

Hamiltonians with Position-Dependent Mass, Deformations and Supersymmetry

C. Quesne¹, B. Bagchi², A. Banerjee², V.M. Tkachuk³

¹Physique Nucléaire Théorique et Physique Mathématique,
Université Libre de Bruxelles, Campus de la Plaine CP229,
Boulevard du Triomphe, B-1050 Brussels, Belgium

²Department of Applied Mathematics, University of Calcutta,
92 Acharya Prafulla Chandra Road, Kolkata 700 009, India

³Ivan Franko Lviv National University, Chair of Theoretical Physics,
12 Drahomanov Street, Lviv UA-79005, Ukraine

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Abstract. A new method for generating exactly solvable Schrödinger equations with a position-dependent mass is proposed. It is based on a relation with some deformed Schrödinger equations, which can be dealt with by using a supersymmetric quantum mechanical approach combined with a deformed shape-invariance condition. The solvability of the latter is shown to impose the form of both the deformed superpotential and the position-dependent mass. The conditions for the existence of bound states are determined. A lot of examples are provided and the corresponding bound-state spectrum and wave functions are reviewed

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1 Introduction

Schrödinger equations with a position-dependent mass (PDM) play an important role in many physical problems. They appear in the energy-dependent functional approach to quantum many-body systems [1] (e.g., nuclei, quantum liquids, ³He clusters, metal clusters) and are very useful in the description of electronic properties of condensed-matter systems [2] (e.g., compositionally-graded crystals, quantum dots, liquid crystals). The PDM presence may also reflect other unconventional effects, such as deformation of the canonical commutation relations or curvature of the underlying space [3] or else pseudo-Hermiticity of the Hamiltonian [4].

Several exactly solvable, quasi-exactly solvable or conditionally exactly solvable PDM Schrödinger equations have been constructed using point canonical

transformations, Lie algebraic methods or supersymmetric quantum mechanical (SUSYQM) and shape-invariance (SI) techniques (see [5, 6] and references quoted therein). Most of them can be obtained from known constant-mass models by changes of variable and of function. As a consequence the spectrum is left unchanged although the potential is given by a complicated mass-deformed expression.

Here we present a new method [7], generalizing the work in [3]. It is based on a relation with deformed Schrödinger equations and uses a SUSYQM method combined with a deformed SI condition. The resulting spectra will contrast with those obtained in the constant-mass case.

2 PDM and Deformed Schrödinger Equations

As is well known, the noncommutativity of a PDM with the momentum operator results in an ambiguity in the kinetic energy operator definition coming from the selected ordering. On choosing the von Roos general two-parameter form of T [8], which has an inbuilt Hermiticity and contains all the plausible forms as special cases, and setting $\hbar = 2m_0 = 1$, the PDM Schrödinger equation can be written as

$$\left[-\frac{1}{2} \left(M^{\xi'} \frac{d}{dx} M^{\eta'} \frac{d}{dx} M^{\zeta'} + M^{\zeta'} \frac{d}{dx} M^{\eta'} \frac{d}{dx} M^{\xi'} \right) + V \right] \psi = E\psi. \quad (1)$$

Here $V(\mathbf{a}; x)$ is the potential, $M(\boldsymbol{\alpha}; x)$ is the dimensionless form of the mass function $m(\boldsymbol{\alpha}; x) = m_0 M(\boldsymbol{\alpha}; x)$, \mathbf{a} and $\boldsymbol{\alpha}$ denote two sets of parameters, and the von Roos ambiguity parameters ξ' , η' , ζ' are constrained by the condition $\xi' + \eta' + \zeta' = -1$.

On setting $M(\boldsymbol{\alpha}; x) = [f(\boldsymbol{\alpha}; x)]^{-2}$ and $f(\boldsymbol{\alpha}; x) = 1 + g(\boldsymbol{\alpha}; x)$, where $f(\boldsymbol{\alpha}; x)$ is some positive-definite function and $g(\boldsymbol{\alpha}; x) = 0$ corresponds to the constant-mass case, Eq. (1) becomes

$$\left[-\frac{1}{2} \left(f^{\xi} \frac{d}{dx} f^{\eta} \frac{d}{dx} f^{\zeta} + f^{\zeta} \frac{d}{dx} f^{\eta} \frac{d}{dx} f^{\xi} \right) + V \right] \psi = E\psi \quad (2)$$

with $\xi + \eta + \zeta = 2$. The ambiguity parameters ξ , η , ζ (denoted collectively by $\boldsymbol{\xi}$) can be transferred from the kinetic energy term to an effective potential

$$V_{\text{eff}}(\mathbf{b}; x) = V + \tilde{V}, \quad \tilde{V} = \rho f f'' + \sigma f'^2, \quad (3)$$

so that Eq. (2) acquires the form

$$H\psi \equiv \left[- \left(\sqrt{f} \frac{d}{dx} \sqrt{f} \right)^2 + V_{\text{eff}} \right] \psi = E\psi. \quad (4)$$

In (3), a prime denotes derivative with respect to x , $\rho = \frac{1}{2}(1 - \xi - \zeta)$, $\sigma = (\frac{1}{2} - \xi)(\frac{1}{2} - \zeta)$ and $\mathbf{b} = (\mathbf{a}, \boldsymbol{\alpha}, \boldsymbol{\xi})$.

Equation (4) can be reinterpreted as a deformed Schrödinger equation with $H = \pi^2 + V_{\text{eff}}$, where $\pi = \sqrt{f} p \sqrt{f}$ satisfies the deformed commutation relation $[x, \pi] = i f$ and $f(\alpha; x)$ acts as a deforming function.

3 General Procedure

The first step in our procedure consists in taking for V_{eff} any known SI potential under parameter translation. This means that the initial potential V will be determined by inverting (3) as $V = V_{\text{eff}} - \tilde{V}$, so that its parameters \mathbf{a} will be $\mathbf{a} = (\mathbf{b}, \alpha, \xi)$.

We then consider H as the first member $H_0 = H$ of a hierarchy of Hamiltonians

$$H_i = A^+(\alpha, \lambda_i) A^-(\alpha, \lambda_i) + \sum_{j=0}^i \epsilon_j, \quad i = 0, 1, 2, \dots, \quad (5)$$

where the first-order operators

$$A^\pm(\alpha, \lambda_i) = \mp \sqrt{f(\alpha; x)} \frac{d}{dx} \sqrt{f(\alpha; x)} + W(\lambda_i; x) \quad (6)$$

satisfy a deformed SI condition

$$A^-(\alpha, \lambda_i) A^+(\alpha, \lambda_i) = A^+(\alpha, \lambda_{i+1}) A^-(\alpha, \lambda_{i+1}) + \epsilon_{i+1} \quad (7)$$

for $i = 0, 1, 2, \dots$. Here ϵ_i and λ_i , $i = 0, 1, 2, \dots$, are some constants. Solving Eq. (7) means that it is possible to find a superpotential $W(\lambda; x)$, a deforming function $f(\alpha; x)$ and some constants λ_i , ϵ_i , $i = 0, 1, 2, \dots$, with $\lambda_0 = \lambda$, such that

$$V_{\text{eff}}(\mathbf{b}; x) = W^2(\lambda; x) - f(\alpha; x) W'(\lambda; x) + \epsilon_0 \quad (8)$$

and

$$\begin{aligned} & W^2(\lambda_i; x) + f(\alpha; x) W'(\lambda_i; x) \\ &= W^2(\lambda_{i+1}; x) - f(\alpha; x) W'(\lambda_{i+1}; x) + \epsilon_{i+1}, \quad i = 0, 1, 2, \dots \end{aligned} \quad (9)$$

As a consequence, the (deformed) SUSY partner H_1 of H will be characterized by a potential $V_{\text{eff},1}(\mathbf{b}, \alpha, \lambda; x) = V_{\text{eff}}(\mathbf{b}; x) + 2f(\alpha; x) W'(\lambda; x)$. Note that in Eqs. (8) and (9), the additional terms with respect to the undeformed case are proportional to gW' .

To find a solution for W , f , ϵ_i and λ_i , our strategy consists in (i) assuming that the deformation does not affect the form of W but only brings about a change in its parameters λ (which will now also depend on α), and (ii) choosing $g(\alpha; x)$ in such a way that the function $g(\alpha; x) W'(\lambda; x)$ contains the same kind of terms as those already present in the undeformed case, i.e., $W^2(\lambda; x)$ and $W'(\lambda; x)$.

Once we have found a solution to Eqs. (8) and (9), the bound-state energy spectrum and corresponding wave functions can be found as in conventional SUSYQM. They may be written as

$$E_n(\boldsymbol{\alpha}, \boldsymbol{\lambda}) = \sum_{i=0}^n \epsilon_i \quad (10)$$

and

$$\psi_n(\boldsymbol{\alpha}, \boldsymbol{\lambda}; x) \propto \frac{1}{\sqrt{f(\boldsymbol{\alpha}; x)}} \varphi_n(\boldsymbol{\alpha}, \boldsymbol{\lambda}; x) \exp\left(-\int^x \frac{W(\boldsymbol{\lambda}_n; \tilde{x})}{f(\boldsymbol{\alpha}; \tilde{x})} d\tilde{x}\right), \quad (11)$$

where $\varphi_n(\boldsymbol{\alpha}, \boldsymbol{\lambda}; x)$ fulfils the equation

$$\begin{aligned} \varphi_{n+1}(\boldsymbol{\alpha}, \boldsymbol{\lambda}; x) &= -f(\boldsymbol{\alpha}; x)\varphi_n'(\boldsymbol{\alpha}, \boldsymbol{\lambda}_1; x) \\ &+ [W(\boldsymbol{\lambda}_{n+1}; x) + W(\boldsymbol{\lambda}; x)]\varphi_n(\boldsymbol{\alpha}, \boldsymbol{\lambda}_1; x) \end{aligned} \quad (12)$$

with $\varphi_0(\boldsymbol{\alpha}, \boldsymbol{\lambda}; x) = 1$. They remain however formal solutions till one has checked that they satisfy appropriate physical conditions. In the PDM or deformed case, such conditions are twofold: (i) square integrability on the interval of definition of V_{eff} , as in the conventional case, and (ii) Hermiticity of π or, equivalently, of H in the corresponding Hilbert space. The latter imposes that $|\psi|^2 f = \psi^2/\sqrt{M}$ vanishes at the end points of the interval. This extra condition may have some relevant effects whenever the PDM vanishes there.

4 Classes of Superpotentials and Corresponding $g(\boldsymbol{\alpha}; x)$

One can show that all the known potentials which are SI under parameter translation fall into three classes. For all of them, we have determined the general form of the deforming function allowing Eqs. (8) and (9) to remain solvable in accordance with the general strategy reviewed in Sec. 3.

In terms of some parameter-independent function $\phi(x)$ and of two parameters λ, μ making up the set $\boldsymbol{\lambda}$, the results are given by

$$\begin{aligned} \text{Class 1: } W(\boldsymbol{\lambda}; x) &= \lambda\phi(x) + \mu, \\ \phi'(x) &= A\phi^2(x) + B\phi(x) + C, \\ g(\boldsymbol{\alpha}; x) &= \frac{A'(\boldsymbol{\alpha})\phi^2(x) + B'(\boldsymbol{\alpha})\phi(x) + C'(\boldsymbol{\alpha})}{A\phi^2(x) + B\phi(x) + C}, \\ \text{Class 2: } W(\boldsymbol{\lambda}; x) &= \lambda\phi(x) + \frac{\mu}{\phi(x)}, \\ \phi'(x) &= A\phi^2(x) + B, \\ g(\boldsymbol{\alpha}; x) &= \frac{A'(\boldsymbol{\alpha})\phi^2(x) + B'(\boldsymbol{\alpha})}{A\phi^2(x) + B}, \end{aligned} \quad (13)$$

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$$\begin{aligned} \text{Class 3: } W(\lambda; x) &= \frac{\lambda\phi(x) + \mu}{\sqrt{A\phi^2(x) + B}}, \\ \phi'(x) &= [C\phi(x) + D]\sqrt{A\phi^2(x) + B}, \\ g(\alpha; x) &= \frac{C'(\alpha)\phi(x) + D'(\alpha)}{C\phi(x) + D}. \end{aligned}$$

Here A, B, C, D and $A'(\alpha), B'(\alpha), C'(\alpha), D'(\alpha)$ are some numerical and α -dependent constants, respectively.

In all three cases, the integral on the right-hand side of Eq. (11) can be explicitly carried out by simple integration techniques. Furthermore, by changes of variable and of function, $\varphi_n(\alpha, \lambda; x)$ in the same equation can be transformed into an n th-degree polynomial $P_n(\alpha, \lambda; y)$ as follows:

$$\begin{aligned} \text{Class 1: } \varphi_n(\alpha, \lambda; x) &= P_n(\alpha, \lambda; y), & y &= \phi(x), \\ \text{Class 2: } \varphi_n(\alpha, \lambda; x) &= y^{-n/2}P_n(\alpha, \lambda; y), & y &= \phi^{-2}(x), \\ \text{Class 3: } \varphi_n(\alpha, \lambda; x) &= (Ay^2 + B)^{-n/2}P_n(\alpha, \lambda; y), & y &= \phi(x). \end{aligned} \quad (14)$$

Such polynomials are related to deformed classical orthogonal polynomials and satisfy the equations:

$$\begin{aligned} \text{Class 1: } P_{n+1}(\alpha, \lambda; y) &= -\{[A + A'(\alpha)]y^2 \\ &+ [B + B'(\alpha)]y + C + C'(\alpha)\} \dot{P}_n(\alpha, \lambda_1; y) \\ &+ [(\lambda_{n+1} + \lambda)y + \mu_{n+1} + \mu]P_n(\alpha, \lambda_1; y), \\ \text{Class 2: } P_{n+1}(\alpha, \lambda; y) &= 2y\{A + A'(\alpha) + [B + B'(\alpha)]y\} \dot{P}_n(\alpha, \lambda_1; y) \\ &+ \{\lambda_{n+1} + \lambda - n[A + A'(\alpha)] + [\mu_{n+1} + \mu - n(B + B'(\alpha))]\}y \\ &\times P_n(\alpha, \lambda_1; y), \\ \text{Class 3: } P_{n+1}(\alpha, \lambda; y) &= \{[C + C'(\alpha)]y + D + D'(\alpha)\} \\ &\times \left[-(Ay^2 + B)\dot{P}_n(\alpha, \lambda_1; y) + nAyP_n(\alpha, \lambda_1; y) \right] \\ &+ [(\lambda_{n+1} + \lambda)y + \mu_{n+1} + \mu]P_n(\alpha, \lambda_1; y). \end{aligned} \quad (15)$$

5 Some Simple Examples

5.1 Particle in a box and trigonometric Pöschl-Teller potential

Let us consider the superpotential

$$W(\lambda; x) = \lambda \tan x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}. \quad (16)$$

In the undeformed case, for $\lambda = A > 1$, it gives rise to the trigonometric Pöschl-Teller potential [9]

$$V_{\text{eff}}(A; x) = A(A - 1) \sec^2 x, \quad (17)$$

whose bound-state energies and wave functions [10] are given by $E_n = (A+n)^2$ and $\psi_n(x) \propto (\cos x)^A C_n^{(A)}(\sin x)$, $n = 0, 1, 2, \dots$. The particle-in-a-box problem being the limiting case of Pöschl-Teller for $A \rightarrow 1$ corresponds to

$$V_{\text{eff}}(x) = \begin{cases} 0 & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \infty & \text{if } x = \pm \frac{\pi}{2} \end{cases}. \quad (18)$$

The superpotential (16) belongs to class 1 with $\phi(x) = \tan x$ and $\mu = 0$. On choosing $A = C = 1$, $B = 0$, $A'(\alpha) = \alpha$, and $B'(\alpha) = C'(\alpha) = 0$, we get

$$g(\alpha; x) = \alpha \sin^2 x, \quad (19)$$

which for $-1 < \alpha \neq 0$ leads to a positive-definite deforming function $f(\alpha; x)$.

In the particle-in-a-box problem, one easily finds that Eqs. (8) and (9) are fulfilled provided $\lambda_i = (i + 1)(1 + \alpha)$ and $\epsilon_i = (2i + 1)(1 + \alpha)$, $i = 0, 1, 2, \dots$. The corresponding energy eigenvalues and wave functions are given by

$$E_n(\alpha, \lambda) = (1 + \alpha)(n + 1)^2, \quad (20)$$

$$\psi_n(\alpha, \lambda; x) \propto \frac{(\cos x)^{n+1}}{(1 + \alpha \sin^2 x)^{(n+2)/2}} P_n(\alpha, \lambda; \tan x), \quad (21)$$

respectively. Here $P_n(\alpha, \lambda; y)$ satisfies the equation

$$P_{n+1}(\alpha, \lambda; y) = -[1 + (1 + \alpha)y^2] \dot{P}_n(\alpha, \lambda_1; y) + (n + 3)(1 + \alpha)y P_n(\alpha, \lambda_1; y), \quad (22)$$

where a dot denotes derivative with respect to y . For any $n = 0, 1, 2, \dots$, the wave functions (21) are square integrable and ensure the Hermiticity of π . Hence in the presence of deformation (19), the particle-in-a-box problem still has an infinite number of bound states making up a quadratic spectrum. As can be checked, Eqs. (20) and (21) go over to the undeformed energies and wave functions since $P_n(\alpha, \lambda; \tan x)$ becomes proportional to $\sec^n x C_n^{(1)}(\sin x)$ for $\alpha \rightarrow 0$.

For the trigonometric Pöschl-Teller potential (17), the results are similar although more complicated. The energies and associated wave functions are then given by

$$\begin{aligned} E_n(\alpha, \lambda) &= (\lambda + n)^2 - \alpha(\lambda - n^2) \\ &= \left[\frac{1}{2}(\Delta + 1) + n\right]^2 + \alpha n(n + 1) - \frac{1}{4}\alpha^2, \end{aligned} \quad (23)$$

$$\begin{aligned} \psi_n(\alpha, \lambda; x) &\propto (\cos x)^{\frac{\lambda}{1+\alpha}+n} (1 + \alpha \sin^2 x)^{-\frac{1}{2}(\frac{\lambda}{1+\alpha}+n+1)} \\ &\quad \times P_n(\alpha, \lambda; \tan x), \end{aligned} \quad (24)$$

where $\lambda = \frac{1}{2}(1 + \alpha + \Delta)$, $\Delta \equiv \sqrt{(1 + \alpha)^2 + 4A(A - 1)}$ and $P_n(\alpha, \lambda; y)$ satisfies the equation

$$P_{n+1}(\alpha, \lambda; y) = -[1 + (1 + \alpha)y^2]\dot{P}_n(\alpha, \lambda_1; y) + [2\lambda + (n + 1)(1 + \alpha)]yP_n(\alpha, \lambda_1; y) \quad (25)$$

with $\lambda_1 = \lambda + 1 + \alpha$. All functions $\psi_n(\alpha, \lambda; x)$, $n = 0, 1, 2, \dots$, are physically acceptable as bound-state wave functions.

The starting potential in the PDM Schrödinger equation (1) can be written as $V = V_{\text{eff}} - \tilde{V}$, where V_{eff} is given by (17) or (18), while

$$\tilde{V}(\alpha, \rho, \sigma; x) = -(\rho + \sigma)\alpha^2 \cos^2 2x + \rho\alpha(2 + \alpha) \cos 2x + \sigma\alpha^2. \quad (26)$$

5.2 Free particle and hyperbolic Pöschl-Teller potential

In the undeformed case, the hyperbolic counterpart of the superpotential (16), namely

$$W(\lambda; x) = \lambda \tanh x, \quad -\infty < x < \infty, \quad (27)$$

corresponds for $\lambda = A > 0$ to the hyperbolic Pöschl-Teller potential [11]

$$V_{\text{eff}}(A; x) = -A(A + 1) \operatorname{sech}^2 x, \quad (28)$$

whose $n_{\text{max}} + 1$ bound-state energies and wave functions are given by $E_n = -(A - n)^2$ and $\psi_n(x) \propto (\operatorname{sech} x)^{A-n} C_n^{(A-n+\frac{1}{2})}(\tanh x)$, $n = 0, 1, \dots, n_{\text{max}}$, with $A - 1 \leq n_{\text{max}} < A$. For $A \rightarrow 0$, we get the free-particle problem as a limiting case.

The superpotential (27) also belongs to class 1 with $\phi(x) = \tanh x$ and $\mu = 0$. On choosing this time $A = -C = -1$, $B = 0$, $A'(\alpha) = \alpha$, and $B'(\alpha) = C'(\alpha) = 0$, we get

$$g(\alpha; x) = \alpha \sinh^2 x, \quad (29)$$

which for $0 < \alpha < 1$ leads to a positive-definite deforming function $f(\alpha; x)$.

In the case of the hyperbolic Pöschl-Teller potential and $g(\alpha; x)$ given in (29), the energies and associated wave functions can be written as

$$\begin{aligned} E_n(\alpha, \lambda) &= -(\lambda - n)^2 - \alpha(\lambda + n^2) \\ &= -\left[\frac{1}{2}(\Delta - 1) - n\right]^2 + \alpha n(n + 1) + \frac{1}{4}\alpha^2, \quad (30) \\ \psi_n(\alpha, \lambda; x) &\propto (\operatorname{sech} x)^{\frac{\lambda}{1-\alpha} - n} (1 + \alpha \sinh^2 x)^{\frac{1}{2}(\frac{\lambda}{1-\alpha} - n - 1)} \\ &\quad \times P_n(\alpha, \lambda; \tanh x) \quad (31) \end{aligned}$$

where $\lambda = \frac{1}{2}(\alpha - 1 + \Delta)$, $\Delta \equiv \sqrt{(1 - \alpha)^2 + 4A(A + 1)}$ and $P_n(\alpha, \lambda; y)$ satisfies an equation similar to (25). However, it turns out that although for any $n = 0, 1, 2, \dots$, $\psi_n(\alpha, \lambda; x)$ is square integrable on the real line, it does not

satisfy the condition $|\psi|^2 f \rightarrow 0$ at the boundaries $x \rightarrow \pm\infty$. Hence, with a deformed function corresponding to (29), the hyperbolic Pöschl-Teller potential has no bound state.

This result can be extended to the free-particle problem. It contrasts with what was obtained in [5] in another context and illustrates the strong dependence of the bound-state spectrum on the mass environment.

6 Results and Comments

We have used the procedure illustrated in Sect. 5 to obtain a deforming function satisfying Eqs. (8) and (9), as well as the resulting bound-state energies and wave functions, for all the SI potentials contained in Table 4.1 of [12]. Below we list the results obtained for g , E_n , and \tilde{V} .

Shifted oscillator:

$$V_{\text{eff}} = \frac{1}{4}\omega^2 \left(x - \frac{2b}{\omega} \right)^2,$$

$$g = \alpha x^2 + 2\beta x, \quad \alpha > \beta^2 \geq 0,$$

$$E_n = \left(n + \frac{1}{2} \right) \Delta + \left(n^2 + n + \frac{1}{2} \right) \alpha + b^2$$

$$- \left(\frac{[(2n+1)\Delta + (2n^2+2n+1)\alpha]\beta - b\omega}{\Delta + (2n+1)\alpha} \right)^2,$$

$$\Delta \equiv \sqrt{\omega^2 + \alpha^2}, \quad n = 0, 1, 2, \dots,$$

$$\tilde{V} = 2(\rho + 2\sigma)\alpha x(\alpha x + 2\beta) + 2\rho\alpha + 4\sigma\beta^2.$$

Three-dimensional oscillator:

$$V_{\text{eff}} = \frac{1}{4}\omega^2 x^2 + \frac{l(l+1)}{x^2}, \quad 0 \leq x < \infty,$$

$$g = \alpha x^2, \quad \alpha > 0,$$

$$E_n = \Delta \left(2n + l + \frac{3}{2} \right) + \alpha \left[2(n+l+1)(2n+1) + \frac{1}{2} \right],$$

$$\Delta \equiv \sqrt{\omega^2 + \alpha^2}, \quad n = 0, 1, 2, \dots,$$

$$\tilde{V} = 2(\rho + 2\sigma)\alpha^2 x^2 + 2\rho\alpha.$$

Coulomb:

$$V_{\text{eff}} = -\frac{e^2}{x} + \frac{l(l+1)}{x^2}, \quad 0 \leq x < \infty,$$

$$g = \alpha x, \quad \alpha > 0,$$

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$$E_n = - \left(\frac{e^2 - \alpha[n^2 + (l+1)(2n+1)]}{2(n+l+1)} \right)^2,$$

$n = 0, 1, \dots, n_{\max}$, where n_{\max} = largest integer such that
 $n^2 + (l+1)(2n+1) < \frac{e^2}{\alpha}$ if $\alpha < \frac{e^2}{l+1}$,

$$\tilde{V} = \sigma\alpha^2.$$

Morse:

$$V_{\text{eff}} = B^2 e^{-2x} - B(2A+1)e^{-x}, \quad A, B > 0,$$

$$g = \alpha e^{-x}, \quad \alpha > 0,$$

$$E_n = -\frac{1}{4} \left(\frac{2B(2A+1) - [(2n+1)\Delta + (2n^2 + 2n + 1)\alpha]}{\Delta + (2n+1)\alpha} \right)^2,$$

$\Delta \equiv \sqrt{4B^2 + \alpha^2}$, $n = 0, 1, \dots, n_{\max}$, where n_{\max} =
largest integer smaller than A and such that $\alpha < \alpha_{\max}(n_{\max})$ with

$$\alpha_{\max}(0) = \frac{4A(A+1)B}{2A+1},$$

$$\alpha_{\max}(n) = \frac{B(2A+1)(2n^2 + 2n + 1)}{2n^2(n+1)^2}$$

$$- \frac{B(2n+1)[(2A+1)^2 + 4n^2(n+1)^2]^{1/2}}{2n^2(n+1)^2}, \quad n = 1, 2, \dots,$$

$$\tilde{V} = (\rho + \sigma)\alpha^2 e^{-2x} + \rho\alpha e^{-x}.$$

Eckart:

$$V_{\text{eff}} = A(A-1) \text{csch}^2 x - 2B \coth x, \quad A \geq \frac{3}{2}, \quad B > A^2,$$

$$0 \leq x < \infty,$$

$$g = \alpha e^{-x} \sinh x, \quad -2 \leq \alpha \neq 0,$$

$$E_n = -(A+n)^2 - \left(\frac{B - \frac{1}{2}\alpha[(2n+1)A + n^2]}{A+n} \right)^2$$

$$- \alpha[(2n+1)A + n^2],$$

$n = 0, 1, 2, \dots$ if $\alpha = -2$,
 $n = 0, 1, \dots, n_{\max}$ if $\alpha > -2$, where n_{\max} = largest integer
such that $(A+n)^2 < \frac{2B + \alpha A(A-1)}{2+\alpha}$,

$$\tilde{V} = (\rho + \sigma)\alpha^2 e^{-4x} - \rho\alpha(2+\alpha)e^{-2x}.$$

Scarf I:

$$V_{\text{eff}} = (B^2 + A^2 - A) \sec^2 x - B(2A - 1) \tan x \sec x,$$

$$0 < B < A - 1, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2},$$

$$g = \alpha \sin x, \quad 0 < |\alpha| < 1,$$

$$E_n = -\frac{1}{4}(2n + 1 + \Delta_+ + \Delta_-)^2 + \alpha \left(n + \frac{1}{2} \right) (\Delta_+ - \Delta_-)$$

$$- \alpha^2 \left(n^2 + n + \frac{1}{2} \right),$$

$$\Delta_{\pm} \equiv \sqrt{\frac{1}{4}(1 \mp \alpha)^2 + (A \pm B)(A \pm B - 1)}, \quad n = 0, 1, 2, \dots,$$

$$\tilde{V} = -(\rho + \sigma)\alpha^2 \sin^2 x - \rho\alpha \sin x + \sigma\alpha^2.$$

Rosen-Morse I:

$$V_{\text{eff}} = A(A - 1) \csc^2 x + 2B \cot x, \quad A \geq \frac{3}{2}, \quad 0 \leq x \leq \pi,$$

$$g = \sin x(\alpha \cos x + \beta \sin x), \quad \frac{|\alpha|}{2} < \sqrt{1 + \beta}, \quad \beta > -1,$$

$$E_n = (A + n)^2 - \left(\frac{B + \frac{1}{2}\alpha[(2n + 1)A + n^2]}{A + n} \right)^2 + \beta[(2n + 1)A + n^2],$$

$$n = 0, 1, 2, \dots,$$

$$\tilde{V} = (\rho + \sigma) \left[\frac{1}{2}(\alpha^2 - \beta^2) \cos 4x + \alpha\beta \sin 4x \right]$$

$$+ \rho(2 + \beta)(-\alpha \sin 2x + \beta \cos 2x) + (-\rho + \sigma) \frac{1}{2}(\alpha^2 + \beta^2).$$

Three potentials considered in [12] are missing from the list: Scarf II because no nontrivial values of the parameters may ensure positive definiteness of $f(\alpha; x)$, Rosen-Morse II and generalized Pöschl-Teller because they do not have any bound state in the deformed case.

For the remaining potentials, strikingly distinct influences of deformation or mass parameters on bound-state energy spectra are observed. In some cases (shifted oscillator, three-dimensional oscillator, Scarf I and Rosen-Morse I), the infinite number of bound states of conventional quantum mechanics remains infinite after the onset of deformation. Similarly, for Morse potential and for Eckart potential with $\alpha \neq -2$, one keeps a finite number of bound states. For the Coulomb potential, however, the infinite number of bound states is converted

into a finite one, while for Eckart potential with $\alpha = -2$, the finite number of bound states becomes infinite. It is also remarkable that, whenever finite, the bound-state number becomes dependent on the deforming parameter.

For the potentials V to be used in the PDM Schrödinger equation (1), we get either the same shape as V_{eff} (shifted oscillator, three-dimensional oscillator, Coulomb and Morse) or a different shape (remaining potentials). In the first case, the mass and ambiguity parameters only lead to a renormalization of the potential parameters and/or an energy shift.

7 Conclusion

In this communication, we have shown how to generate new exactly solvable PDM (resp. deformed) Schrödinger equations with a bound-state spectrum different from that of the corresponding constant-mass (resp. undeformed) Schrödinger equations and we have illustrated our method by several examples. In addition, we have demonstrated the importance of the Hermiticity condition on the deformed momentum operator for the existence of bound states.

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