

On an Exact Cosmological Solution of Higher Spin Gauge Theory

E. Sezgin¹, P. Sundell²

¹George P. and Cynthia W. Mitchell Institute for Fundamental Physics,
Texas A&M University, College Station, TX 77843-4242, USA

²Department for Theoretical Physics, Uppsala University,
Box 803, 751 08 Uppsala, Sweden

Received 11 October 2006

Abstract. We review our recent exact solution to four-dimensional higher spin gauge theory invariant under a higher spin extension of $SO(3, 1)$ and we comment on its cosmological interpretation. We find an effective Einstein-scalar field theory that admits this solution, and we highlight the significance of the Einstein frame and what we call higher spin frame in the cosmological interpretation of the solution.

PACS number: 11.15.-q

1 Introduction

Four-dimensional interacting higher spin gauge theory is an extension of ordinary gravity by an infinite tower of higher-rank symmetric tensor gauge fields as well as particular lower spin fields. The simplest model – the minimal bosonic model – consists of physical fields of rank $s = 0, 2, 4, \dots$ with the rank-2 field playing the role of a metric and the rank-0 field playing the role of a particular matter sector. The field equations can be given in a generally covariant weak-field expansion, in which all physical fields except the metric are treated as small fluctuations. The metric field equation contains a negative cosmological term, and the theory admits the anti-de Sitter spacetime as an unbroken vacuum solution, with radius set by the fundamental length scale.

A generic feature of higher spin gauge theory is that the field equations are strongly coupled in the sense of derivative expansion, which means that the weak-field expansion is limited to the perturbative study of solutions with small curvatures as well as small scalar field fluctuations. In particular, the scalar field potential is blurred by equally sized higher-derivative corrections. Moreover, in the metric sector there is in general torsion. Finally, there is in general no known

consistent truncation of the equations down to the lower-spin sector, as lower-spin fields serve as sources for higher spin fields, and there is no independent coupling constant that can be identified with the higher spin fields.

Given this state of affairs, while full higher spin gauge theories have been known in $D \leq 4$ since the early work of Vasiliev [1], not much is known about their exact solutions beyond the anti-de Sitter vacuum. The equations assume, however, a remarkably simple form when written in terms of master fields – conjectured in [2] to be a topological open twistor string describing the phase-space, or deformation, quantization of the scalar $SO(3, 2)$ -singleton – which are integrable in the sense that the gauge fields, contained in a master one-form \widehat{A}_μ , and thus the space-time geometry can be given *algebraically* in terms of the Weyl tensors and matter fields contained in a master zero-form $\widehat{\Phi}$. This formulation becomes especially powerful when $\widehat{\Phi}$ is fixed completely by symmetries, such as the cases examined in [3] that are invariant under 3, 4, 6 dimensional groups.

Here we shall review the resulting $SO(3, 1)$ invariant exact solution found in [3] and comment on its cosmological interpretation. In particular, we shall point out the significance of the Einstein frame and what we call higher spin frame in the cosmological interpretation of the solution. The later frame naturally arises in higher spin field equations and it has bosonic torsion, while the Einstein frame is more natural for the cosmological interpretation. Indeed, as we shall show here, the latter frame avoids a big crunch singularity, provided the standard language appropriate to gravity is used, in which the notions of horizons and singularities are based on the geodesic equation motion of ordinary test particles. A more rigorous understanding requires, however, a higher spin covariant counterpart, which is still not available. Nonetheless it is our hope that our interpretation captures some significant features of the ultimate story. We shall also compare these results with those of [4] where an AdS cosmological solution of a consistently truncated sector of gauged $D = 4$, $\mathcal{N} = 8$ supergravity has been examined, and a big crunch singularity occurs.

2 The Master Equations and the Gauge Function

The minimal bosonic model is an extension of AdS gravity with spin $s = 0, 2, 4, \dots$ fields, each occurring once. These are exactly the massless representations which occur in the symmetric tensor product of two ultra-short fundamental representations of $SO(3, 2)$ known as singletons. The occurrence of a scalar field is noteworthy and it is a universal feature of all higher spin gauge theories.

Master fields denoted by (A_μ, Φ) arise naturally in the corresponding frame-like, or unfolded, formulation as follows. Firstly, the vierbein e_μ^a (whose relation to the Einstein frame is discussed in Section 4), the Lorentz connection ω_μ^{ab} , and their higher spin analogs $W_{\mu, a_1 \dots a_{s-1}, b_1 \dots b_t}$, $0 \leq t \leq s - 1$, $s = 4, 6, \dots$, with $W_{(a_1, a_2 \dots a_s)}$ defining the physical spin- s field, make up the adjoint master

one-form

$$A_\mu(x, y, \bar{y}) = \frac{1}{2i}(e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab} + \dots) = e_\mu + \omega_\mu + W_\mu + K_\mu, \quad (1)$$

where K_μ is a field re-definition required for manifest $SO(3, 1)$ invariance, to be described below, and (M_{ab}, P_a) are the $SO(3, 2)$ generators which can be realized in terms of $SL(2, C)$ -doublet oscillators y_α and $\bar{y}_\alpha = (y_\alpha)^\dagger$ as

$$M_{ab} = -\frac{1}{8} \left[(\sigma_{ab})^{\alpha\beta} y_\alpha y_\beta + (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} \right], \quad P_a = \frac{1}{4} (\sigma_a)^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}. \quad (2)$$

Second, the scalar field ϕ , the spin-2 Weyl tensor $C_{ab,cd}$ (which we take to be symmetric in its pairs of indices), its higher spin analogs and all possible derivatives of these fields fit into a twisted-adjoint master zero-form

$$\begin{aligned} \Phi(x, y, \bar{y}) &= \phi + iP^a \nabla_a \phi + i^2 P^a P^b \nabla_a \nabla_b \phi + \dots \\ &+ M^{ab} M^{cd} (C_{ac,bd} + \dots) \\ &+ \text{spin } s = 4, 6, \dots \text{ sectors}, \end{aligned} \quad (3)$$

where combinatorial coefficients are suppressed. The master fields A_μ and Φ are extended – or deformation quantized – into full master fields \widehat{A} and $\widehat{\Phi}$ obeying the constraints

$$\begin{aligned} \widehat{F} &\equiv \widehat{d}\widehat{A} + \widehat{A} \star \widehat{A} = \frac{i}{4} \widehat{\Phi} \star \left(b dz^\alpha dz_\alpha e^{iy^\alpha z_\alpha} + \bar{b} d\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}} e^{-i\bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}} \right), \\ \widehat{D}\widehat{\Phi} &\equiv \widehat{d}\widehat{\Phi} + \widehat{A} \star \widehat{\Phi} - \widehat{\Phi} \star \pi(\widehat{A}) = 0, \end{aligned} \quad (4)$$

with $\widehat{d} = d + \widehat{d}'$ where $d = dx^\mu \partial_\mu$ and $\widehat{d}' = (dz^\alpha \partial_\alpha + \text{h.c.})$ are exterior derivatives on a spacetime \mathcal{M} and a non-commutative twistor space \mathcal{Z} , respectively. The parameter $b = 1$ in Type A model, in which the scalar ϕ is even under parity, and $b = i$ in the Type B model, in which ϕ is odd under parity. The extended master fields are maps from $\mathcal{M} \times \mathcal{Z}$ to the space of functions on \mathcal{Z} , viz.

$$\begin{aligned} \widehat{A} &= dx^\mu \widehat{A}_\mu(x, z, \bar{z}; y, \bar{y}) + dz^\alpha \widehat{A}_\alpha(x, z, \bar{z}; y, \bar{y}) + d\bar{z}^{\dot{\alpha}} \widehat{A}_{\dot{\alpha}}(x, z, \bar{z}; y, \bar{y}), \\ \widehat{\Phi} &= \widehat{\Phi}(x, z, \bar{z}; y, \bar{y}), \quad A_\mu = \widehat{A}_\mu|_{\mathcal{Z}=0}, \quad \Phi = \widehat{\Phi}|_{\mathcal{Z}=0}. \end{aligned} \quad (5)$$

where $(x^\mu, z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})$ coordinatize $\mathcal{M} \times \mathcal{Z} \times \mathcal{Z}$ and the associative \star -product is defined by

$$\begin{aligned} \widehat{f}(y, \bar{y}; z, \bar{z}) \star \widehat{g}(y, \bar{y}; z, \bar{z}) &= \int \frac{d^2 \xi d^2 \eta d^2 \bar{\xi} d^2 \bar{\eta}}{(2\pi)^4} e^{i\eta^\alpha \xi_\alpha + i\bar{\eta}^{\dot{\alpha}} \bar{\xi}_{\dot{\alpha}}} \times \\ &\times \widehat{f}(y + \xi, \bar{y} + \bar{\xi}; z + \xi, \bar{z} - \bar{\xi}) \widehat{g}(y + \eta, \bar{y} + \bar{\eta}; z - \eta, \bar{z} + \bar{\eta}). \end{aligned} \quad (6)$$

The minimal master fields satisfy the additional discrete symmetry conditions

$$\tau(\widehat{A}) = -\widehat{A}, \quad \widehat{A}^\dagger = -\widehat{A}, \quad \tau(\widehat{\Phi}) = \bar{\pi}(\widehat{\Phi}), \quad (b\widehat{\Phi})^\dagger = \pm b\pi(\widehat{\Phi}), \quad (7)$$

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where $\tau(\widehat{f}(y, \bar{y}, z, \bar{z})) = \widehat{f}(iy, i\bar{y}, -iz, -i\bar{z})$ and $\pi(\widehat{f}) = \widehat{f}(-y, \bar{y}; -z, \bar{z})$. The sign in the equation involving b corresponds to $\phi(x) = \widehat{\Phi}|_{Y=Z=0}$ transforming under parity (acting in tangent space) into $\pm\phi(x)$ with $+$ in Type A model and $-$ in Type B model. By convention, we take $b = 1$ and $b = i$ in the Type A and B models, respectively, so that $\widehat{\Phi}^\dagger = \pi(\widehat{\Phi})$ and $\phi^\dagger = \phi$.

The gauge transformation are given by

$$\delta_{\widehat{\epsilon}} \widehat{A} = \widehat{D} \widehat{\epsilon}, \quad \delta_{\widehat{\epsilon}} \widehat{\Phi} = -\widehat{\epsilon} \star \widehat{\Phi} + \widehat{\Phi} \star \pi(\widehat{\epsilon}). \quad (8)$$

A close examination of the Lorentz transformations of the full master fields [5,6] given by

$$\widehat{\epsilon}_L = \frac{1}{4i} \Lambda^{\alpha\beta} (x) \widehat{M}_{\alpha\beta} - \text{h.c.}, \quad (9)$$

$$\widehat{M}_{\alpha\beta} = y_\alpha y_\beta - z_\alpha z_\beta + \frac{1}{2} \{ \widehat{S}_\alpha, \widehat{S}_\beta \}_*, \quad \widehat{S}_\alpha = z_\alpha - 2i \widehat{A}_\alpha, \quad (10)$$

shows that e_μ^a and W_μ , defined in (1), transform canonically under the Lorentz transformation provided

$$K_\mu = \frac{1}{4i} \omega_\mu^{\alpha\beta} \widehat{S}_\alpha \star \widehat{S}_\beta|_{Z=0} - \text{h.c.} = i\omega_\mu^{\alpha\beta} (\widehat{A}_\alpha \star \widehat{A}_\beta)|_{Z=0} - \text{h.c.}, \quad (11)$$

where the gauge condition $\widehat{A}_\alpha|_{\widehat{\Phi}=0} = 0$ has been assumed. Thus, locally, a space-time field configuration $\phi(x)$, $g_{\mu\nu}(x)$ and $\phi_{\mu_1 \dots \mu_s}(x)$ ($s = 4, 6, \dots$) can be unfolded and packed into a twisted-adjoint initial condition $\Phi(x; y, \bar{y})|_{x=0}$, which is deformed into

$$\widehat{\Phi}'(z, \bar{z}; y, \bar{y}) = \widehat{\Phi}|_{x=0}, \quad \widehat{A}'_\alpha(z, \bar{z}; y, \bar{y}) = \widehat{A}_\alpha(x, z, \bar{z}; y, \bar{y})|_{x=0}. \quad (12)$$

This can be made precise by solving the constraints $\widehat{F}_{\mu\nu} = 0$, $\widehat{F}_{\mu\alpha} = 0$ and $\widehat{D}_\mu \widehat{\Phi}$ using a gauge function $\widehat{L} = \widehat{L}(x, z, \bar{z}; y, \bar{y})$,

$$\widehat{A}_\mu = \widehat{L}^{-1} \star \partial_\mu \widehat{L}, \quad \widehat{A}_\alpha = \widehat{L}^{-1} \star (\widehat{A}'_\alpha + \partial_\alpha) \widehat{L}, \quad \widehat{\Phi} = \widehat{L}^{-1} \star \widehat{\Phi}' \star \pi(\widehat{L}), \quad (13)$$

and determine the remaining Z -dependence from

$$\widehat{F}'_{\alpha\beta} \equiv 2\partial_{[\alpha} \widehat{A}'_{\beta]} + [\widehat{A}'_\alpha, \widehat{A}'_\beta]_* = -\frac{ib}{2} \epsilon_{\alpha\beta} \widehat{\Phi}' \star \kappa, \quad (14)$$

$$\widehat{F}'_{\alpha\beta} \equiv \partial_\alpha \widehat{A}'_\beta - \partial_\beta \widehat{A}'_\alpha + [\widehat{A}'_\alpha, \widehat{A}'_\beta]_* = 0, \quad (15)$$

$$\widehat{D}'_\alpha \widehat{\Phi}' \equiv \partial_\alpha \widehat{\Phi}' + \widehat{A}'_\alpha \star \widehat{\Phi}' + \widehat{\Phi}' \star \pi(\widehat{A}'_\alpha) = 0, \quad (16)$$

given $\widehat{\Phi}'|_{Z=0} \equiv C'(y, \bar{y})$ and fixing the gauges $\widehat{A}'_\alpha|_{C'=0} = 0$ and $\widehat{L}|_{C'=0} = L(x; y, \bar{y})$, in turn implying $\partial_\alpha \widehat{L} = 0$, that is, $\widehat{L} = L(x; y, \bar{y})$.

In case $\widehat{\Phi}'$ is invariant under a symmetry group G_r with full parameters $\widehat{\mathcal{C}}'$, and assuming that $\widehat{\Phi}'$ and $\widehat{\mathcal{C}}'$ have well-defined perturbative expansions in C' of the form $\widehat{\Phi}' = C' + \widehat{\Phi}'_{(2)} + \dots$ and $\widehat{\mathcal{C}}' = \epsilon' + \widehat{\mathcal{C}}'_{(1)} + \dots$, where $\epsilon' = \epsilon'(y, \bar{y})$ is an adjoint representation of G_r , then it follows that C' must obey $\epsilon' \star C' - C' \star \epsilon' = 0$. For G_6 , the latter condition admits two-parameter solution spaces except at the special point [3]

$$G_6 = SO(3, 1) : \quad \epsilon' = \frac{1}{4i} \Lambda^{\alpha\beta} M_{\alpha\beta} - \text{h.c.}, \quad C' = \frac{\nu}{b}, \quad (17)$$

where ν/b is a constant real deformation parameter (which requires ν to be real and purely imaginary in the Type A and Type B models, respectively). Next we turn to the promotion of this linearized solution into the exact solution given in [3] – which is presently the only known exact solution to Vasiliev’s four-dimensional higher spin gauge theory other than AdS spacetime.

3 The $SO(3, 1)$ Invariant Exact Solution

To describe the $SO(3, 1)$ invariant solution in spacetime it is convenient to use the stereographic coordinate on AdS_4 with inverse radius λ , viz.

$$e_{(0)}^{\alpha\dot{\alpha}} = -\frac{\lambda(\sigma^a)^{\alpha\dot{\alpha}} dx_a}{h^2}, \quad \omega_{(0)}^{\alpha\beta} = -\frac{\lambda^2(\sigma^{ab})^{\alpha\beta} dx_a dx_b}{h^2},$$

$$h = \sqrt{1 - \lambda^2 x^2}, \quad x^2 = x^a x^b \eta_{ab}, \quad (18)$$

in turn corresponding via $L^{-1} \star dL = \frac{1}{4i}(\omega_{(0)}^{\alpha\beta} y_\alpha y_\beta + \bar{\omega}_{(0)}^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + 2e_{(0)}^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}})$ to the gauge function [7]

$$L(x; y, \bar{y}) = \frac{2h}{1+h} \exp \left[\frac{i\lambda x^a (\sigma_a)^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}}}{1+h} \right], \quad (19)$$

with $L^{-1}(x; y\bar{y}) = L(-x; y\bar{y})$. The full $SO(3, 1)$ invariance condition (17) then becomes

$$[\widehat{M}'_{\alpha\beta}, \widehat{\Phi}']_\pi = 0, \quad \widehat{D}'_\alpha \widehat{M}'_{\beta\gamma} = 0, \quad (20)$$

where $\widehat{M}'_{\alpha\beta}$ are defined by (10) with internal connection given by \widehat{A}'_α . Using also the τ -invariance condition on \widehat{A}'_α , it follows that

$$\widehat{\Phi}' = f(u, \bar{u}), \quad \widehat{S}'_\alpha = z_\alpha S(u, \bar{u}), \quad u = y^\alpha z_\alpha, \quad \bar{u} = u^\dagger = \bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}, \quad (21)$$

where f is a real function. The internal constraints $\widehat{F}'_{\alpha\dot{\alpha}} = 0$ and $\widehat{D}'_\alpha \widehat{\Phi}' = 0$ are then solved by

$$\widehat{\Phi}'(u, \bar{u}) = \frac{\nu}{b}, \quad S(u, \bar{u}) = S(u), \quad (22)$$

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where ν/b is the constant introduced in (17). The remaining constraint (14) then takes the form $[\widehat{S}'^\alpha, \widehat{S}'_\alpha]_\star = 4i(1 - \nu e^{iu})$. To solve this equation, following [8], we use the integral representation

$$S(u) = 1 + \int_{-1}^1 dt q(t) e^{\frac{i}{2}(1+t)u}, \quad (23)$$

where $t \in [-1, 1]$, as can be seen from perturbation theory. The equation for S then takes the form of an integral equation that can be solved by means of algebraic techniques invented in [8] (see also [3] for a slight refinement of the basis of functions on $[-1, 1]$). The result reads

$$\begin{aligned} q(t) &= -\frac{\nu}{4} \left(F \left(\frac{\nu}{2} \log \frac{1}{t^2} \right) + t F \left(-\frac{\nu}{2} \log \frac{1}{t^2} \right) \right), \\ F(\zeta) &\equiv {}_1F_1 \left[\frac{1}{2}; 2; \zeta \right] = 1 + \frac{\zeta}{4} + \frac{\zeta^2}{16} + \dots, \end{aligned} \quad (24)$$

and the internal primed solution is thus given by

$$\begin{aligned} \widehat{\Phi}' &= \frac{\nu}{b}, \\ \widehat{A}'_\alpha &= \frac{i\nu}{8} z_\alpha \int_{-1}^1 dt e^{\frac{i}{2}(1+t)u} \left[F \left(\frac{\nu}{2} \log \frac{1}{t^2} \right) + t F \left(-\frac{\nu}{2} \log \frac{1}{t^2} \right) \right]. \end{aligned} \quad (25)$$

Expanding $\exp(\frac{itu}{2})$ yields integrals that converge at $t = 0$ and $t = \pm 1$, and \widehat{A}'_α is a power-series expansion in u with coefficients that are functions of ν that are analytic at $\nu = 0$ and with different analytic structure on the real and imaginary axis.

The physical scalar field and the (auxiliary) Weyl tensors are obtained by unpacking $\Phi = \widehat{\Phi}|_{Z=0}$ according to (3). From

$$\widehat{\Phi} = L^{-1} \star \widehat{\Phi}' \star \pi(L) = \frac{\nu}{b} L^{-1} \star L^{-1} = \frac{\nu}{b} (1 - \lambda^2 x^2) \exp[-i\lambda x^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}}], \quad (26)$$

it follows that

$$\phi(x) = \frac{\nu}{b} h^2 = \frac{\nu}{b} (1 - \lambda^2 x^2), \quad (27)$$

while all Weyl tensors vanish. The above expressions are valid for $\lambda^2 x^2 < 1$. The gauge fields are obtained by unpacking $A_\mu = L^{-1} \star \partial_\mu L = e_\mu^{(0)} + \omega_\mu^{(0)}$ using the decomposition (1), *i.e.* $A_\mu \equiv e_\mu + \omega_\mu + W_\mu + K_\mu$ with $K_\mu = i(\omega_\mu^{\alpha\beta} L^{-1} \star \widehat{A}'_\alpha \star \widehat{A}'_\beta \star L + \text{h.c.})$ given by

$$K_\mu = \frac{Q}{4i} \omega_\mu^{\alpha\beta} \left[(1 + a^2)^2 y_\alpha y_\beta + 4(1 + a^2) a_\alpha^{\dot{\alpha}} y_\beta \bar{y}_{\dot{\alpha}} + 4a_\alpha^{\dot{\alpha}} a_\beta^{\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} \right] - \text{h.c.}, \quad (28)$$

where $a_{\alpha\dot{\alpha}} = (1+h)^{-1}\lambda x_{\alpha\dot{\alpha}}$ and

$$Q = -\frac{1}{4}(1-a^2)^2 \int_{-1}^1 dt \int_{-1}^1 dt' \frac{q(t)q(t')(1+t)(1+t')}{(1-tt'a^2)^4}. \quad (29)$$

Decomposing $Q = Q_+(a^2) + Q_-(a^2)$, $Q_{\pm}(-a^2) = \pm Q_{\pm}(a^2)$, one finds

$$Q_+ = -\frac{(1-a^2)^2}{4} \sum_{p=0}^{\infty} \binom{-4}{2p} a^{4p} \left(\sqrt{1 - \frac{\nu}{2p+1}} - \sqrt{1 + \frac{\nu}{2p+3}} \right)^2 \quad (30)$$

$$Q_- = \frac{(1-a^2)^2}{4} \sum_{p=0}^{\infty} \binom{-4}{2p+1} a^{4p+2} \left(\sqrt{1 - \frac{\nu}{2p+3}} - \sqrt{1 + \frac{\nu}{2p+3}} \right)^2,$$

which have branch cuts along the real axis for $\text{Re } \nu \leq -3$ and $\text{Re } \nu \geq 1$. From (1) and (28) it follows that all higher spin gauge fields vanish,

$$W_{\mu}^{a_1 \dots a_{s-1}} = 0, \quad s = 4, 6, \dots, \infty \quad (31)$$

while the vierbein and Lorentz connection are given by

$$\omega^{\alpha\beta} = f\omega_{(0)}^{\alpha\beta}, \quad e^a = f_1 dx^a + \lambda^2 f_2 dx^b x_b x^a, \quad (32)$$

$$f = \frac{1 + (1-a^2)^2 \bar{Q}}{|1 + (1+a^2)^2 Q|^2 - 16a^4 |Q|^2}, \quad (33)$$

$$f_1 + \lambda^2 x^2 f_2 = \frac{2}{h^2}, \quad f_2 = \frac{8(Qf + \bar{Q}\bar{f})}{h^2(1+h)^2}, \quad (34)$$

where $a^2 = (1-h)/(1+h)$ and we recall that $h = \sqrt{1-\lambda^2 x^2}$ so that $a^2 \in [-1, 1]$ as x^a varies over the stereographic coordinate chart. For the Type A model, the function Q is real, and we have the simplifications

$$f_1 = \frac{2f}{h^2} [1 + (1-a^2)^2 Q], \quad f_2 = \frac{16Qf}{h^2(1+h)^2},$$

$$f = [1 + (1 + 6a^2 + a^4)Q]^{-1}. \quad (35)$$

which are valid also to order ν^2 in the Type B model. Expanding $Q(a^2, \nu) = \sum_{n=2}^{\infty} \nu^n Q_n(a^2)$, the coefficients $Q_n(a^2)$ with $n \geq 4$ are bounded while $Q_{2,3}(a^2)$ diverge logarithmically at $a^2 = -1$, as can be seen from

$$Q_2 = \frac{(1-a^2)^2}{48a^4} \left[1 - \frac{2a^2}{(1-a^2)^2} + \frac{(1-a^2)^2}{2a^2} \log \frac{1-a^2}{1+a^2} \right], \quad (36)$$

$$Q_3 = \frac{(1-a^2)^2}{96a^6} \left[a^2 + (1-a^4) Li_2^{(-)}(a^2) + (1+a^4) \log \frac{1-a^2}{1+a^2} \right], \quad (37)$$

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where $Li_2^{(-)}(z) = \frac{1}{2}(Li_2(z) - Li_2(-z)) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)^2}$. At $a^2 = 1$, the double integral in (29) diverges at $t = t' = \pm 1$ while the pre-factor vanishes, producing the finite residue

$$\lim_{a^2 \rightarrow 1} Q = \lim_{a^2 \rightarrow 1} Q_2 = -\frac{\nu^2}{24}. \quad (38)$$

Thus, for $\nu \ll 1$, we can approximate $Q \simeq Q_2$ for $-1 \leq a^2 \leq 1$.

To obtain a globally well-defined solution, one introduces a second gauge function $\tilde{L} = L(\tilde{x}; y, \tilde{y})$ where $\lambda^2 \tilde{x}^2 < 1$, and $\tilde{x}^a = x^a / (\lambda^2 x^2)$ for $\lambda^2 x^2 < 0$. The two local representatives have the same functional form, with x^a replaced by \tilde{x}^a , and they are related in the overlap region by a simultaneous reparametrization *and* locally defined gauge transformation with gauge function $\tilde{L}^{-1} \star L$. This implies a Z_2 -duality transformation, which acts on the scalar field as [3]

$$\tilde{\phi}(\tilde{x}) = \frac{\nu \phi(x)}{\phi(x) - \nu}, \quad \lambda^2 x^2 = (\lambda^2 \tilde{x}^2)^{-1} < 0. \quad (39)$$

4 Holographic and Cosmological Interpretation

The solution consists of a scalar field profile on a Weyl-flat metric, which can be written as (from here on we set $\lambda = 1$ for notational simplicity)

$$ds^2 = \frac{4\Omega^2 (d(g_1 x))^2}{(1 - g_1^2 x^2)^2}, \quad (40)$$

$$\Omega = \frac{(1 - g_1^2 x^2) f_1}{2g_1}, \quad g_1 = \exp\left(\frac{1}{2} \int_1^{x^2} \frac{f_2(t) dt}{f_1(t)}\right). \quad (41)$$

The spacetime decomposes into three-dimensional $SO(3, 1)$ orbits describing local foliations of AdS_4 with dS_3 and H_3 spaces in the regions $x^2 > 0$ and $x^2 < 0$, respectively. In the coordinates

$$x^2 > 0 : x^0 = \sinh \tau \tan \frac{\psi}{2}, \quad x^i = n^i \cosh \tau \tan \frac{\psi}{2}, \quad (42)$$

$$x^2 < 0 : x^0 = \cosh \psi \tan \frac{\tau}{2}, \quad x^i = n^i \sinh \psi \tan \frac{\tau}{2}, \quad (43)$$

with $n^i n^i = 1$, our solution takes the form

$$x^2 > 0 : ds^2 = d\psi^2 + \eta^2 \sinh^2 \psi (-d\tau^2 + \cosh^2 \tau d\Omega_2), \quad (44)$$

$$\phi = \frac{\nu}{b} \operatorname{sech}^2 \frac{\psi}{2}, \quad (45)$$

$$x^2 < 0 : ds^2 = -d\tau^2 + \eta^2 \sin^2 \tau (d\psi^2 + \sinh^2 \psi d\Omega_2), \quad (46)$$

$$\phi = \frac{\nu}{b} \sec^2 \frac{\tau}{2}, \quad (47)$$

where

$$\eta = \frac{f_1 h^2}{2}, \quad a^2 = \begin{cases} \tanh^2 \frac{\psi}{4} & x^2 > 0 \\ -\tan^2 \frac{\tau}{4} & x^2 < 0 \end{cases}. \quad (48)$$

In the Type A model, and to order ν^2 in the Type B model, we have

$$\eta = \frac{1 + (1 - a^2)^2 Q}{1 + (1 + 6a^2 + a^4) Q}. \quad (49)$$

The solution has non-trivial torsion

$$T^a \equiv de^a + \omega^a_b \wedge e^b = -e^a \wedge d \log \eta, \quad (50)$$

that can be removed by going to an Einstein frame via a Weyl rescaling

$$\tilde{e}^a = \eta^{-1} e^a, \quad \tilde{\omega}_{ab} = \omega_{ab}. \quad (51)$$

The resulting torsion free Einstein metric reads

$$d\tilde{s}^2 = \frac{4\tilde{\Omega}^2 d\tilde{x}^2}{(1 - \tilde{x}^2)^2}, \quad \tilde{\Omega} = \frac{\Omega}{\eta}, \quad (52)$$

or in terms of foliations,

$$x^2 > 0 : d\tilde{s}^2 = d\tilde{\psi}^2 + \sinh^2 \psi (-d\tau^2 + \cosh^2 \tau d\Omega_2), \quad (53)$$

$$d\tilde{\psi} = \frac{d\psi}{\eta}, \quad (54)$$

$$x^2 < 0 : d\tilde{s}^2 = -d\tilde{\tau}^2 + \sin^2 \tau (d\psi^2 + \sinh^2 \psi d\Omega_2), \quad (55)$$

$$d\tilde{\tau} = \frac{d\tau}{\eta}, \quad (56)$$

We propose that the Weyl rescaling (51) can be generalized to a background covariant transformation taking higher spin frame (1), which in general has torsion T^a depending on $\hat{\Phi}$, to an Einstein frame \tilde{e}^a in which $\tilde{T}^a = 0$. Although the transformation may be complicated in general, it should reduce to the above Weyl rescaling on the $SO(3, 1)$ invariant solution. For consistency, it must therefore be possible to write $\eta = \eta(x^2)$ as a local background covariant functional independent of ν . Indeed, as found in [3] there exist zero-forms $\mathcal{C}_{(2n)}^- = \mathcal{C}_{(2n)}^-[\hat{\Phi}]$ that reduce on the $SO(3, 1)$ invariant solution to ν^{2n} , which can then be used to define the Weyl rescaling covariantly by taking $\eta = \eta((1 - \phi)/\mathcal{C})$ by choosing, for example, $\mathcal{C}[\hat{\Phi}] = \sqrt{\mathcal{C}_{(2)}^-}$.

In the asymptotic region $x^2 \rightarrow 1$, the scale factor $\tilde{\Omega} \rightarrow 1$ and the scalar field

$$\phi = \frac{\nu}{b} \left(\xi - \frac{1}{2} \xi^2 + \dots \right), \quad (57)$$

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where the radial coordinate ξ is defined by $\tanh^2(\psi/2) = e^{-\xi}$, and the unperturbed AdS_4 metric reads $ds_{(0)}^2 = (dr^2 + ds_{dS_3}^2)/\sinh^2(\xi/2)$. In global coordinates, $ds_{(0)}^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega^2$, one instead finds

$$\phi = \frac{2\nu}{b} \left(\frac{1}{r \sin t} - \frac{1}{r^2 \sin^2 t} + \dots \right). \quad (58)$$

In general, if $\phi = \alpha z + \beta z^2 + \dots$ and $ds_{(0)}^2 = (dz^2 + d\sigma^2)/(\lambda(z))^2$ where λ has a simple zero at $z = 0$, then the relation $\beta = \beta(\alpha)$ describes a deformation of the holographically dual field theory, which has been conjectured to be the $O(N)$ model and the Gross-Neveu model in the cases of the Type A and Type B models, respectively [9–12]. In our case, we find

$$\beta = -k\alpha^2, \quad k = \frac{b}{2\nu}, \quad (59)$$

corresponding to a marginal triple-trace deformation of the ultraviolet fixed points of the $O(N)$ and GN models built from the scalar Konishi operator along the lines discussed in [4]. Interestingly, by considering a quantum mechanical approximation, the deformation was found in [4] to generate a bounce in the expectation value of the Konishi operator. Indeed, this is in qualitative agreement with the scalar field profile traced out by the bulk scalars $\phi(x)$ for $\lambda^2 x^2 > -1$ and $\tilde{\phi}(\tilde{x})$ for $\lambda^2 \tilde{x}^2 > -1$. In [4] this deformation was considered as a simplified model, obtained essentially by neglecting the non-abelian structure on the D2 brane, meant to capture qualitatively the behavior of an analogous marginal triple-trace deformation of the three-dimensional CFT on coinciding membranes forming the holographic dual of a $SO(3, 1)$ -invariant instanton of gauged $\mathcal{N} = 8$ supergravity. Here we instead consider it as the actual holographic dual of our solution to the higher-spin gauge theory.

The minimal bosonic models we have studied here are consistent truncations of the higher spin gauge theory based on $shs(8|4) \supset osp(8|4)$ [13], which contains, respectively, $35_+ + 35_-$ scalars and pseudo-scalars in the supergravity multiplet and $1 + 1$ scalar and pseudo-scalar in an $s_{\max} = 4$ multiplet, which we refer to as the Konishi multiplet. While our solutions in the Type A and Type B models utilize the Konishi scalar and pseudo-scalar, respectively [14], the solution of [4] activates instead one of the scalars residing in the supergravity multiplet. Therefore a meaningful comparison of the solutions requires two steps. First, the construction of a new solution in which one of the supergravity scalars in the higher spin gauge theory is activated. Second, the higher spin symmetries must be spontaneously broken down to standard diffeomorphisms.

The breaking of higher spin symmetries requires Goldstone modes which can either be fundamental [9, 15, 16] or composite [17, 18]. In the former scenario, the symmetries are broken classically by the stringy dilaton and the Goldstone modes are massive multi-singleton states. This corresponds to a non-abelian

D2-brane deformation of the holographic dual, whereby the higher spin multiplets are separated from the supergravity multiplet by a large mass-gap. In the latter scenario, on the other hand, the non-abelian structures are not activated, and the virtual processes are instead implemented on the field theory side in the form of double-trace-like “sewing operations” [9, 17]. Correspondingly, radiative corrections in the bulk induce small mass gaps provided the Konishi scalars are subjected to suitable boundary conditions [18]. It should, of course, also be possible to quantize the theory while preserving all symmetries by imposing other boundary conditions (reflecting the conformal dimensions at the free fixed point).

This suggests that M theory on $AdS_4 \times S^7$ with N units of seven-form flux has two phases: a supergravity phase, where all higher spin symmetries are strongly broken, and a higher spin phase, where all symmetries are either unbroken or weakly broken by radiative corrections as described above. In the supergravity phase there are two mass-scales: the Planck scale and the membrane scale, given by powers of N such that the latter is much smaller than the former. Since N is the only free parameter, one may therefore speculate that the supergravity phase arises for energies much smaller than the membrane scale, while the higher spin phase arises for energies much larger than the membrane scale and much smaller than the Planck scale, such that the membrane is effectively tensionless while the bulk theory is nonetheless weakly coupled. The resulting spectrum should contain massless as well as massive fields, that we expect arise from the tensionless membrane along the lines discussed in [2]. We think of our solution as exact in the classical $N \rightarrow \infty$ limit of the higher spin phase.

With the caveats mentioned in the Introduction in mind, having to do with the lack of an understanding of higher-spin covariant geometry, we next proceed to examine some salient features of the standard geometry of our solution. For $x^2 \geq 0$, all the scale factors remain finite and non-vanishing. At $x^2 = 0$, the scale factors $\eta \sinh \psi$ and $\eta \sin \tau$ have ψ and τ derivatives equal to 1, respectively, which means that the DW region is “glued” smoothly to the FRW regions (without deficit or excess angle). For $\nu \ll 1$ and $1 + a^2 \ll 1$, we can approximate $Q \simeq -\frac{\nu^2}{6} \log \frac{1}{1+a^2}$. Thus, for $\tau \sim \tau_{\text{crit}}$, given by

$$\sin \tau_{\text{crit}} \simeq e^{\frac{-3}{2|\nu^2|}} \quad (60)$$

the scale factor η behaves as

$$\eta \sim \left[\frac{|\nu^2|}{3} e^{\frac{3}{2|\nu^2|}} (\tau_{\text{crit}} - \tau) \right]^\epsilon, \quad \epsilon = \begin{cases} +1 & \text{A model} \\ -1 & \text{B model} \end{cases} \quad (61)$$

Thus, in the Type A model it takes infinite proper time (measured in Einstein frame) to reach the critical point defined in (60), where we note that the scalar field takes the value

$$\phi_{\text{crit}} \simeq \frac{4\nu}{b} e^{\frac{3}{|\nu^2|}}. \quad (62)$$

On the other hand, in the Type B model it takes finite proper time to pass this point and eventually reach $\tau = \pi$, which is the surface where $\phi \rightarrow +\infty$. Beyond this surface h^2 is negative, as can be seen either by going to global coordinates or taking $x^2 > 1$, which makes the gauge function L and hence the solution formally ill-defined (at $\tau = \pi$ the first derivatives of the FRW scale factors $\eta \sin \tau$ and $\sin \tau$ with respect to τ and $\tilde{\tau}$, respectively, are equal to $+1$, while their higher derivatives blow up, so that the scale factors are not real analytic at $\tau = \pi$). Thus, the $SO(3, 1)$ invariant cosmology is singularity free in the Type A model, in the sense that it takes infinite proper time to reach the critical point, while it hits the singularity at $\tau = \pi$ in finite proper time in the Type B model. This phenomenon may eventually be understood starting from the microscopic origin of the Vasiliev equations based on topological open phase-space strings [2]. These probe the $SO(3, 1)$ invariant phase-space geometry described by $\hat{\Phi}'$ and \hat{A}'_α , which appears to be singularity free in both the Type A and Type B models.

5 The Effective Einstein-Scalar Field Theory

The qualitative features of the solution to the higher spin gauge theory can be reproduced by a standard scalar-coupled gravity model. To construct an “effective” action whose field equations admit the solution presented above, we proceed as follows. We begin by parametrizing the Lagrangian as

$$e^{-1}\mathcal{L} = K \left(R(\omega) - \frac{1}{2}G\partial_\mu\phi\partial^\mu\phi - V \right), \quad (63)$$

where K, G, V are functions of ϕ to be determined. In this section we set $\lambda = 1$, which can be easily re-instated by dimensional analysis, for notational simplicity. We work in first order formalism and therefore treat the spin connection ω as an independent field. Thus, the field equations are

$$R_{\mu\nu}(e) = \frac{1}{2}G\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}Vg_{\mu\nu}, \quad (64)$$

$$T_{\mu\nu}{}^a = e_{[\mu}^a\partial_{\nu]} \log K, \quad (65)$$

where the torsion tensor is defined as usual by $T^a = de^a + \omega^a{}_b \wedge e^b$, and $R_{\mu\nu}(e)$ is the symmetric part of $R_{\mu,\nu}(\omega) = R_{\mu\rho}{}^{ab}(\omega)e_b^{\rho}e_{\nu b}$, and as such, it is the standard Ricci tensor in terms of torsion-free spin connection, or equivalently, the symmetric Christoffel symbol. In obtaining (64), we have used (65) to show that $R_{[\mu,\nu]} = 0$. The equation (65) follows from the variation of the action with respect to the spin connection. As for the scalar field equation, it follows by taking the divergence (64) and using the conservation law $D^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$.

Substituting our solution into the field equations, after considerable algebra we

find for the Type A model that

$$K = \frac{4}{(f_1 h^2)^2}, \quad (66)$$

$$G = 2K \left(f_1 \frac{\partial f}{\partial h^2} + \frac{1}{2} f_2 f \right), \quad (67)$$

$$V = -6K [f^2 + f(1-f)h^2] + \frac{1}{2} h^4 (1-h^2) G, \quad (68)$$

where it is understood that $h^2 = \phi/\nu$. Next, we observe that the Hilbert-Einstein term can be written in terms of torsion free connection $\omega_\mu{}^{ab}(e)$ by using the relation $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + e_\mu{}^{[a} \partial^b] \frac{\partial \log K}{\partial \phi}$, and subsequently we can go over to Einstein frame by rescaling the metric as

$$g_{\mu\nu} = K^{-1} \bar{g}_{\mu\nu}. \quad (69)$$

Note that, evaluated on the solution, $K^{-1} = \eta^2$, with η given in (49). Dropping the bar for simplicity in notation, we get the action

$$I = \int d^4x \sqrt{-g} \left(R(\omega(e)) - \frac{1}{2} G \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} K^{-1} V \right), \quad (70)$$

Evidently the potential takes a highly complicated form, reflecting the higher derivative scalar field self-couplings in the full theory, which can be rewritten as contact terms on the $SO(3,1)$ invariant solution. Since the potential is obtained to accommodate our exact solution, it offers a highly limited information regarding the structure of the full action or field equations of the higher spin gauge theory. Nonetheless, the potential is not fixed by picking just any forms of metric and scalar field configurations we like (though this approach may be of some utility in its own right for gaining insights to some aspects of gravitational instantons, as shown, for example, in [19]), but rather it is a consequence of a solution that is dictated *a priori* by a well defined higher spin gauge theory.

6 Concluding Remarks

Higher spin gauge theory, which at a first superficial glance may appear to be more complex than ordinary gravity, in fact exhibits a remarkable simplicity in that the field equations can be solved by means of purely algebraic methods. Thus, nontrivial exact solutions can be given even without knowing the action nor the equations of motion in a form in which the spacetime fields and their couplings are explicitly displayed. One may speculate that Vasiliev's equations are somehow exactly solvable in phase space (see [8] for concrete work along these lines) so that any solution could be obtained algebraically starting from the knowledge of the Weyl zero-form at a point in space time. Ultimately, once

the connection to ordinary gravity has been made more explicit, one may hope that these basic properties of higher spin gauge theory could also shed light on similar issues in ordinary gravity.

As for the cosmological applications of our exact solution, which is the only known time dependent solution of the higher spin gauge theory at present, a further development of the theory is needed to provide a geometric formulation with manifest higher spin symmetries, and to facilitate the description of geodesic equation of motion and harmonic analysis. Recalling that the higher spin gauge theory is believed to emerge from tensionless limit of strings and branes, it is tempting to envisage a new cosmological model in which the very early universe is described by tensionless strings and branes, and that as the universe cools down, the higher spin symmetries first break down by mechanism mentioned earlier to the usual symmetries associated with spin $s \leq 2$ massless fields, and subsequently the universe evolves in the more familiar fashion that involves inflation and other phenomena that we can describe by means of matter coupled supergravity theories embedded in the tensionful broken phase of string theory. Matter couplings can be described in higher spin gauge theory even though the formalism for achieving this has not been adequately developed so far. Concerning inflation in the context of asymptotically AdS spacetimes, the idea that such spacetimes may contain an inflationary de Sitter regions (see, for example, [20] and references therein) may also be entertained in the context of a cosmological model based on higher spin gauge theory. In particular, it would be interesting to determine whether the massless Konishi scalars that are present only in the tensionless limit have a role to play in any inflation scenario.

Acknowledgments

We thank Thomas Hertog, Gary Horowitz, Jason Kumar, Hong Lu, Chris Pope and Misha Vasiliev for useful discussions. P. Sundell would like to thank the George P. and Cynthia W. Mitchell Institute for Fundamental Physics for hospitality. The work of E.S. is supported in part by NSF grant PHY-0314712 and the work of P.S. is supported in part by INTAS Grant 03-51-6346.

References

- [1] M.A. Vasiliev (1990) *Phys. Lett.* **B243** 378.
- [2] J. Engquist and P. Sundell, “Brane partons and singleton strings”, hep-th/0508124.
- [3] E. Sezgin and P. Sundell, “An exact solution of 4D higher spin gauge theory”, hep-th/0508158.
- [4] T. Hertog and G. T. Horowitz (2004) *JHEP* **0407** 073; (2005) *JHEP* **0504** 005.
- [5] M.A. Vasiliev, “Higher spin gauge theories: star-product and AdS space”, hep-th/9910096.
- [6] E. Sezgin and P. Sundell (2002) *JHEP* **0207** 055.

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- [7] K.I. Bolotin and M.A. Vasiliev (2000) *Phys. Lett.* **B479** 421.
- [8] S.F. Prokushkin and M.A. Vasiliev (1999) *Nucl. Phys.* **B545** 385.
- [9] E. Sezgin and P. Sundell (2002) *Nucl. Phys.* **B644** 303; (2003) *Erratum-ibid.* **B660** 403.
- [10] I.R. Klebanov and A.M. Polyakov (2002) *Phys. Lett.* **B550** 213.
- [11] R.G. Leigh and A.C. Petkou (2003) *JHEP* **0306** 011.
- [12] E. Sezgin and P. Sundell (2005) *JHEP* **0507** 044.
- [13] E. Sezgin and P. Sundell, *JHEP*, **9811**, 016 (1998).
- [14] J. Engquist, E. Sezgin and P. Sundell (2002) *Class. Quant. Grav.* **19** 6175.
- [15] B. Sundborg (2001) *Nucl. Phys. Proc. Suppl.* **102** 113.
- [16] N. Beisert, M. Bianchi, J.F. Morales and H. Samtleben (2004) *JHEP* **0407** 058.
- [17] I.R. Klebanov and A.M. Polyakov (2002) *Phys. Lett.* **B550** 213.
- [18] L. Girardello, M. Porrati and A. Zaffaroni (2003) *Phys. Lett.* **B561** 289.
- [19] S. Mukherjee, B.C. Paul, N. Dadhich and A. Kshirsagar (1992) *Phys. Rev.* **D45** 2772.
- [20] B. Freivogel, V.E. Hubeny, A. Maloney, R. Myers, M. Rangamani and S. Shenker, “Inflation in AdS/CFT”, hep-th/0510046.