2-1), \tag{4}$$

$$t(\eta,\mu) = \mu\eta, \tag{5}$$

and the range of the coordinates η and μ is given by

$$-\infty < \eta < +\infty, \quad \mu \geq 1. \tag{6}$$

In more explicit form we have

$$2\eta^2 = \sqrt{(\rho^2 - t^2 + 1)^2 + 4t^2} + t^2 - \rho^2 - 1, \tag{7}$$
$$2\mu^2 = \sqrt{(\rho^2 - t^2 + 1)^2 + 4t^2} - t^2 + \rho^2 + 1, \tag{8}$$

and

$$e^{2\Gamma} = \frac{e^{2f}}{\eta^2 + \mu^2}. \tag{9}$$

Taking into account all the above assumptions, we obtain the following system of partial differential equations:

$$- \partial_\eta \left[(1+\eta^2)\partial_\eta\varphi\right] + \partial_\mu \left[(\mu^2 - 1)\partial_\mu\varphi\right] = \frac{1}{2}e^{4\varphi}\left[-(1+\eta^2)(\partial_\eta\chi)^2 + (\mu^2-1)(\partial_\mu\chi)^2\right], \tag{10}$$

$$-\partial_\eta \left[e^{4\varphi}(1+\eta^2)\partial_\eta\chi\right] + \partial_\mu \left[e^{4\varphi}(\mu^2-1)\partial_\mu\chi\right] = 0, \tag{11}$$

$$\frac{\eta}{1+\eta^2}\partial_\mu f + \frac{\mu}{\mu^2-1}\partial_\eta f = \frac{\mu\eta}{(1+\eta^2)(\mu^2-1)} + 2\partial_\eta\varphi\partial_\mu\varphi + \frac{1}{2}e^{4\varphi}\partial_\eta\chi\partial_\mu\chi, \tag{12}$$

$$2\eta\partial_\eta f + 2\mu\partial_\mu f = 2 + (1+\eta^2)\left[2(\partial_\eta\varphi)^2 + \frac{1}{2}e^{4\varphi}(\partial_\eta\chi)^2\right] + (\mu^2-1)\left[2(\partial_\mu\varphi)^2 + \frac{1}{2}e^{4\varphi}(\partial_\mu\chi)^2\right]. \tag{13}$$

We have found the following solution to the above system:

$$f(\eta,\mu) = \frac{1}{2}\ln\left[N^2(A^2\eta^2 + B^2\mu^2 - 1)\right], \tag{14}$$
$$\varphi(\eta,\mu) = -\frac{1}{2}\ln\left[\frac{A^2\eta^2 + B^2\mu^2 - 1}{(A\eta-1)^2 + B^2\mu^2}\right], \tag{15}$$
$$\chi(\eta,\mu) = \frac{2B\mu}{(A\eta-1)^2 + B^2\mu^2}. \tag{16}$$

Here $N \neq 0$, A and B are constants, with $B^2 - A^2 = 1$.

3 Physical Interpretation

In order to discuss the physics behind the found solution, we need to analyze the behavior of the metric, the so-called "C-energy", the Weyl and Ricci scalars near the axis, near the spacial infinity and in the vicinity of the null infinity.

3.1 The metric and the scalar fields

Behavior near the axis: $\rho <<| t |$, $\rho \to 0$.

It is obvious that the norm of the azimuthal Killing vector $\partial/\partial\phi$ is

$$g(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\phi}) = \rho^2, \tag{17}$$

and therefore, the axis $\rho = 0$ is a (quasi)regular surface of space-time. Moreover, near the axis the metric becomes

$$ds^2 \simeq N^2A^2\left(-dt^2 + d\rho^2 + \frac{\rho^2}{N^2A^2}d\phi^2\right) + dz^2. \tag{18}$$

The behavior of the dilaton and axion field near the axis is given by the expressions:

$$\varphi \approx -\frac{1}{2}\ln\left[\frac{A^2(1+t^2)}{(1-At)^2+B^2}\right], \tag{19}$$

$$\chi \approx \frac{2B}{(1-At)^2+B^2}. \tag{20}$$

The deficit angle or surplus is

$$\delta\phi_{axis} = 2\pi - \lim_{\rho\to 0}\frac{\int_0^{2\pi}\sqrt{g_{\phi\phi}}d\phi}{\int_0^{\rho}\sqrt{g_{\rho\rho}}d\rho} = 2\pi\left(1-\frac{1}{\mid NA\mid}\right). \tag{21}$$

Therefore, there is a conical singularity on the axis measured by (21), which signals the existence of a cosmic string. In order to avoid the surplus and negative energy density of the string, we have to take $\mid NA \mid > 1$. The value $\mid NA \mid = 1$ eliminates the deficit angle (*i.e.* the string disappears) and the solution describes cylindrical gravitational waves.

Asymptotic behavior: $\rho >>| t |$, $\rho \to \infty$.

One can find that the asymptotic form of the metric is

$$ds^2 \simeq N^2B^2\left(-dt^2 + d\rho^2 + \frac{\rho^2}{N^2B^2}d\phi^2\right) + dz^2. \tag{22}$$

This explicit form shows that the metric is asymptotically flat[1] with deficit angle or surplus, given by

$$\delta\phi_{\infty} = 2\pi - \lim_{\rho\to\infty}\frac{\int_0^{2\pi}\sqrt{g_{\phi\phi}}d\phi}{\int_0^{\rho}\sqrt{g_{\rho\rho}}d\rho} = 2\pi\left(1-\frac{1}{\mid NB\mid}\right) \tag{23}$$

[1] Of course, this solution is not asymptotically simple in the sense of Penrose.

The asymptotic forms of the dilaton and axion field are given by

$$\varphi \approx \frac{1}{B^2\rho^2}, \tag{24}$$

$$\chi \approx \frac{2}{B\rho}. \tag{25}$$

Since $\mid B \mid > \mid A \mid$, we have

$$\delta\phi_{\infty} > \delta\phi_{axis} \tag{26}$$

which reflects the contribution of the gravitational waves.

The choice $\mid N \mid = \mid B \mid$ would eliminate the asymptotic deficit angle. However, this choice would cause a surplus near the axis and therefore negative string energy density.

Behavior near the null infinity: $\rho = \mid t \mid \to \infty$.

Considering the past and future null infinity, we find the following expressions:

$$\mid g_{00} \mid = e^{2\Gamma} \approx \frac{N^2}{2}(A^2+B^2)\left(1 - \frac{1}{2\rho(A^2+B^2)}\right), \tag{27}$$

$$\chi \approx \frac{2B}{A^2+B^2}\frac{1}{\sqrt{\rho}}, \tag{28}$$

$$\varphi \approx \mp \frac{A}{A^2+B^2}\frac{1}{\sqrt{\rho}} \tag{29}$$

where the upper sign is for the future null infinity. The behavior of the metric component and the scalar fields near the null infinities show that the solution presents gravitational and dilaton-axion waves (disturbances), propagating along the null directions. This is confirmed by the next subsection discussions.

3.2 The C-energy

The so-called C-energy [21] is given by

$$C(t,\rho) = \frac{1}{2}\ln \mid g_{00}g_{zz} \mid = \Gamma(t,\rho). \tag{30}$$

It measures the energy of a cylindrically symmetric gravitational system. More precisely, $C(t,\rho)$ is proportional to the energy density per unit length contained in a cylinder of radius ρ. It is useful to introduce the null coordinates

$$u = t - \rho, \tag{31}$$

$$\upsilon = t + \rho, \tag{32}$$

so that the past null infinity corresponds to $u \to -\infty$ with finite υ, while the future null infinity corresponds to $\upsilon \to +\infty$ with finite u. The quantity $\partial_\upsilon C = \frac{1}{2}\left(\partial_t C - \partial_\rho C\right)$ measures the energy radiated away.

In our case the explicit form of the function $C(t,\rho)$ is

$$C(t,\rho) = \frac{1}{2}\ln\left[\frac{N^2\left(A^2\eta^2 + B^2\mu^2 - 1\right)}{\eta^2+\mu^2}\right]. \tag{33}$$

We also have

$$\partial_u C = -\frac{\sqrt{(1+\eta^2)(\mu^2-1)}\left[\eta\sqrt{\mu^2-1} + \mu\sqrt{1+\eta^2}\right]^2}{2(\eta^2+\mu^2)^2(A^2\eta^2+B^2\mu^2-1)}, \tag{34}$$

$$\partial_\upsilon C = \frac{\sqrt{(1+\eta^2)(\mu^2-1)}\left[\eta\sqrt{\mu^2-1} - \mu\sqrt{1+\eta^2}\right]^2}{2(\eta^2+\mu^2)^2(A^2\eta^2+B^2\mu^2-1)}. \tag{35}$$

Behavior near the axis: $\rho \ll \mid t \mid$, $\rho \to 0$.

In this region one obtains

$$C \approx \frac{1}{2}\ln\left[N^2\left(A^2 + \frac{\rho^2}{(1+t^2)^2}\right)\right], \tag{36}$$

$$\partial_u C \approx -\frac{1}{2A^2(1+t^2)^2}\left(\rho + \frac{2t\rho^2}{1+t^2}\right), \tag{37}$$

$$\partial_\upsilon C \approx \frac{1}{2A^2(1+t^2)^2}\left(\rho - \frac{2t\rho^2}{1+t^2}\right). \tag{38}$$

These explicit expressions show that there is no energy flux at the axis, neither at the time-like infinity $\mid t \mid \to \infty$. The C-energy takes a constant value on the axis itself,

$$C_{axis} = \ln \mid NA \mid . \tag{39}$$

Asymptotic behavior: $\rho \gg \mid t \mid$, $\rho \to \infty$.

One can show that

$$C \approx \ln\left[\mid NB \mid \left(1 + \frac{1}{2B^2\rho^2}\right)\right], \tag{40}$$

$$\partial_u C \approx -\frac{1}{2B^2\rho^3}\left(1 + \frac{2t}{\rho}\right), \tag{41}$$

$$\partial_\upsilon C \approx \frac{1}{2B^2\rho^3}\left(1 - \frac{2t}{\rho}\right). \tag{42}$$

Making use of these formulas, we find that at spacial infinity $\rho \to \infty$ there is no energy flux, and that the C-energy tends to the constant value

$$C_\infty = \ln \mid NB \mid . \tag{43}$$

Behaviour near the null infinity: $\rho = \mid t \mid \to \infty$.

In the region under consideration we have

$$C \approx \frac{1}{2} \ln \left(\frac{1}{2} N^2 \right) \left(A^2 + B^2 - \frac{1}{\rho} \right) . \tag{44}$$

We also find that

$$\partial_u C \approx -\frac{1}{2\,(A^2+B^2)} \left[1 + \frac{1}{2(A^2+B^2)\rho} \right], \tag{45}$$

$$\partial_v C \approx \frac{1}{8(A^2+B^2)\rho^2} \tag{46}$$

near the future null infinity, and

$$\partial_u C \approx -\frac{1}{8(A^2+B^2)\rho^2}, \tag{47}$$

$$\partial_v C \approx \frac{1}{2\,(A^2+B^2)} \left[1 + \frac{1}{2(A^2+B^2)\rho}, \right] \tag{48}$$

for the past null infinity. These expressions imply that at the past and future null infinities there are incoming and outgoing radiations, respectively. Taking also the considerations in the previous subsections into account, we may conclude that our solution describes gravitational and dilaton-axion waves, which originate at the past null infinity, converge towards the axis (the string), and propagate to the future null infinity.

3.3 The Weyl scalars

After elaborate calculations, we find the Weyl scalars for our solution

$$\Psi_0 = -\frac{e^{-2\Gamma}\sqrt{1+\eta^2}\sqrt{\mu^2-1}}{2(\eta^2+\mu^2)^2(A^2\eta^2+B^2\mu^2-1)} \left[\mu\sqrt{1+\eta^2} - \eta\sqrt{\mu^2-1} \right]^2 , \tag{49}$$

$$\begin{aligned} \Psi_2 = \frac{1}{6} \frac{e^{-2\Gamma}}{(\eta^2+\mu^2)[(1-A\eta)^2+B^2\mu^2]^2(A^2\eta^2+B^2\mu^2-1)^2} \times \\ \left[\left[(1-A\eta^2)^2 - B^2\mu^2 \right] (A^2\eta^2 - B^2\mu^2 + A^2 + B^2) \right. \\ \left. + 4B^2\mu^2(1-A\eta)^2(A^2\eta^2 - \mu^2 + A^2 + 1) \right] , \end{aligned} \tag{50}$$

$$\Psi_4 = -\frac{e^{-2\Gamma}\sqrt{1+\eta^2}\sqrt{\mu^2-1}}{2(\eta^2+\mu^2)^2(A^2\eta^2+B^2\mu^2-1)}\left[\mu\sqrt{1+\eta^2}+\eta\sqrt{\mu^2-1}\right]^2. \quad (51)$$

The Weyl scalars are regular everywhere. The same holds for the Ricci scalars since, as one could see, the scalar fields, their derivatives, and the metric components are also regular everywhere. The space-time is therefore free from curvature singularities. The investigation of the Weyl scalars in the three characteristic regions (near the axis, asymptotic region and near the null infinities) confirms the conclusion of the previous subsection.

One can check that the space-time is of Petrov type I.

4 The String Frame

Our considerations were in the Einstein frame so far. That is why we should also comment the picture in the string frame. Both frames are related through the conformal transformation

$$\tilde{g}_{ab} = e^{-2\varphi} g_{ab}. \quad (52)$$

The explicit calculations show that the deficit angles in both frames are the same:

$$\delta\tilde{\phi}_{axis} = \delta\phi_{axis},$$
$$\delta\tilde{\phi}_{\infty} = \delta\phi_{\infty}.$$

The behavior of the string frame C-energy and Weyl scalars in the characteristic regions is qualitatively the same as in the Einstein frame. The physical interpretation therefore also holds in the string frame.

5 Conclusion

In this paper we have presented a new exact cylindrically symmetric solution to the dilaton-axion gravity equations. The solution is asymptotically flat, regular on the axis of symmetry, and free from curvature singularities. It can be interpreted from a physical point of view as a nonrotating cosmic string interacting with gravitational and dilaton-axion waves.

Note Added: During the review process of this paper, the author found more general classes of solutions describing rotating cosmic strings interacting with gravitational and dilaton-axion waves [22].

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