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Geometric Solutions of Upper Triangular Toda Hierarchies

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Abstract. In this paper one considers commutative subalgebras of the $\mathbb{Z} \times \mathbb{Z}$-matrices generated by a maximal commutative subalgebra of the complex $k \times k$-matrices and a shift matrix that commutes with them. Key objects are parameter dependent perturbations of these algebras inside the upper triangular $\mathbb{Z} \times \mathbb{Z}$-matrices such that the perturbed generators satisfy Lax equations with respect to an infinite number of commuting directions. Various appropriate geometric settings are described in which one can actually construct solutions of these hierarchies.

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1 Introduction

Toda hierarchies play a prominent role in string theory and quantum gravity, see e.g. [1], [2], [7] and [5]. In this paper appropriate geometric settings are described in which one can actually construct solutions of so-called upper triangular Toda hierarchies. The content of the various sections is as follows: in the first section one starts with a description of the commutative subalgebras of the $\mathbb{Z} \times \mathbb{Z}$-matrices generated by a maximal commutative subalgebra of the complex $k \times k$-matrices and a shift matrix that commutes with them. Next one presents in this section the main topic: deformations of these algebras inside the upper triangular $\mathbb{Z} \times \mathbb{Z}$-matrices such that the perturbed generators satisfy Lax equations with respect to an infinite number of commuting directions. These nonlinear evolution type equations form the so-called upper triangular Toda hierarchies. Here one also finds a setting for their linearization and a relation for conjugated algebras between solutions of this linearization. The next section gives a wide range of infinite dimensional groups for which one will construct solutions of the hierarchy, it also treats the central decomposition in those groups and gives a description of the relevant commuting flows that fit into them. The last section presents the construction of the solutions.

2 Upper Triangular Toda Hierarchies

Before giving the actual form of the deformation and a description of the non-linear equations they should satisfy, one introduces a number of notations. Let R be a commutative ring with unit element 1. Then the ring of $k \times k-$matrices with coefficients from R is denoted by $M_k(R)$. Likewise one writes $M_{\mathbb{Z}}(R)$ for the R-module of $\mathbb{Z} \times \mathbb{Z}$-matrices with coefficients from R. Basic matrices in $M_{\mathbb{Z}}(R)$ are the $E_{(i,j)}$, i and $j \in \mathbb{Z}$, defined by

$$(E_{(i,j)})_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}1.$$

Any $\mathbb{Z} \times \mathbb{Z}$-matrix B decomposes as a formal sum

$$B = \sum_{i,j \in \mathbb{Z}} b_{(i,j)} E_{(i,j)}, \text{ with } b_{(i,j)} \in R.$$

Like in the finite dimensional case the numbering of the columns of the matrix is chosen to increase from left to right and that of the rows from north to south. For each $k \geq 1$, one can decompose any $\mathbb{Z} \times \mathbb{Z}$-matrix B in $k \times k$-blocks

$$B = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & B_{n-1\,n-1} & B_{n-1\,n} & B_{n-1\,n+1} & \ddots \\ \ddots & B_{n\,n-1} & B_{n\,n} & B_{n\,n+1} & \ddots \\ \ddots & B_{n+1\,n-1} & B_{n+1\,n} & B_{n+1\,n+1} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where the $\alpha\beta$-th matrix coefficient $(B_{ij})_{\alpha\beta}$ of $B_{ij} \in M_k(R)$ is defined by

$$(B_{ij})_{\alpha\beta} = b_{(\alpha-1+ik,\beta-1+jk)}.$$

A central role is played by the matrix

$$\Lambda := \sum_{i \in \mathbb{Z}} E_{(i-1,i)}$$

whose k-th power is linked to the $k \times k$-block decomposition mentioned before. It has the form

$$\Lambda^k = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & \mathrm{Id} & 0 & \ddots \\ \ddots & 0 & 0 & \mathrm{Id} & \ddots \\ \ddots & 0 & 0 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

With every collection $\{d(ks)|s \in \mathbb{Z}\}$ of matrices in $M_k(R)$ one associates a diagonal of $k \times k$-blocks diag$(d(ks))$ in $M_{\mathbb{Z}}(R)$ that is given by the formula

$$\operatorname{diag}(d(ks)) := \sum_{s\in\mathbb{Z}}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} d(ks)_{\alpha\beta}E_{(ks+\alpha-1,ks+\beta-1)}.$$

The $k \times k$-block decomposition of this matrix looks as follows

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & d(kn-k) & 0 & 0 & \ddots \\ \ddots & 0 & d(kn) & 0 & \ddots \\ \ddots & 0 & 0 & d(kn+k) & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Denote the ring of $k \times k$-block diagonal matrices in $M_{\mathbb{Z}}(R)$ by

$$\mathcal{D}_k(R) = \{d = \operatorname{diag}(d(ks))|d(ks) \in M_k(R) \text{ for all } s \in \mathbb{Z}\}.$$

One fixes a ringhomomorphism i_k from $M_k(R)$ into $\mathcal{D}_k(R)$. It consists of taking for an $A \in M_k(R)$ all diagonal blocks of $i_k(A)$ equal to A. Since conjugating elements from $\mathcal{D}_k(R)$ with Λ^k shifts the $k\times k$-blocks upward along the diagonal, one sees that any matrix $A \in M_{\mathbb{Z}}(R)$ can uniquely be written as

$$A = \sum_{j\in\mathbb{Z}} d_j\Lambda^{kj}, \tag{1}$$

with d_j in $\mathcal{D}_k(R)$. One calls $d_j\Lambda^{kj}$ the *j-th $k \times k$-block diagonal* of A. To this decomposition one links two notations: if $A = \sum_{j\in\mathbb{Z}} d_j\Lambda^j$ as in (1) then one writes

$$A_+(k) = \sum_{j\geq 0} d_j\Lambda^{kj} \quad\text{and}\quad A_-(k) = \sum_{j<0} d_j\Lambda^{kj}.$$

Inside $M_{\mathbb{Z}}(R)$ two subspaces are considered that are rings w.r.t. the usual product

Definition 2.1 *An element A in $M_{\mathbb{Z}}(R)$ is called upper $k \times k$-block triangular of level m, if it can be written as*

$$A = \sum_{j\geq m} d_j\Lambda^{kj}, \quad \textit{with} \quad d_j \in \mathcal{D}_k(R).$$

The collection of all these elements is denoted by $UT_m(R)$. One calls m the order of A in Λ^k, if d_m is nonzero. Further one uses the notation

$$UT(R) := \bigcup_{k\in\mathbb{Z}} UT_k(R)$$

for the R-algebra of all upper triangular matrices.

Likewise one introduces the opposite class of matrices

Definition 2.2 *An element A in $M_{\mathbb{Z}}(R)$is called lower $k \times k$-block triangular of level m, if it can be written as*

$$A = \sum_{j \leq m} d_j \Lambda^{kj}, \quad with \;\; d_j \in \mathcal{D}_k(R).$$

The collection of all these elements is denoted by $LT_m(R)$. Like for UT one calls m the order of A in Λ^k, if d_m is nonzero. Similarly, the notation

$$LT(R) := \bigcup_{k \in \mathbb{Z}} LT_k(R)$$

is used for the set of all lower triangular matrices.

In the rest of this paper R will be a commutative $\mathbb{C}$-algebra so that $M_k(\mathbb{C})$ embeds naturally into $M_k(R)$. Let $\mathbf{k}$ be a commutative subalgebra of $M_k(\mathbb{C})$ with basis $\{F_\rho \mid 1 \leq \rho \leq l\}$. In order to maximize the number of directions among the commuting flows one assumes $\mathbf{k}$ to be maximal. Here one can think of the diagonal matrices, but also of algebras like

$$\mathbf{k} = \left\{ k = \sum_{i=0}^{k-1} a_i B^i \text{ with } B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \right\}$$

with a significant nilpotent component. The maximality assumption assures that the identity Id belongs to $\mathbf{k}$.

The upper triangular Toda hierarchy associated with $\mathbf{k}$ was introduced in [4] and is a generalization of a hierarchy considered in [10]. In the hierarchy one considers deformations inside $UT(R)$ of the commutative subalgebra of $M_{\mathbb{Z}}(R)$ generated by the matrices Λ^{-k} and the $\{i_k(F_\alpha) \mid 1 \leq \alpha \leq l\}$. This deformation should first of all preserve the algebraic relations between the generators

$$[\Lambda^{-k}, i_k(F_\alpha)] = 0 \text{ and } i_k(F_\alpha) i_k(F_\beta) = \sum_\gamma d^\gamma_{\alpha\beta} i_k(F_\gamma). \tag{2}$$

As deformation of the matrix Λ^{-k} one takes a matrix of the form

$$\mathcal{M} := \sum_{i \geq -1} m_i \Lambda^{ki} \text{ with } m_{-1} \text{ invertible .}$$

One verifies easily that each such a matrix can be written as

$$\mathcal{M}(W) = W\Lambda^{-k}W^{-1}, \text{ where } W = \sum_{i\geq 0} w_i \Lambda^{ik} \text{ with } w_0 \text{ invertible.} \tag{3}$$

The deformations of the $\{i_k(F_\alpha)\}$ are the

$$V_\alpha = \sum_{i\geq 0} v_{i,\alpha}\Lambda^{ki}, \text{ with } v_{0,\alpha} = w_0 i_k(F_\alpha) w_0^{-1}.$$

As mentioned before they should first of all satisfy

$$[\mathcal{M}, V_\alpha] = 0 \text{ and } V_\alpha V_\beta = \sum_\gamma d^\gamma_{\alpha\beta} V_\gamma. \tag{4}$$

Note that the algebraic relations (4) are automatically satisfied if one takes for all α

$$V_\alpha(W) := W i_k(F_\alpha) W^{-1}. \text{ with } W \text{ as in } (3). \tag{5}$$

One says then that the perturbation $\{\mathcal{M}(W), V_\alpha(W)\}$ has been obtained by *dressing* the trivial solution. Besides the relations (4), the matrices $\{\mathcal{M}, V_\alpha\}$ should satisfy a number of nonlinear differential equations for a set of commuting derivations $\{\partial_{j\beta} \mid j > 0, 1 \leq \beta \leq l\}$ of R

$$\partial_{j\beta}(\mathcal{M}) = [(\mathcal{M}^j V_\beta))_-, \mathcal{M}] =: [C_{j\beta}, \mathcal{M}] \text{ and } \partial_{j\beta}(V_\alpha) = [C_{j\beta}, V_\alpha]. \tag{6}$$

They are called the *Lax equations of the upper triangular Toda hierarchy associated with* $\mathbf{k}$. Note that the above form of the hierarchy is similar to that of the KP-hierarchy and other soliton equations, where the perturbation of the commutative algebra of differential operators corresponding to the trivial solution is taken inside the ring of pseudodifferential operators, see [9] and [6].

These Lax equations can be obtained as the compatibility conditions for a suitable linear system, the so-called *linearization* of the hierarchy. Thereto one considers the free $UT(R)$-module $M^{(0)}$ of oscillating matrices at zero, consisting of the formal product of an element in $UT(R)$ and the exponential factor

$$\varphi_0 := \exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{l} s_{j\beta} i_k(F_\beta)\Lambda^{-jk}) \tag{7}$$

Define the action of $\partial_{j\beta}$ on φ_0 as the partial derivative w.r.t. the parameter $s_{j\beta}$. On elements of $UT(R)$ one lets $\partial_{j\beta}$ act coefficient wise and on the formal product in $M^{(0)}$ the action is defined by imposing the Leibnitz rule and combining the two foregoing actions. The linearization consists then of the following set of equations for a $\varphi := \hat{\varphi}\varphi_0 \in M^{(0)}$

$$\mathcal{M}\varphi = \varphi\Lambda^{-k}, V_\alpha\varphi = \varphi i_k(F_\alpha) \text{ and } \partial_{j\beta}(\varphi) = C_{j\beta}\varphi. \tag{8}$$

The exponential factor φ_0 satisfies these equations for the trivial solution $\mathcal{M} = \Lambda^{-k}$ and $V_\alpha = i_k(F_\alpha)$. If $\hat{\varphi} = \sum_{i\geq 0} w_i \Lambda^{ik}$ with w_0 invertible , then this set of equations implies the Lax equations for the matrices

$$\mathcal{M}(\hat{\varphi}) = \hat{\varphi}\Lambda^{-k}\hat{\varphi}^{-1} \text{ and } V_\alpha(\hat{\varphi}) = \hat{\varphi} i_k(F_\alpha)\hat{\varphi}^{-1} \tag{9}$$

as one will see in section 4. One calls φ then *a wave matrix at zero* for this set of solutions.

Consider a $C \in \mathrm{GL}_k(\mathbb{C})$. Then the $\{CF_\rho C^{-1}\}$ form an appropriate basis of $C\mathbf{k}C^{-1}$ and $i_k(C)\varphi_0 i_k(C^{-1})$ is the exponential factor in this case. By conjugating all the equations in (8) with $i_k(C)$ one sees first of all that the $\{i_k(C)\mathcal{M}i_k(C^{-1}), i_k(C)V_\alpha i_k(C^{-1})\}$ are obtained by dressing the trivial solution $\{\Lambda^{-k}, i_k(CF_\alpha C^{-1})\}$ of the upper triangular Toda hierarchy associated with $C\mathbf{k}C^{-1}$ with $i_k(C)\hat{\varphi} i_k(C^{-1})$. Secondly one verifies directly that for all $j \geq 1$ and all β

$$(i_k(C)\mathcal{M}^j V_\beta i_k(C^{-1}))_- = i_k(C)C_{j\beta} i_k(C^{-1}).$$

In other words $i_k(C)\varphi i_k(C^{-1})$ is a wave function at zero for the set $\{i_k(C)\mathcal{M}i_k(C^{-1}), i_k(C)V_\alpha i_k(C^{-1})\}$ and thus they form a solution of the upper triangular Toda hierarchy associated with $C\mathbf{k}C^{-1}$.

In practice elements of $M^{(0)}$ occur if one requires sufficient convergence conditions of both factors. Here one has a wide range of choices as is illustrated in the sequel, where various geometric settings are described in which this product becomes real and where the equations of the linearization can be shown to hold.

3 The Geometric Setting

The infinite dimensional groups that enable the construction of solutions consist of collections of operators on a suitable Hilbert space. Let S^1 be the unit circle in the complex plane. Consider the Hilbert space $H = L^2(S^1, \mathbb{C}^k)$ of square integrable $\mathbb{C}^k$-valued functions on S^1. The elements of this space are denoted by

$$h = \sum_{n\in\mathbb{Z}} a(n)z^n, \text{ where } a(n) \in \mathbb{C}^k \text{ for all } n \in \mathbb{Z}.$$

Let $(\cdot \mid \cdot)$ be the standard inner product on $\mathbb{C}^k$. The inner product on H is then given by

$$< \sum_{n\in\mathbb{Z}} a(n)z^n \mid \sum_{n\in\mathbb{Z}} b(n)z^n >:= \sum_{n\in\mathbb{Z}} (a(n) \mid b(n)).$$

If $\{f_i \mid 0 \leq i \leq k-1\}$ denotes the standard basis of $\mathbb{C}^k$, where f_i has a one on the $i+1$-th entry and zeros elsewhere, then on gets the orthonormal basis $\{e_i \mid i \in \mathbb{Z}\}$ of H by putting for all $j \in \mathbb{Z}$ and all $s, 0 \leq s \leq k-1$,

$$e_{s+kj} := f_s z^j.$$

To each bounded operator $B \in B(H)$ one associates the $\mathbb{Z} \times \mathbb{Z}$-matrix $[B] = ([B]_{(l,k)})$ w.r.t. this basis. E.g. if $A \in \mathrm{gl}_k(\mathbb{C})$, then multiplying from the left with A defines a bounded map $M_A : H \mapsto H$ such that the $k \times k$-block decomposition of its $\mathbb{Z} \times \mathbb{Z}$-matrix looks like

$$[M_A] = i_k(A) = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & A & 0 & 0 & \ddots \\ \ddots & 0 & A & 0 & \ddots \\ \ddots & 0 & 0 & A & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Inside H one considers a number of subspaces. For each $i \in \mathbb{Z}$, let $H^{(i)}$ be the complex subspace of H spanned by the

$$\{f_s z^i \mid 0 \leq s \leq k-1\}.$$

The projection $H \mapsto H^{(i)}$ given by $\sum_{j \in \mathbb{Z}} h(j) z^j \mapsto h(i) z^i$ is denoted by $p^{(i)}$. The space H decomposes as the direct sum

$$H = \oplus_{i \in \mathbb{Z}} H^{(i)}$$

and this determines for each bounded linear operator $B \in B(H)$ the associated block decomposition $B = (B_{ij})$, where $B_{ij} := p^{(i)} \circ B \mid H^{(j)}$ and correspondingly the matrix decomposition $[B] = ([B_{ij}])$ in $k \times k$-blocks. Another subspace that will play a role in the sequel is the subspace H_j, $j \in \mathbb{Z}$, defined by

$$H_j = \oplus_{i \leq j} H^{(i)}.$$

with its orthogonal projection $p_j := \oplus_{i \leq j} p^{(i)}$. One will use a special notation for the decomposition of any element $b \in B(H)$ w.r.t. the splitting $H = H_j \oplus H_j^{\perp}$, namely

$$b = \begin{pmatrix} b_{++}(j)) & b_{+-}(j) \\ b_{-+}(j) & b_{--}(j) \end{pmatrix}. \tag{10}$$

For two Hilbert spaces H_1 and H_2 and any integer $p \geq 1$ one writes $S_p(H_1, H_2)$ or shortly S_p for the Schatten ideals, see [8], of bounded operators $A : H_1 \mapsto H_2$ such that

$$||A||_p^p := \mathrm{trace}(A^* A)^{\frac{p}{2}} < \infty.$$

For each such a p one introduces the group $G(p)$ by

$$G(p) = \left\{ g = (g_{ij}) \in \mathrm{GL}(H) \;\middle|\; \begin{matrix} \oplus_{i<j} g_{ij} \in S_p \\ \oplus_{i<j} (g^{-1})_{ij} \in S_p \end{matrix} \right\}.$$

It consists of the invertible elements in the Banach algebra

$$\mathcal{G}(p) = \left\{ b = (b_{ij}) \in B(H) \,\middle|\, \oplus_{i<j} b_{ij} \in S_p \right\}$$

equipped with the norm $||\cdot||_{res}$ defined by

$$||b||_{res} = ||(b_{ij})||_{res} := ||b|| + || \oplus_{i<j} b_{ij}||_p.$$

Here $||\cdot||$ is the operator norm and $||\cdot||_p$ the Schatten norm. This turns $G(p)$ in a natural way into a Banach Lie group with $\mathcal{G}(p)$ as its Lie algebra. For each $N \geq 1$, one has the subgroup

$$G_N := \left\{ g = (g_{ij}) \in G(p) \,\middle|\, \begin{array}{c} g_{ij} = 0 \text{ for } i < j \text{ and } |j| > N \\ g_{ij} = 0 \text{ for } i < -N \text{ and } -N \leq j \leq N \\ g_{jj} \text{ is invertible for } |j| > N \end{array} \right\} \quad (11)$$

By using the fact that in each $\mathrm{GL}(n, \mathbb{C})$ any element can be linked to the identity by a continuous path, one proves the same for the group G_N, in other words G_N is connected. Since each element in $G(p)$ differs from an element in G_N for some N by a small operator in the Schatten class S_p one may conclude

Lemma 1 *The group $G(p)$ is connected.*

Next the *big cell* in $G(p)$ will be discussed. The Lie algebra $\mathcal{G}(p)$ is the sum of the Lie subalgebras

$$\mathcal{P} := \left\{ p = (p_{ij}) \in \mathcal{G}(p) \,\middle|\, p_{ij} = 0 \text{ for all } i > j \right\}$$

and

$$\mathcal{U}_- := \left\{ u = (u_{ij}) \in \mathcal{G}(p) \,\middle|\, u_{ij} = 0 \text{ for all } i \leq j \right\}.$$

Their corresponding Lie groups are

$$P := \left\{ p = (p_{ij}) \in G(p) \,\middle|\, p_{ij} = 0 \text{ and } (p^{-1})_{ij} = 0 \text{ for all } i > j \right\}$$

and

$$U_- := \left\{ u = (u_{ij}) \in G(p) \,\middle|\, \begin{array}{c} u_{ij} = 0 \text{ for all } i < j \\ u_{ii} = \mathrm{Id} \text{ for all } i \in \mathbb{Z} \end{array} \right\}.$$

As the map from $\mathcal{U}_- \times \mathcal{P}$ to $G(p)$ defined by

$$(u, p) \mapsto \exp(u) \exp(p)$$

is a local diffeomorphism at $(0, 0)$, the set $U_- P$ is an open subset of G. It can be characterized in a similar fashion as in the finite dimensional case

Proposition 3.1 *Let $\Omega \subset G(p)$ be the collection of all $g \in G(p)$ such that $g_{++}(i)$ is invertible for all $i \in \mathbb{Z}$. Then Ω is equal to U_-P.*

Clearly U_-P is contained in Ω. To prove the remaining inclusion one decomposes the space

$$H = H_{-N-1} \oplus \sum_{i=-N}^{N} H^{(i)} \oplus H_N^{\perp}$$

and shows that for each $g \in \Omega$ there is a sufficiently large N such that all the $g_{jj}, j < -N$, and all the $g_{jj}, j > N$, are invertible. Inside the groups $\mathrm{GL}(H_{-N-1})$ and $\mathrm{GL}(H_N^{\perp})$ the required decompositions for $g_{++}(-N-1)$ and $g_{--}(N)$ can then be found by convergent recursive procedures. By combining this with the finite dimensional result and by using the fact that Ω is invariant under left translations from U_- and right translations from P one gets for a $g \in \Omega$ the splitting $g = u_-p$. The big cell Ω is a dense open subset of $G(p)$ since the group $G(p)$ is connected.

Next the relevant commuting flows will be discussed. There is a natural group of commuting flows associated with $\mathbf{k}$ that embeds into $\mathrm{GL}(H)$. Let namely U be any open connected neighbourhood in the complex plane of the unit circle S^1. Then one writes $\Gamma(U, \mathbf{k})$ for the set of holomorphic maps $\gamma : U \mapsto \mathbf{k}$ such that

$$\det(\gamma(u)) \neq 0 \text{ for all } u \in U.$$

It is a group for the pointwise multiplication in $\mathrm{GL}_k(\mathbb{C})$. If two such neighbourhoods U_1 and U_2 satisfy $U_2 \subset U_1$ then one has a natural embedding of $\Gamma(U_1, \mathbf{k})$ into $\Gamma(U_2, \mathbf{k})$ and the inductive limit is denoted by $\Gamma(\mathbf{k})$. Each $\gamma \in \Gamma(\mathbf{k})$ has a Fourier series

$$\sum_{i \in \mathbb{Z}} \gamma_i z^i, \text{ with } \gamma_i \in \mathbf{k}.$$

It is convenient to embed $\Gamma(\mathbf{k})$ into $\mathrm{GL}(H)$ by letting it act on H through multiplication from the left with this series. It defines then the bounded operator $M_\gamma : H \mapsto H$. n general the operators $M_\gamma : H \mapsto H$ do not belong to $G(p)$. E.g. let $\gamma(0)$ denote the element $u \mapsto \mathrm{Id}\, u^{-1}$, then $[M_{\gamma(0)}] = \Lambda^k$ and $M_{\gamma(0)}$ does not belong to $G(p)$. Since the elements of the commutator subgroup of $M_{\gamma(0)}$ have the form

$$\sum_{i \in \mathbb{Z}} M_{A_i} M_{\gamma(0)}^i, \text{ with } A_i \in \mathrm{gl}_k(\mathbb{C}),$$

one sees that $\Gamma(\mathbf{k})$ contains pretty much all the directions that commute with $\mathbf{k}$ and $M_{\gamma(0)}$.

The next step is to determine a subgroup of these flows that embeds into $G(p)$. By splitting the algebra $\mathbf{k}$ into its semi simple and nilpotent part and by using the fact that holomorphic vector bundles over $\mathbb{P}^1(\mathbb{C})$ are a direct sum of line bundles, see [3], one arrives at the following decomposition of $\Gamma(\mathbf{k})$

Proposition 3.2 *There is a subgroup* $\Delta(\mathbf{k})$ *isomorphic to* $\mathbb{Z}^r$*, where* r *is the dimension of the semi simple part of* $\Gamma(\mathbf{k})$*, such that* $\Gamma(\mathbf{k})$=$\Gamma_+(\mathbf{k})\,\Delta(\mathbf{k})\,\Gamma_-(\mathbf{k})$*, where*

$$\Gamma_+(\mathbf{k}) = \{\gamma \mid \gamma = \exp(\sum_{s\leq 0}\gamma_s z^s), \text{ with } \gamma_s \in \mathbf{k} \text{ for all } s \leq 0\}$$

and

$$\Gamma_-(\mathbf{k}) = \{\gamma \mid \gamma = \exp(\sum_{s>0}\gamma_s z^s), \text{ with } \gamma_s \in \mathbf{k} \text{ for all } s > 0\}.$$

The subgroup $\Gamma_-(\mathbf{k})$ embeds into the subgroup U_- of $G(p)$ as the inverse of $\gamma_- = \exp(\sum_{s>0}\gamma_s z^s) \in \Gamma_+(\mathbf{k})$ is equal to $\exp(\sum_{s>0} -\gamma_s z^s)$. If one takes a basis $\{F_\beta \mid 1 \leq \beta \leq l\}$ of $\mathbf{k}$, then there is for each element γ_- of $\Gamma_-(\mathbf{k})$ an $N > 1$ such that

$$\gamma_- = \exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{l} s_{j\beta}F_\beta z^j), s_{j\beta} \in \mathbb{C}, \sum_{j,\beta}|s_{j\beta}|N^i < \infty.$$

In other words, the $\{s_{j\beta}\}$ are the coordinates on $\Gamma(\mathbf{k})_-$ w.r.t. the directions $\{F_\beta z^j\}$.

4 Construction of Solutions

One starts with an element $g \in G(p)$. Inside the group of commuting flows $\Gamma_-(\mathbf{k})$ one considers

$$\Gamma_-(g,\mathbf{k}) = \{\gamma_- \in \Gamma_-(\mathbf{k}) \mid g\gamma_-^{-1} \in \Omega\}.$$

By approximating an element $g \in G(p)$ with an element from G_N for a sufficiently large N one shows that the condition defining $\Gamma_-(g,\mathbf{k})$ translates into the nonzero set of suitable nonvanishing determinant functions and since the group $G(p)$ is connected there holds

Lemma 2 *The set* $\Gamma_-(g,\mathbf{k})$ *is an open dense subset of* $\Gamma_-(\mathbf{k})$.

Now one chooses for R the ring of holomorphic functions on $\Gamma_-(g,\mathbf{k})$ and as the set of derivations of R one takes the

$$\{\partial_{j\beta} := \frac{\partial}{\partial s_{j\beta}} \mid j \geq 1, 1 \leq \beta \leq l\}.$$

Since the element $\gamma_- \in \Gamma_-(g,\mathbf{k})$, there holds $g\gamma_-^{-1} = u_-(g,\gamma_-)p(g,\gamma_-)$, with one component $p(g,\gamma_-) \in P$ and the other $u_-(g,\gamma_-) \in U_-$.

From the construction of this decomposition in proposition 3.1 one deduces that both $[p(g,\gamma_-)]$ and $[p(g,\gamma_-)]_{-1}$ belong to $UT_0(R)$. Similarly, the matrix $[u_-(g,\gamma_-)]$ and its inverse are in $LT_0(R)$. Thus one has obtained an element $[p(g,\gamma_-)][\gamma_-] =: \hat{\Phi}\varphi_0 = \Phi$ of $M^{(0)}$ for which the product is well-defined. Note that on the matrix level one has the relation

$$[u_-(g,\gamma_-)]^{-1}[g] = [p(g,\gamma_-)][\gamma_-]. \tag{12}$$

Consider now the derivative of $[p(g,\gamma)][\gamma_-]$ w.r.t. the parameter $s_{j\beta}$. On one hand there holds

$$\begin{aligned}\partial_{j\beta}(\hat{\Phi}[\gamma_-]) &= \partial_{j\beta}(\hat{\Phi})[\gamma_-] + \hat{\Phi} i_k(F_\beta)\Lambda^{-jk}[\gamma_-]\\ &= \{\partial_{j\beta}(\hat{\Phi})\hat{\Phi}^{-1} + \mathcal{M}(\hat{\Phi})^j V_\beta(\hat{\Phi})\}\Phi.\end{aligned} \tag{13}$$

On the other hand, if one writes $\Psi := [u_-(g,\gamma_-)]^{-1}$, then equation (12) results in

$$\partial_{j\beta}(\Phi) = \partial_{j\beta}(\Psi)[g] = \{\partial_{j\beta}(\Psi)\Psi^{-1}\}\Phi. \tag{14}$$

As $M^{(0)}$ is a free $UT(R)$-module, the equations (13) and (14), yield

$$\partial_{j\beta}(\hat{\Phi})\hat{\Phi}^{-1} + \mathcal{M}(\hat{\Phi})^j V_\beta(\hat{\Phi}) = \partial_{j\beta}(\Psi)\Psi^{-1}.$$

Since the right hand side of this equation belongs to $LT_{-1}(R)$, the left hand side to $UT_{-j}(R)$ and the matrix $\partial_{j\beta}(\hat{\Phi})\hat{\Phi}^{-1}$ to $UT_0(R)$ one may conclude that

$$\partial_{j\beta}(\Psi)\Psi^{-1} = (\mathcal{M}(\hat{\Phi})^j V_\beta(\hat{\Phi}))_- \text{ and } \partial_{j\beta}(\hat{\Phi})\hat{\Phi}^{-1} = -(\mathcal{M}(\hat{\Phi})^j V_\beta(\hat{\Phi}))_+$$

In particular Φ satisfies the linearization equations for the matrices $(\mathcal{M}(\hat{\Phi}), V_\alpha(\hat{\Phi}))$. To get the Lax equations one applies the derivation $\partial_{j\beta}$ to the defining equations of $\mathcal{M}(\hat{\Phi})$ and $V_\alpha(\hat{\Phi})$

$$\begin{aligned}\partial_{j\beta}(\mathcal{M}(\hat{\Phi})) &= \partial_{j\beta}(\hat{\Phi})\Lambda^{-k}\hat{\Phi} + \hat{\Phi}\Lambda^{-k}\partial_{j\beta}(\hat{\Phi}^{-1})\\ &= \partial_{j\beta}(\hat{\Phi})\hat{\Phi}^{-1}\hat{\Phi}\Lambda^{-k} - \hat{\Phi}\Lambda^{-jk}\hat{\Phi}^{-1}\partial_{j\beta}(\hat{\Phi})\hat{\Phi}^{-1}\\ &= [\partial_{j\beta}(\hat{\Phi})\hat{\Phi}^{-1}, \mathcal{M}(\hat{\Phi})] = [C_{j\beta} - \mathcal{M}(\hat{\Phi})^j V_\beta(\hat{\Phi}), \mathcal{M}(\hat{\Phi})]\\ &= [C_{j\beta}, \mathcal{M}(\hat{\Phi})]\end{aligned}$$

and similarly for $V_\alpha(\hat{\Phi})$

$$\begin{aligned}\partial_{j\beta}(V_\alpha(\hat{\Phi})) &= \partial_{j\beta}(\hat{\Phi}) i_k(F_\alpha)\hat{\Phi} + \hat{\Phi} i_k(F_\alpha)\partial_{j\beta}(\hat{\Phi}^{-1})\\ &= [\partial_{j\beta}(\hat{\Phi})\hat{\Phi}^{-1}, V_\alpha(\hat{\Phi})] = [C_{j\beta}, V_\alpha(\hat{\Phi})].\end{aligned}$$

If one replaces $g \in G$ by $u_0 g$ with an arbitrary $u_0 \in U_-$, then the upper triangular component does not change: $p(u_0 g, \gamma_-) = p(g,\gamma_-)$ and consequently the corresponding solutions of the hierarchy are the same. One resumes the results obtained in a

Theorem 4.1 *The matrix* $[p(g,\gamma_-)][\gamma_-]$ *is a wave function at zero for the set of matrices* $(\mathcal{M}([p(g,\gamma_-)]), V_\beta([p(g,\gamma_-)]))$. *In particular these matrices form a solution of the upper triangular Toda hierarchy associated with* $\mathbf{k}$. *For each* $u_0 \in U_-$ *one has*

$$\mathcal{M}([p(g,\gamma_-)]) = \mathcal{M}([p(u_0 g,\gamma_-)]) \text{ and } V_\beta([p(g,\gamma_-)]) = V_\beta([p(u_0 g,\gamma_-)]).$$

One concludes with the geometric description of the transform between solutions related to conjugated algebras as given in section 2. Take again a $C \in \mathrm{GL}_k(\mathbb{C})$. Note that for $\gamma_- \in \Gamma_-(\mathbf{k})$ one has $[M_C\gamma_- M_C^{-1}] = i_k(C)\varphi_0 i_k(C)^{-1}$. and that each M_C is in the normalizer of both groups P and U_-. Therefore conjugating the relation $g\gamma_-^{-1} = u_-(g,\gamma_-)p(g,\gamma_-)$ with M_C and using theorem 4.1 results in

Corollary 4.2 *The solution of the upper triangular Toda hierarchy associated with* $C\mathbf{k}C^{-1}$ *corresponding to* $M_C g M_C^{-1}$ *in the above construction is the transform as indicated in section 2 of the solution of the upper triangular Toda hierarchy associated with* $\mathbf{k}$ *corresponding to* g.

5 Conclusion

It will be clear that besides with the Schatten classes one could define many more similar groups in which the above construction can be carried out. The class of the Hilbert-Schmidt operators (p=2) is particularly apt to define suitable line bundles which permit the introduction of the so-called τ-functions. This topic will discussed at a different occasion.

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