

Algebraic Solution of the Supersymmetric Hydrogen Atom

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Abstract. The $\mathcal{N} = 2$ supersymmetric extension of the SCHRÖDINGER-HAMILTONIAN with $1/r$ -potential in d dimension is constructed. The system admits a supersymmetrized LAPLACE-RUNGE-LENZ vector which extends the rotational $SO(d)$ symmetry to a hidden $SO(d+1)$ symmetry. It is used to determine the discrete eigenvalues with their degeneracies and the corresponding bound state wave functions.

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1 Classical Motion in Newton/Coulomb Potential

For a closed system of two non-relativistic point masses interacting via a central force the angular momentum \mathbf{L} of the relative motion is conserved and the motion is always in the plane perpendicular to \mathbf{L} . If the force is derived from a $1/r$ -potential, there is an additional conserved quantity: the LAPLACE-RUNGE-LENZ¹ vector,

$$\mathbf{C} = \frac{1}{m} \mathbf{p} \times \mathbf{L} - \frac{e^2}{r} \mathbf{r}.$$

This vector is perpendicular to \mathbf{L} and points in the direction of the semi-major axis. For the hydrogen atom the corresponding Hermitian vector operator has the form

$$\mathbf{C} = \frac{1}{2m} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{e^2}{r} \mathbf{r} \quad (1)$$

with reduced mass m of the proton-electron system. By exploiting the existence of this conserved vector operator, PAULI calculated the spectrum of the hydrogen

¹A more suitable name for this constant of motion would be HERMANN-BERNOULLI-LAPLACE vector, see [1].

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atom by purely algebraic means [2, 3]. He noticed that the angular momentum \mathbf{L} together with the vector operator

$$\mathbf{K} = \sqrt{\frac{-m}{2H}} \mathbf{C}, \quad (2)$$

which is well-defined and Hermitian on bound states with negative energies, generate a hidden $SO(4)$ symmetry algebra,

$$\begin{aligned} [L_a, L_b] &= i\epsilon_{abc}L_c, \\ [L_a, K_b] &= i\epsilon_{abc}K_c, \\ [K_a, K_b] &= i\epsilon_{abc}L_c, \end{aligned} \quad (3)$$

and that the HAMILTON-Operator can be expressed in terms of $\mathcal{C}_{(2)} = \mathbf{K}^2 + \mathbf{L}^2$, one of the two second-order CASIMIR operators of this algebra, as follows

$$H = -\frac{me^4}{2} \frac{1}{\mathcal{C}_{(2)} + \hbar^2}. \quad (4)$$

One also notices that the second CASIMIR operator $\tilde{\mathcal{C}}_{(2)} = \mathbf{L} \cdot \mathbf{K}$ vanishes and arrives at the bound state energies by purely group theoretical methods. The existence of the conserved vector \mathbf{K} also explains the accidental degeneracy of the hydrogen spectrum.

We generalize the COULOMB-problem to d dimensions by keeping the $1/r$ -potential. Distances are measured in units of the reduced COMPTON wavelength, such that the SCHRÖDINGER-operator takes the form

$$H = p^2 - \frac{\eta}{r}, \quad p_a = \frac{1}{i} \partial_a, \quad a = 1, \dots, d. \quad (5)$$

η is twice the fine structure constant. Energies are measured in units of $mc^2/2$.

The Hermitian generators $L_{ab} = x_a p_b - x_b p_a$ of the rotation group satisfy the familiar $so(d)$ commutation relations

$$[L_{ab}, L_{cd}] = i(\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{ad}L_{bc} - \delta_{bc}L_{ad}). \quad (6)$$

It is not very difficult to guess the generalization of the LAPLACE-RUNGE-LENZ vector (1) in d dimensions [4],

$$C_a = L_{ab}p_b + p_b L_{ab} - \frac{\eta x_a}{r}. \quad (7)$$

These operators commute with H in (5) and form a $SO(d)$ -vector,

$$[L_{ab}, C_c] = i(\delta_{ac}C_b - \delta_{bc}C_a). \quad (8)$$

The commutator of C_a and C_b is proportional to the angular momentum,

$$[C_a, C_b] = -4iL_{ab}H. \quad (9)$$

Now one proceeds as in three dimensions and defines on the negative energy subspace of $L_2(\mathbb{R}^d)$ the Hermitian operators

$$K_a = \frac{1}{2} \frac{C_a}{\sqrt{-H}} \quad \text{with} \quad [K_a, K_b] = iL_{ab}. \quad (10)$$

The operators $\{L_{ab}, K_a\}$ form a closed symmetry algebra and can be combined to form generators L_{AB} of the orthogonal group¹ $SO(d+1)$,

$$L_{AB} = \left(\begin{array}{c|c} L_{ab} & K_a \\ \hline -K_b & 0 \end{array} \right). \quad (11)$$

They obey the commutation relations (6) with indices running from 1 to $d+1$.

One finds a relation similar to (4) by solving

$$C_a C_a = -4K_a K_a H = \eta^2 + (2L_{ab}L_{ab} + (d-1)^2) H$$

for the Hamiltonian,

$$H = p^2 - \frac{\eta}{r} = -\frac{\eta^2}{(d-1)^2 + 4C_{(2)}}. \quad (12)$$

$C_{(2)}$ is the second-order CASIMIR operator of the dynamical symmetry group,

$$C_{(2)} = \frac{1}{2} L_{AB} L_{AB} = \frac{1}{2} L_{ab} L_{ab} + K_a K_a. \quad (13)$$

It remains to find the admitted irreducible representations of $SO(d+1)$. In three dimensions they are fixed by the condition $\tilde{C}_{(2)} = 0$ on the CASIMIR operator not entering the relation (4). In $d = 2n-1$ and $d = 2n$ dimensions there are n CASIMIR operators of the dynamical symmetry group and we expect $n-1$ conditions. The analysis in [5] lead to the following results:

- Only the completely symmetric representations of $SO(d+1)$ are realized.
- As in three dimensions the energies, degeneracies and eigenfunctions are determined by group-theoretic methods.

2 SUSY Quantum Mechanics

The HILBERT-Space of a supersymmetric system is the sum of its bosonic and fermionic subspaces, $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$. Let A be a linear operator $\mathcal{H}_F \rightarrow \mathcal{H}_B$. We shall use a block notation such that the vectors in \mathcal{H}_B have upper and those in \mathcal{H}_F lower components,

$$|\psi\rangle = \begin{pmatrix} |\psi_B\rangle \\ |\psi_F\rangle \end{pmatrix}.$$

¹For scattering states ($E > 0$) a similar redefinition leads to generators of the Lorentz group $SO(d, 1)$. Here we are interested in bound states and will not further discuss this possibility.

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Then the nilpotent *supercharge* and its adjoint take the forms

$$\mathcal{Q} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Q}^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix} \implies \{\mathcal{Q}, \mathcal{Q}\} = 0. \quad (14)$$

The block-diagonal super-HAMILTONian

$$H \equiv \{\mathcal{Q}, \mathcal{Q}^\dagger\} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_B & 0 \\ 0 & H_F \end{pmatrix}, \quad (15)$$

commutes with the supercharge and the (fermion) number operator

$$N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Bosonic states have $N = 0$ and fermionic states $N = 1$. The supercharge and its adjoint decrease and increase this conserved number by one.

In most applications in quantum mechanics A is a first order differential operator

$$A = i\partial_x + iW(x) \quad (16)$$

and yields the isospectral partner-HAMILTONians

$$H_B = p^2 + V_B, \quad H_F = p^2 + V_F, \quad \text{with } V_{B/F} = W^2 \pm W'. \quad (17)$$

Such one-dimensional systems were introduced by NICOLAI and WITTEN some time ago [6, 7]. See the texts [8, 9] for a discussion of such models and in particular their relation to isospectral deformations and integrable systems.

3 SQM in Higher Dimensions

Supersymmetric quantum mechanical systems also exist in higher dimensions [7, 10]. The construction is motivated by the following rewriting of the supercharge

$$\mathcal{Q} = \psi \otimes A \quad \text{and} \quad \mathcal{Q}^\dagger = \psi^\dagger \otimes A^\dagger$$

containing the *fermionic* operators

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with anti-commutation relations

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0 \quad \text{and} \quad \{\psi, \psi^\dagger\} = \mathbb{1}.$$

In [10] this construction has been generalized to d dimensions. Then one has d fermionic annihilation operators ψ_k and d creation operators ψ_k^\dagger ,

$$\{\psi_k, \psi_\ell\} = \{\psi_k^\dagger, \psi_\ell^\dagger\} = 0 \quad \text{and} \quad \{\psi_k, \psi_\ell^\dagger\} = \delta_{k\ell}, \quad k, \ell = 1, \dots, d. \quad (18)$$

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For the supercharge one makes the Ansatz

$$Q = i \sum \psi_k (\partial_k + W_k(\mathbf{x})).$$

It is *nilpotent* (i.e. $Q^2 = 0$) if and only if $\partial_k W_\ell - \partial_\ell W_k = 0$ holds true. Locally this integrability condition is equivalent to the existence of a potential $\chi(x)$ with $W_k = \partial_k \chi$. Thus we are lead to the following *nilpotent* supercharge

$$Q = e^{-\chi} Q_0 e^\chi \quad \text{with} \quad Q_0 = i \sum \psi_k \partial_k. \quad (19)$$

It acts on elements of the HILBERT-space

$$\mathcal{H} = L_2(\mathbb{R}^d) \otimes \mathbb{C}^{2^d},$$

which is graded by the 'fermion-number' operator $N = \sum \psi_a^\dagger \psi_a$,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_d, \quad N|_{\mathcal{H}_p} = p\mathbb{1}. \quad (20)$$

A state in \mathcal{H}_p has the expansion

$$\Psi = \sum f_{a_1 \dots a_p}(x) |a_1 \dots a_p\rangle, \quad |a_1 \dots a_p\rangle = \psi_{a_1}^\dagger \dots \psi_{a_p}^\dagger |0\rangle \quad (21)$$

with antisymmetric $f_{a_1 \dots a_p}$. Q decreases N by one and its adjoint increases it by one. It follows that the super-HAMILTONIAN

$$\begin{aligned} H = \{Q, Q^\dagger\} &= H_0 \otimes \mathbb{K}_{2^d} - 2 \sum \psi_k^\dagger \psi_\ell \partial_k \partial_\ell \chi \\ &= H_d \otimes \mathbb{K}_{2^d} + 2 \sum \psi_k \psi_\ell^\dagger \partial_k \partial_\ell \chi \end{aligned} \quad (22)$$

preserves the 'fermion-number'. The operators in the extreme sectors,

$$\begin{aligned} H_0 &\equiv H|_{\mathcal{H}_0} = -\Delta + (\nabla \chi, \nabla \chi) + \Delta \chi \\ H_d &\equiv H|_{\mathcal{H}_d} = -\Delta + (\nabla \chi, \nabla \chi) - \Delta \chi. \end{aligned} \quad (23)$$

are ordinary SCHRÖDINGER-operators, whereas the restriction of H to any other sector is a matrix-SCHRÖDINGER-operator,

$$H_p \equiv H|_{\mathcal{H}_p} : 2^{\binom{d}{p}} \times 2^{\binom{d}{p}} - \text{matrix.}$$

Due to the nilpotency of Q and $[Q, H] = 0$ one has a HODGE-type decomposition of the HILBERT-space [5],

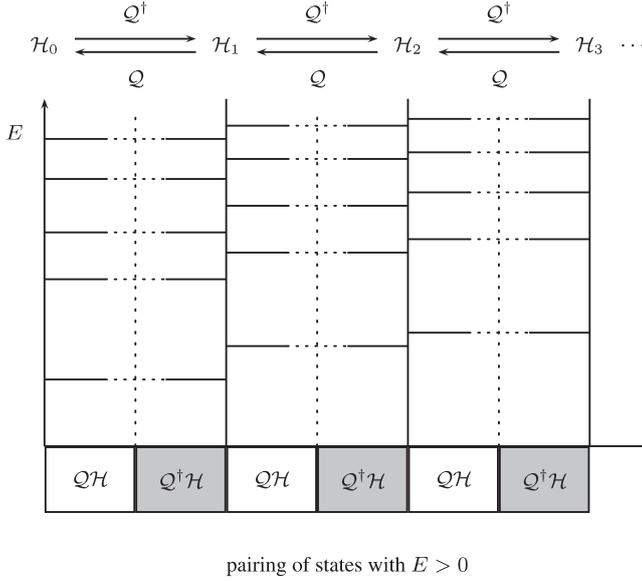
$$\mathcal{H} = Q\mathcal{H} \oplus Q^\dagger\mathcal{H} \oplus \text{Ker } H. \quad (24)$$

Actually, the graded HILBERT-space is a Q -complex of the following structure,

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$$\mathcal{H}_0 \begin{array}{c} \xrightarrow{Q^\dagger} \\ \xleftarrow{Q} \end{array} \mathcal{H}_1 \begin{array}{c} \xrightarrow{Q^\dagger} \\ \xleftarrow{Q} \end{array} \mathcal{H}_2 \begin{array}{c} \xrightarrow{Q^\dagger} \\ \xleftarrow{Q} \end{array} \cdots \begin{array}{c} \xrightarrow{Q^\dagger} \\ \xleftarrow{Q} \end{array} \mathcal{H}_d$$

Similarly as in the one-dimensional case one has a pairing of all H -eigenstates with non-zero energy. Every excited state is degenerate and the eigenfunctions are mapped into each other by Q and its adjoint. The situation is depicted in the following figure,



4 The Supersymmetric H-Atom

We supersymmetrized the H-atom along these lines and showed that it admits supersymmetric generalizations of the angular momentum and LAPLACE-RUNGE-LENZ vector [5]. As for the ordinary COULOMB problem the hidden $SO(d+1)$ -symmetry allows for a purely algebraic solution. Here we discuss the construction for the 3-dimensional system and sketch the generalization to arbitrary dimensions.

To construct the supersymmetrized H-atom in 3 dimensions we choose $\chi = -\lambda r$ in (19) and obtain the super-HAMILTONIAN [5]

$$H = (-\Delta + \lambda^2)\mathcal{K}_8 - \frac{2\lambda}{r}B, \quad B = \mathcal{K} - N + S^\dagger S, \quad S = \hat{x} \cdot \psi \quad (25)$$

on the HILBERT-space

$$\mathcal{H} = L_2(\mathbb{R}^3) \times \mathbb{C}^8 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3. \quad (26)$$

We defined the triplet ψ containing the 3 annihilation operators ψ_1, ψ_2, ψ_3 . States in \mathcal{H}_0 are annihilated by S and states in \mathcal{H}_3 by S^\dagger . With $\{S^\dagger, S\} = \not\equiv$ we find the following HAMILTON-operators in these extreme subspaces,

$$\begin{aligned} H_0 &= -\Delta + \lambda^2 - \frac{2\lambda}{r}, \\ H_3 &= -\Delta + \lambda^2 + \frac{2\lambda}{r}. \end{aligned}$$

H_0 describes the proton-electron and H_3 the proton-positron system.

The conserved angular momentum contains a spin-type term,

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{x} \wedge \mathbf{p} - i\psi^\dagger \wedge \psi. \quad (27)$$

The operators \mathbf{x} and ψ are both vectors such that S and B in (25) commute with this total angular momentum. To find the SUSY extension of the RUNGE-LENZ vector is less simple. It reads [5]

$$\mathbf{C} = \mathbf{p} \wedge \mathbf{J} - \mathbf{J} \wedge \mathbf{p} - 2\lambda \hat{\mathbf{x}} B \quad (28)$$

with \mathbf{J} from (27) and B from (25). The properly normalized vector

$$\mathbf{K} = \frac{1}{2} \frac{\mathbf{C}}{\sqrt{\lambda^2 - H}} \quad (29)$$

together with \mathbf{J} form an $SO(4)$ symmetry algebra on the subspace of bound states for which $H < \lambda^2$.

To solve for the spectrum we would like to find a relation similar to (4). However, one soon realizes that there is no algebraic relation between the conserved operators $\not\equiv, \mathbf{N}, \mathbf{J}^2, \mathbf{K}^2$ and H . However, we can prove the equation

$$\begin{aligned} \lambda^2 \mathcal{C}_{(2)} &= \mathbf{K}^2 H + (\mathbf{J}^2 + (1 - \mathbf{N})^2) \mathcal{Q} \mathcal{Q}^\dagger \\ &+ (\mathbf{J}^2 + (2 - \mathbf{N})^2) \mathcal{Q}^\dagger \mathcal{Q}, \end{aligned} \quad (30)$$

where $\mathcal{C}_{(2)}$ is the second-order CASIMIR (4). This relation is sufficient to obtain the energies since each of the three subspaces in the HODGE-decomposition (24) is left invariant by H and thus we may diagonalize it on each subspace separately. Since $H|_{\mathcal{Q}\mathcal{H}} = \mathcal{Q}\mathcal{Q}^\dagger$ and $H|_{\mathcal{Q}^\dagger\mathcal{H}} = \mathcal{Q}^\dagger\mathcal{Q}$ we can solve (30) for H in both subspaces,

$$\begin{aligned} H|_{\mathcal{Q}\mathcal{H}} &= \lambda^2 \frac{\mathcal{C}_{(2)}}{(1 - \mathbf{N})^2 + \mathcal{C}_{(2)}}, \\ H|_{\mathcal{Q}^\dagger\mathcal{H}} &= \lambda^2 \frac{\mathcal{C}_{(2)}}{(2 - \mathbf{N})^2 + \mathcal{C}_{(2)}}. \end{aligned} \quad (31)$$

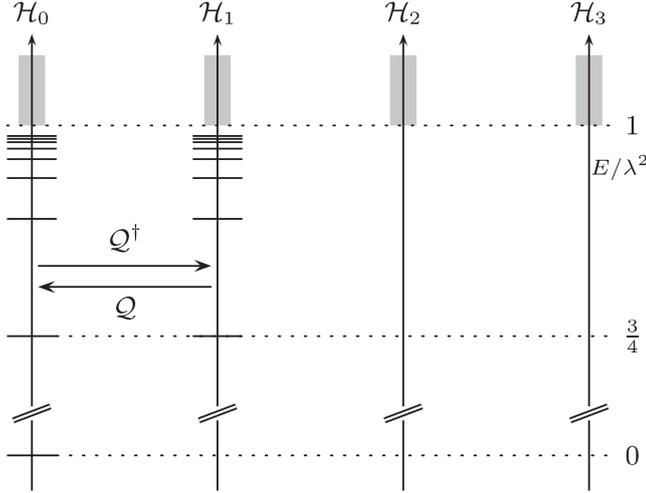
States with zero energy are annihilated by both \mathcal{Q} and \mathcal{Q}^\dagger , and according to (30) the second-order CASIMIR must vanish on these modes, such that

$$\mathcal{C}_{(2)}|_{\text{Ker } H} = 0.$$

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We conclude that every supersymmetric ground state of H is an $SO(4)$ singlet.

In the figure below we have plotted the spectrum of the supersymmetric H -atom in 3 dimensions. The bound states reside in the sectors with fermion numbers 0 and 1. In the sectors with fermion numbers 2 and 3 there are only scattering states. All bound states transform according to the symmetric representations of $SO(4)$. This is particular to 3 dimensions. The energies with degeneracies and the wave functions for all bound states can be found in [5].



realization of $so(4) \Rightarrow$ all bound states

5 Higher Dimensions

The super-HAMILTONIAN (22) with $\chi = -\lambda r$ describes a supersymmetrized COULOMB-problem in d dimensions. As in 3 dimensions it can be solved with the help of a supersymmetrized angular momentum and RUNGE-LENZ vector generating a dynamical symmetry $SO(d+1)$. The supersymmetric extension of the angular momenta reads

$$J_{ab} = L_{ab} + S_{ab} \quad \text{with} \quad S_{ab} = \frac{1}{i} \left(\psi_a^\dagger \psi_b - \psi_b^\dagger \psi_a \right). \quad (32)$$

The supercharge, HAMILTONIAN and $\mathcal{S} = \hat{x} \cdot \psi$ are scalars with respect to the rotations generated by the J_{ab} . The supersymmetric extension of LAPLACE-RUNGE-LENZ vector

$$C_a = J_{ab} p_b + p_b J_{ab} - 2\lambda \hat{x}_a B \quad (33)$$

and the super-HAMILTONIAN

$$H = -\Delta + \lambda^2 - \frac{2\lambda}{r} B \quad (34)$$

both contain the scalar operator

$$B = \frac{1}{2}(d-1)\mathcal{K} - \mathbf{N} + \mathcal{S}^\dagger \mathcal{S}. \quad (35)$$

Again the FOCK-BARGMANN symmetry group $SO(d+1)$ is generated by

$$L_{AB} = \begin{pmatrix} L_{ab} & K_a \\ -K_b & 0 \end{pmatrix}, \quad K_a = \frac{C_a}{\sqrt{4(\lambda^2 - H)}},$$

and the second-order CASIMIR

$$\mathcal{C}_{(2)} = \frac{1}{2}J_{AB}J_{AB}, \quad (36)$$

together with λ, d, \mathbf{N} enter the formulas for

$$H|_{\mathcal{QH}} \quad \text{and} \quad H|_{\mathcal{Q}^\dagger\mathcal{H}}.$$

The analysis parallels the one in 3 dimensions. To find the allowed representations one uses the branching-rules from the dynamical symmetry $SO(d+1)$ to the rotational symmetry $SO(d)$ generated by the J_{ab} . Only those representation for which the YOUNG-diagram has exactly one row and exactly one column give rise to normalizable states. The construction of the bound state wave function uses the realization of the CARTAN- and step operators H_α, E_α as differential operators. This way one finds the highest weight state in each representation [5].

6 Conclusions

We have succeeded in supersymmetrizing the celebrated construction of PAULI, FOCK and BARGMANN. For the COULOMB-problem with extended $\mathcal{N} = 2$ supersymmetry we have found the conserved angular momentum and conserved RUNGE-LENZ vector. Together they generate the FOCK-BARGMANN symmetry group $SO(d+1)$. A general relation of the type

$$\mathcal{Q}\mathcal{Q}^\dagger = f_1(\lambda, d, \mathbf{N}, \mathcal{C}_{(2)}) \quad \text{and} \quad \mathcal{Q}^\dagger\mathcal{Q} = f_2(\lambda, d, \mathbf{N}, \mathcal{C}_{(2)}) \quad (37)$$

has been derived which allows for an algebraic treatment of the supersymmetrized hydrogen atom in d dimensions. The energies depend on the fine structure constant, the dimension of space, the fermion number and the second order CASIMIR-operator. The bound states transform according to particular irreducible $SO(d+1)$ -representations. The allowed representations, the explicit form of the bound states and their energies have been determined.

We have not discussed the scattering problem. It is well-known how to extend supersymmetric methods from bound to scattering states in supersymmetric quantum mechanical systems [12]. Thus one may expect that the construction generalizes to the scattering problem, for which the non-compact dynamical symmetry group will be $SO(d, 1)$.

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ITZYKSON and BANDER [13] distinguished between the infinitesimal and the global method to solve the COULOMB problem. The former is based on the LAPLACE-RUNGE-LENZ vector and is the method used here. In the second method one performs a stereographic projection of the d -dimensional momentum space to the unit sphere in $d+1$ dimensions which in turn implies a $SO(d+1)$ symmetry group. It would be interesting to perform a similar global construction for the supersymmetrized systems.

Every multiplet of the dynamical symmetry group appears several times [5] and there is a new 'accidental' degeneracy: in higher dimensions some eigenvalues of the Hamiltonian appear in many different particle-number sectors. It may very well be, that the algebraic structures discussed in the present work have a more natural setting in the language of superalgebras or the $SO(d, 2)$ -setting in [4]. We have not investigated this questions.

There exist earlier results on the supersymmetry of both the non-relativistic and relativistic hydrogen atom. In [14] the RUNGE-LENZ vector or its relativistic generalization, the JOHNSON-LIPPMANN operator, enter the expressions for the supercharges belonging to the *ordinary* SCHRÖDINGER- or DIRAC-operators with $1/r$ potential. This should be contrasted with the present work, where the COULOMB-problem is only a particular channel of a manifestly supersymmetric matrix-SCHRÖDINGER operator. Our HAMILTONians incorporate both the proton-electron and the proton-positron systems as particular subsectors.

The supercharge (19) and super-HAMILTONian (22) describe a wide class of supersymmetric systems, ranging from the supersymmetric oscillator in d dimensions to lattice WESS-ZUMINO-models with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetries in 2 dimensions [11]. In passing we mention, that the supercharge in d dimension is actually a dimensionally reduced DIRAC operator in $2d$ dimensions. During the reduction process the ABELian gauge potential A_μ in $2d$ dimensions transforms into the potential χ in (19), see [11].

More generally, one may ask for which gauge- and gravitational background field the DIRAC-operator admits an extended supersymmetry. This question has been answered in full generality in [15]. For example, on a 4-dimensional hyper-KÄHLER space with self-dual gauge field the DIRAC-operator admits an $\mathcal{N} = 4$ supersymmetry. The extended supersymmetry may be used to construct possible zero-modes of the DIRAC-operator. Earlier results on the supersymmetries of DIRAC-type operators can be found in [16], for example. COMTET and HORVATHY investigated the solutions of the DIRAC-equation in the hyper-KÄHLER TAUB-NUT gravitational instanton [17]. The spin 0 case can be solved with the help of a KEPLER-type dynamical symmetry [18] and the fermion case by relating it to the spin 0 problem with the help of supersymmetry.

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