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q - Quaternions and Deformed $su(2)$ Instantons on the Quantum Euclidean Space $\mathbb{R}_q^4$

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Abstract. We briefly report on our recent results regarding the introduction of a notion of a q-quaternion and the construction of instanton solutions of a would-be deformed $su(2)$ Yang-Mills theory on the corresponding $SO_q(4)$-covariant quantum space. As the solutions depend on some noncommuting parameters, this indicates that the moduli space of a complete theory will be a noncommutative manifold.

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1 Introduction

The search for instantonic solutions has become a central point of investigation of Yang-Mills gauge theories on noncommutative manifolds after the discovery [20] that deforming $\mathbb{R}^4$ into the canonical noncommutative Euclidean space $\mathbb{R}_\theta^4$ regularizes the zero-size singularities of the instanton moduli space (see also [24]). Among the available deformations of $\mathbb{R}^4$ there is also the Faddeev-Reshetikhin-Takhtadjan noncommutative Euclidean space $\mathbb{R}_q^4$ covariant under $SO_q(4)$ [6], and it is therefore tempting to investigate this issue on it. There is still no satisfactory formulation [16] of gauge field theory on quantum group covariant noncommutative spaces (shortly: quantum spaces) like $\mathbb{R}_q^4$. One main reason is the lack of a proper (*i.e.* cyclic) trace to define gauge invariant observables (action, *etc*). Another one is the $\star$-structure of the differential calculus, which for real q is problematic. Probably a satisfactory formulation will be possible within a generalization of the standard framework of noncommutative geometry [5]. Here we leave these two issues aside and just ask for nontrivial solutions of the deformed (anti)self-duality equations.

As known, great simplifications in the search and classification of instantons in Yang-Mills theory on $\mathbb{R}^4$ occur when the latter is promoted to the quaternion algebra $\mathbb{H}$. We have recently introduced [13, 14] the notion of a q-deformed quaternion as the defining matrix of a copy of $SU_q(2) \times GL^+(1)$, showing that its entries are the coordinates of $\mathbb{R}^4_q$. Then adopting the $SO_q(4)$-covariant differential calculus on $\mathbb{R}^4_q$ [4] and the corresponding Hodge duality map [10, 11] in q-quaternion language we have found [13] solutions A of the (anti)self-duality equations, in the form of 1-form valued 2×2 matrices, that closely resemble their undeformed counterparts (instantons) in $su(2)$ Yang-Mills theory on $\mathbb{R}^4$. [The (still missing) complete gauge theory might however be a deformed $u(2)$ rather than $su(2)$ Yang-Mills theory.] The "coordinates of the center" of the instanton are nevertheless noncommuting parameters, differently from the Nekrasov-Schwarz theory. We have also found some multi-instantons solutions: they are again parametrized by noncommuting parameters playing the role of "size" and "coordinates of the center" of the (anti)instantons. This indicates that the moduli space of a complete theory will be a noncommutative manifold.

Here we briefly report on these results.

2 The q-Quaternion (Hopf) Algebra $C(\mathbb{H}_q)$

Any element X in the (undeformed) quaternion algebra $\mathbb{H}$ is given by

$$X = \mathrm{x}_1 + \mathrm{x}_2 i + \mathrm{x}_3 j + \mathrm{x}_4 k,$$

with $\mathrm{x} \in \mathbb{R}^4$ and imaginary i, j, k fulfilling

$$i^2 = j^2 = k^2 = -1, \quad ijk = -1.$$

Replacing i, j, k by Pauli matrices $\times$ imaginary unit i,

$$X \leftrightarrow x \equiv \begin{pmatrix} \mathrm{x}_1 + \mathrm{x}_4\mathrm{i} & \mathrm{x}_3 + \mathrm{x}_2\mathrm{i} \\ -\mathrm{x}_3 + \mathrm{x}_2\mathrm{i} & \mathrm{x}_1 - \mathrm{x}_4\mathrm{i} \end{pmatrix} =: \begin{pmatrix} \alpha & \gamma \\ -\gamma^\star & \alpha^\star \end{pmatrix}$$

(where $\alpha, \gamma \in \mathbb{C}$), and the quaternionic product becomes represented by matrix multiplication. Therefore $\mathbb{H}$ essentially consists of all complex 2×2 matrices of this form.

This can be q-deformed as follows. We just pick the pioneering definition of the (Hopf) $*$-algebra $C\Big(SU_q(2)\Big)$ [28, 29] without imposing the $\det_q$=1 condition: for $q \in \mathbb{R}$ consider the unital associative $\star$-algebra $\mathcal{A} \equiv C(\mathbb{H}_q)$ generated by elements $\alpha, \gamma, \alpha^\star, \gamma^\star$ fulfilling the commutation relations

$$\begin{aligned} &\alpha\gamma = q\gamma\alpha, \qquad \alpha\gamma^\star = q\gamma^\star\alpha, \qquad \gamma\alpha^\star = q\alpha^\star\gamma, \\ &\gamma^\star\alpha^\star = q\alpha^\star\gamma^\star, \qquad [\alpha, \alpha^\star] = (1-q^2)\gamma\gamma^\star \qquad [\gamma^\star, \gamma] = 0. \end{aligned} \tag{1}$$

Introducing the matrix

$$x \equiv \begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix} := \begin{pmatrix} \alpha & -q\gamma^\star \\ \gamma & \alpha^\star \end{pmatrix}$$

we can rewrite these commutation relations as

$$\hat{R}x_1x_2 = x_1x_2\hat{R} \tag{2}$$

and the conjugation relations as $x^{\alpha\beta\star} = \epsilon^{\beta\gamma}x^{\delta\gamma}\epsilon_{\delta\alpha}$, *i.e.*

$$x^\dagger = \bar{x}\,, \qquad \text{where } \bar{a} := \epsilon^{-1}a^T\epsilon \quad \forall a \in M_2. \tag{3}$$

Here we have used the braid matrix and the ϵ-tensor of $GL_q(2)$ [and $SU_q(2)$],

$$\epsilon := \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} = -q\epsilon^{-1} \qquad \hat{R}^{\alpha\beta}_{\gamma\delta} = q\delta^\alpha_\gamma\delta^\beta_\delta + \epsilon^{\alpha\beta}\epsilon_{\gamma\delta}. \tag{4}$$

[with $\epsilon \equiv (\epsilon_{\alpha\beta})$ and $\epsilon^{-1} \equiv (\epsilon^{\alpha\beta})$]. So $\mathcal{A} := C(\mathbb{H}_q)$ can be endowed also with a bialgebra structure (we are not excluding the possibility that $x \equiv \mathbf{0}_2$), more precisely a real section of the bialgebra $C\left(M_q(2)\right)$ of 2×2 quantum matrices [6]. Since the coproduct

$$\Delta(x^{\alpha\gamma}) = (ax)^{\alpha\gamma}$$

is an algebra map, the matrix product ax of any two matrices a, x with mutually commuting entries and fulfilling (2-3) again fulfills the latter. Therefore we shall call any such matrix x a q-*quaternion*, and $\mathcal{A} := C(\mathbb{H}_q)$ the q-quaternion bialgebra.

As well-known, the so-called 'q-determinant' of x

$$|x|^2 \equiv \det{}_q(x) := x^{11}x^{22} - qx^{12}x^{21} = \alpha^\star\alpha + \gamma^\star\gamma \sim x^{\alpha\alpha'}x^{\beta\beta'}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}, \tag{5}$$

is central, manifestly nonnegative-definite and group-like. It is zero iff $x \equiv \mathbf{0}_2$. If we assume $x \neq \mathbf{0}_2$ and extend $C(\mathbb{H}_q)$ by the new (central, positive-definite) generator $|x|^{-1}$ one finds that x is invertible with inverse

$$x^{-1} = \frac{\bar{x}}{|x|^2}. \tag{6}$$

$C(\mathbb{H}_q)$ becomes a Hopf $\star$-algebra [a real section of $C\left(GL_q(2)\right)$] which we shall call "q-quaternion Hopf algebra". The matrix elements of $T := \frac{x}{|x|}$ fulfill the relations (2) and

$$T^\dagger = T^{-1} = \overline{T}, \qquad \det{}_q(T) = \mathbf{1}, \tag{7}$$

namely generate as a quotient algebra $C\left(SU_q(2)\right)$ [28, 29], therefore $\mathcal{A} := C(\mathbb{H}_q)$ is the (Hopf) $\star$-algebra of functions on the quantum group $SU_q(2) \times GL^+(1)$, as in the $q = 1$ case.

As a $\star$-algebra, $\mathcal{A} := C(\mathbb{H}_q)$ coincides with the algebra of functions on the $SO_q(4)$-covariant quantum Euclidean space $\mathbb{R}^4_q$ of [6], identifying their generators as

$$x^1 = qx^{11}, \quad x^2 = x^{12}, \quad x^3 = -qx^{21}, \quad x^4 = x^{22}.$$

The (left) coaction of $SO_q(4)$ is obtained from the two-fold coproduct $\Delta^{(2)}(x) = axb$ of $SU_q(2)$ (recall that $SO_q(4) = SU_q(2) \otimes SU_q(2)'/\mathbb{Z}_2$) by

$$x \to a\,x\,b^T \tag{8}$$

(b^T means the transpose of b).

A different matrix version (with no interpretation in terms of q-deformed quaternions) of a $SU_q(2) \times SU_q(2)$ covariant quantum Euclidean space was proposed in [18].

3 Other Preliminaries

The $SO_q(4)$-covariant **differential calculus** (d, Ω^*) **on** $\mathbb{R}^4_q \sim \mathbb{H}_q$ [4] is obtained imposing covariant homogeneous bilinear commutation relations (10) between the x^i and the differentials $\xi^i := dx^i$. Partial derivatives are introduced through the decomposition $d = \xi^a \partial_a = \xi^{\alpha\alpha'} \partial_{\alpha\alpha'}$. All other commutation relations are derived by consistency. The complete list is given by

$$\mathsf{P}_a{}^{ij}_{hk} x^h x^k = 0, \tag{9}$$

$$x^h \xi^i = q \hat{\mathsf{R}}^{hi}_{jk} \xi^j x^k, \tag{10}$$

$$(\mathsf{P}_s + \mathsf{P}_t)^{ij}_{hk} \xi^h \xi^k = 0, \tag{11}$$

$$\mathsf{P}_a{}^{ij}_{hk} \partial_j \partial_i = 0, \tag{12}$$

$$\partial_i x^j = \delta^j_i + q \hat{\mathsf{R}}^{jh}_{ik} x^k \partial_h, \tag{13}$$

$$\partial^h \xi^i = q^{-1} \hat{\mathsf{R}}^{hi}_{jk} \xi^j \partial^k. \tag{14}$$

$\hat{\mathsf{R}}$ $\equiv$braid matrix of $SO_q(4)$; $\mathsf{P}_s, \mathsf{P}_a, \mathsf{P}_t \equiv$ deformations of the symmetric trace-free, antisymmetric and trace projectors appearing in the projector decomposition of $\hat{\mathsf{R}}$.

$$q\hat{\mathsf{R}} = \hat{R} \otimes \hat{R}.$$

The Laplacian $\Box \equiv \partial \cdot \partial := \partial_k g^{hk} \partial_h$ is $SO_q(4)$-invariant and commutes the ∂_i. In $\mathcal{H}$ there exists a special invertible element Λ, such that

$$\Lambda x^i = q^{-1} x^i \Lambda, \qquad \Lambda \partial^i = q \partial^i \Lambda, \qquad \Lambda \xi^i = \xi^i \Lambda.$$

Definitions:

- $\bigwedge^* \equiv \natural$-graded algebra generated by the ξ^i, where grading $\natural$ $\equiv$degree in ξ^i; any component $\bigwedge^p$ with $\natural = p$ carries an irreducible representation of $U_q so(4)$ and has the same dimension as in the $q = 1$ case.
- $\mathcal{DC}^* \equiv \natural$-graded algebra generated by x^i, ξ^i, ∂_i. Elements of $\mathcal{DC}^p$ are differential-operator-valued p-forms.

- $\Omega^* \equiv \natural$-graded subalgebra generated by the ξ^i, x^i. By definition $\Omega^0 = \mathcal{A}$ itself, and both Ω^* and Ω^p are $\mathcal{A}$-bimodules. Also, we shall denote Ω^* enlarged with $\Lambda^{\pm 1}$ as $\tilde{\Omega}^*$, and the subalgebra generated by $T^{\alpha\alpha'} := x^{\alpha\alpha'}/|x|$, $dT^{\alpha\alpha'}$ as Ω^*_S (the latter is 4-dim!).
- $\mathcal{H} \equiv$subalgebra generated by the x^i, ∂_i. By definition, $\mathcal{DC}^0 = \mathcal{H}$, and both $\mathcal{DC}^*$ and $\mathcal{DC}^p$ are $\mathcal{H}$-bimodules.

The restricted (but still 4-dimensional!) differential calculus (Ω^*_S, d) coincides with the Woronowicz 4D- on $C\,(SU_q(2))$.

The special 1-form

$$\theta := \frac{1}{1-q^{-2}}|x|^{-2}\,d|x|^2 = \frac{q^{-2}}{q^2-1}\xi^{\alpha\alpha'}\frac{x^{\beta\beta'}}{|x|^2}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}$$

plays the role of "Dirac Operator" [5] of the differential calculus,

$$d\omega_p = [-\theta, \omega_p\}, \qquad \omega_p \in \Omega^p\,.$$

However, $d(f^\star) \neq (df)^\star$, and moreover there is no $\star$-structure $\star : \Omega^* \to \Omega^*$, but only a $\star$-structure

$$\star : \mathcal{DC}^* \to \mathcal{DC}^*$$

[22], with a rather nonlinear character (the latter has been recently [12] recast in a much more suggestive form).

The **Hodge map** [10,11] is a $SO_q(4)$-covariant, $\mathcal{A}$-bilinear map $* : \tilde{\Omega}^p \to \tilde{\Omega}^{4-p}$ such that $*^2 = \mathrm{id}$, defined by

$$^*(\xi^{i_1}...\xi^{i_p}) = q^{-4(p-2)}c_p\,\xi^{i_{p+1}}...\xi^{i_4}\varepsilon_{i_4...i_{p+1}}{}^{i_1...i_p}\Lambda^{2p-4},$$

where $\varepsilon^{hijk} \equiv q$-epsilon tensor and c_p are suitable normalization factors. Actually this extends to a $\mathcal{H}$-bilinear map $* : \mathcal{DC}^p \to \mathcal{DC}^{4-p}$ with the same features. For $p = 2$ Λ-powers disappear and one even gets a map $* : \Omega^2 \to \Omega^2$ defined by

$$^*\xi^i\xi^j = \frac{1}{[2]_q}\xi^h\xi^k\varepsilon_{kh}{}^{ij}\omega_{ji}. \tag{15}$$

Ω^2 (resp. $\mathcal{DC}^2$) splits into the direct sum of $\mathcal{A}$- (resp. $\mathcal{H}$-) bimodules

$$\Omega^2 = \check{\Omega}^2 \oplus \check{\Omega}^{2\prime} \qquad \text{(resp. } \mathcal{DC}^2 = \check{\mathcal{DC}}^2 \oplus \check{\mathcal{DC}}^{2\prime})$$

of the eigenspaces of $*$ with eigenvalues $1, -1$ respectively, whose elements are "self-dual and anti-self-dual 2-forms". $\check{\Omega}^2$ (resp. $\check{\mathcal{DC}}^2$) is generated by the self-dual exterior forms

$$f^{\alpha\beta} := (\xi\bar{\xi})^{\alpha\beta} \tag{16}$$

through (left or right) multiplication by elements of $\mathcal{A}$ (resp. $\mathcal{H}$). $f^{\alpha\beta}$ span a (3,1) corepresentation space of $SU_q(2) \otimes SU_q(2)'$.

One can find 1-form-valued matrices a such that

$$d\,a^{\alpha\beta} = f^{\alpha\beta}; \tag{17}$$

a is uniquely determined to be

$$a^{\alpha\beta} = \mathcal{P}_{s\,\gamma\delta}^{\;\alpha\beta}(\xi\epsilon^{-1}x^T)^{\gamma\delta} \tag{18}$$

if we require $a^{\alpha\beta}$ to transform as $f^{\alpha\beta}$, *i.e.* in the (3,1) dimensional corepresentation of $SU_q(2) \times SU_q(2)'$, whereas will be defined up to d-exact terms of the form

$$\tilde{a} = a + \mathbf{1}_2\, dM(|x|^2)$$

if we just require $\tilde{a}^{\alpha\beta}$ to be in the $(3,1) \oplus (1,1)$ reducible representation. In particular, the 1-form valued matrix $dT\overline{T}$ belongs to the latter. In the $q = 1$ limit (18) becomes

$$a^{\alpha\beta} = \left(\xi\epsilon^{-1}x^T\right)^{(\alpha\beta)} = -\left\{Im(\xi\,\bar{x})\right\}^{\alpha\beta}.$$

Similarly, anti-self-dual $\check{\Omega}^{2\prime}$, $\check{\mathcal{DC}}^{2\prime}$ are generated by

$$f'^{\alpha'\beta'} := (\bar{\xi}\xi)^{\alpha'\beta'} \tag{19}$$

and one can find 1-forms $a'^{\alpha'\beta'}$ such that $d\,a'^{\alpha'\beta'} = f'^{\alpha'\beta'}$, *etc.*

Integration over $\mathbb{R}_q^4$ [8, 9, 25] can be introduced by the decomposition

$$\int_{\mathbb{R}_q^4} d^4x = \int_0^\infty d|x| \int_{|x|\cdot S_q^3} dT^3 .$$

Integration over the radial coordinate has to fulfill the scaling property $\int_0^\infty d|x|\, g(|x|) = \int_0^\infty d(q|x|)\, g(q|x|)$. Integration over the quantum sphere S_q^3 is determined up to normalization by the requirement of $SO_q(4)$-invariance. The algebra of functions on the quantum sphere S_q^3 is generated by the $T^{\alpha\beta} := x^{\alpha\beta}/|x|$.

This integration over $\mathbb{R}_q^4$ fulfills all the main properties of Riemann integration over $\mathbb{R}^4$, including Stokes' theorem, except the cyclic property.

4 Noncommutative Gauge Theories: Standard Framework

The standard framework [5,7,17] for noncommutative gauge theories (*i.e.* gauge theories on noncommutative manifolds) closely mimics that for commutative ones. In $U(n)$ gauge theory the gauge transformations U are unitary $\mathcal{A}$-valued ($\mathcal{A}$ being the algebra of functions on the noncommutative manifold) $n\times n$ unitary matrices, $U \in M_n(\mathcal{A}) \equiv M_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{A}$. The gauge potential $A \equiv (A^{\dot\alpha}_{\dot\beta})$ is a antihermitian 1-form-valued $n \times n$ matrix, $A \in M_n(\Omega^1(\mathcal{A}))$. The definition of the field strength $F \in M_n(\Omega^2(\mathcal{A}))$ associated to A is as usual $F := dA + AA$. At the right-hand side the product AA has to be understood both as a (row by column) matrix product and as a wedge product. Even for $n = 1$, $AA \neq 0$, contrary to the commutative case. The Bianchi identity $DF := dF + [A, F] = 0$ is automatically satisfied and the Yang-Mills equation reads as usual $D^*F = 0$. Because of the Bianchi identity, the latter is automatically satisfied by any solution of the (anti)self-duality equations

$$^*F = \pm F. \tag{20}$$

The Bianchi identity, the Yang-Mills equation, the (anti)self-duality equations, the flatness condition $F = 0$ are preserved by gauge transformations

$$A^U = U^{-1}(AU + dU), \qquad \Rightarrow \qquad F^U = U^{-1}FU.$$

As usual, $A = U^{-1}dU$ implies $F = 0$. Up to normalization factors, the gauge invariant ‘action’ S and ‘Pontryagin index’ $\mathcal{Q}$ are defined by

$$S = \text{Tr}(F\,{}^*F), \qquad\qquad \mathcal{Q} = \text{Tr}(FF), \tag{21}$$

where Tr stands for a positive-definite trace combining the $n \times n$-matrix trace with the integral over the noncommutative manifold (as such, Tr has to fulfill the cyclic property). If integration $\int$ fulfills itself the cyclic property then this is obtained by simply choosing $\text{Tr} = \int \text{tr}$, where tr stands for the ordinary matrix trace. S is automatically nonnegative.

In the present $\mathcal{A} \equiv C(\mathbb{R}^4_q) = C(\mathbb{H}_q)$ case there are **2 main problems**:

1. Integration over $\mathbb{R}^4_q$ fulfills a *deformed* cyclic property [25].
2. $d(f^\star) \neq (df)^\star$, and there is no $\star$-structure $\star : \Omega^* \to \Omega^*$, but only a $\star$-structure $\star : \mathcal{DC}^* \to \mathcal{DC}^*$ [22], with a nonlinear character.

A solution to both problems might be obtained

1. allowing for $\mathcal{DC}^1$-valued A ($\Rightarrow \mathcal{DC}^2$-valued F’s), and/or
2. realizing $\text{Tr}(\cdot)$ by in the form $\text{Tr}_q(\cdot)$:=$\text{Tr}(W\cdot)$, with W some suitable positive definite $\mathcal{H}$-valued (*i.e.* pseudo-differential-operator-valued) $n \times n$ matrix (this implies a change in the hermitian conjugation of differential operators), or even a more general form.

This hope is based on our results [12]: 1) the $\star$-structure $\star : \mathcal{DC}^* \to \mathcal{DC}^*$ can be recast in a more suggestive form of similarity transformations (involving the realization as pseudodifferential operators of the ribbon element $\tilde{w}$ and of the "vector field generators" $\tilde{Z}^i_j$ of the central extension of $U_q so(4)$ with dilatations); 2) d and the exterior coderivative $\delta := - * d*$ become conjugated of each other

$$(\alpha_p, d\beta_{p\text{-}1}) = (\delta\alpha_p, \beta_{p\text{-}1}), \qquad (d\beta_{p\text{-}1}, \alpha_p) = (\beta_{p\text{-}1}, \delta\alpha_p)$$

if one defines

$$(\alpha_p, \beta_p) = \int_{\mathbb{R}^4_q} \alpha_p^\star \; {}^* \; \tilde{w}'^{1/2}\beta_p \, ,$$

where $\tilde{w}'$ is the realization of $\tilde{w}$ as a pseudodifferential operator.

5 The (Anti)Instanton Solution

We first recall the *commutative ($q=1$) solution* of the self-duality equation ${}^*F = F$: the instanton solution of [3] in t'Hooft [26] and in ADHM [2] quaternion notation (see [1] for an introduction) reads:

$$\begin{aligned} A &= dx^i\, \sigma^a \, \underbrace{\eta^a_{ij} x^j \frac{1}{\rho^2 + r^2/2}}_{A^a_i}, \\ &= -Im\left\{\xi\, \frac{\bar{x}}{|x|^2}\right\} \frac{1}{1+\rho^2 \frac{1}{|x|^2}} \\ &= -(dT)\overline{T} \frac{1}{1+\rho^2 \frac{1}{|x|^2}} \end{aligned} \tag{22}$$

$$F = \xi\bar{\xi}\,\rho^2 \frac{1}{(\rho^2 + |x|^2)^2}, \tag{23}$$

where $r^2 := x \cdot x = 2|x|^2$, η^a_{ij} are the so-called 't Hooft η-symbols and ρ is the size of the instanton (here centered at the origin). The third equality is based on the identity

$$\xi\, \frac{\bar{x}}{|x|^2} = (dT)\overline{T} + I_2 \frac{d|x|^2}{2|x|^2}$$

and the observation that the first and second term at the RHS are respectively antihermitian and hermitian, *i.e.* the imaginary and the real part of the quaternion at the LHS.

Noncommutative ($q \neq 1$) solutions of ${}^*F = F$. Looking for A directly in the form $A = \xi\bar{x}\, l/|x|^2 + \theta\, I_2\, n$, where l, n are functions of x only through $|x|$, one finds a family of solutions parametrized by ρ^2 (a nonnegative constant, or more generally a further generator of the algebra) and by the function l itself. The freedom in the choice of l should disappear upon imposing the proper (and still missing) antihermiticity condition on A, as it occurs in the $q=1$ case. For the moment, out of this large family we just pick one which has the right $q \to 1$ limit and closely resembles the undeformed solutions (22-23).

$$\begin{aligned} A &= -(dT)\overline{T}\,\tfrac{1}{1+\rho^2\frac{1}{|x|^2}}, \\ F &= q^{-1}\xi\bar{\xi}\tfrac{1}{|x|^2+\rho^2}\rho^2\tfrac{1}{q^2|x|^2+\rho^2}. \end{aligned} \tag{24}$$

The matrix elements $A^{\alpha\beta}$ span a $(3,1)\oplus(1,1)$ dimensional corepresentation of $SU_q(2)\times SU_q(2)'$, suggesting as the 'fiber' of the gauge group in the complete theory a (possibly deformed) $U(2)$ [instead of a $SU(2)$].

One can shift the 'center of the instanton' away from the origin by the replacement (or 'braided coaddition' [19])

$$x \to x - y,$$

where the 'coordinates of the center' y^i generate a new copy of $\mathcal{A}$, 'braided' with the original one (see below). Therefore the instanton moduli space must be a noncommutative manifold!

By the scaling and translation invariance of integration over $\mathbb{R}^4_q$, if we could find a 'good' pseudodifferential operator W to define gauge invariant "action" and "topological charge" by

$$\mathcal{Q} := \int_{\mathbb{R}^4_q} \mathrm{tr}(WF\,F) = \int_{\mathbb{R}^4_q} \mathrm{tr}(WF\,{}^*F) = S.$$

The latter would, as in the commutative case, equal a constant independent of ρ, y (which by the choice of the normalization of the integral we can make 1).

In the $q=1$ case multi-instanton solution is explicitly written down in the so-called 'singular gauge'. Note that as in the $q=1$ case $T=x/|x|$ is unitary and singular at $x=0$. So it can play the role of a 'singular gauge transformation'. In fact A can be obtained through the gauge transformation $A = T(\hat{A}\overline{T} + d\overline{T})$ from the singular gauge potential

$$\begin{aligned} \hat{A} &= \overline{T}dT\frac{1}{1+|x|^2\frac{1}{\rho^2}} = -\frac{1}{1+|x|^2\frac{1}{q^2\rho^2}}(d\overline{T})T \\ &= -\frac{1}{1+|x|^2\frac{1}{q^2\rho^2}}\left[q^{-1}\bar{\xi}\frac{x}{|x|^2} - \frac{q^{-3}}{1+q}\xi^{\alpha\alpha'}\frac{x^{\beta\beta'}}{|x|^2}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}\right]. \end{aligned} \tag{25}$$

$\hat{A}$ can be expressed also in the form

$$\hat{A} = \phi^{-1}\hat{\mathcal{D}}\phi, \qquad \phi := 1 + q^2\rho^2\frac{1}{|x|^2},$$

where $\hat{\mathcal{D}}$ is the first-order-differential-operator-valued 2×2 matrix obtained from the square bracket in (25) by the replacement $x^{\alpha\alpha'}/|x|^2 \to q^2\partial^{\alpha\alpha'}$:

$$\hat{\mathcal{D}} := q\bar{\xi}\partial - \frac{q^{-1}}{q+1}d \tag{26}$$

(for simplicity we are here assuming that ρ^2 commutes with $\xi\partial$). ϕ is harmonic:

$$\Box\phi = 0.$$

This is the analog of the $q = 1$ case, and is useful for the construction of multi-instanton solutions.

The **anti-instanton solution** is obtained just by converting unbarred into barred matrices, or conversely, as in the $q = 1$ case. For instance, from (24) we obtain the anti-instanton solution in the regular gauge

$$\begin{aligned} A' &= -(d\overline{T})T\,\tfrac{1}{1+\rho^2\frac{1}{|x|^2}}, \\ F' &= q^{-1}\bar{\xi}\xi\tfrac{1}{|x|^2+\rho^2}\rho^2\tfrac{1}{q^2|x|^2+\rho^2}. \end{aligned} \tag{27}$$

6 Multi-Instanton Solutions

We have found solutions of the self-duality equation corresponding to n instantons in the “singular gauge” [26, 27] in the form

$$\hat{A} = \phi^{-1}\hat{\mathcal{D}}\phi, \tag{28}$$

where ϕ is the harmonic scalar function

$$\phi = 1 + \rho_1^2\frac{1}{(x-y_1)^2} + \rho_2^2\frac{1}{(x-y_1-y_2)^2} + ... + \rho_n^2\frac{1}{(x-y_1-...-y_n)^2} \tag{29}$$

as in the commutative case. In the commutative limit

$$\begin{aligned} &\rho_\mu \equiv \text{size of the } \mu\text{-th instanton}, \\ &v_\mu^i := \sum_{\nu=1}^{\mu} y_\nu^i \equiv i\text{-th coordinate of the } \mu\text{-th instanton}. \end{aligned}$$

are constants ($\mu = 1, 2, ..., n$). In the noncommutative setting the new generators ρ_μ^2, y_ν^i have to fulfill the following nontrivial commutation relations:

$$\begin{aligned} &\rho_\nu^2\rho_\mu^2 = q^2\,\rho_\mu^2\rho_\nu^2 && \nu < \mu \\ &\rho_\nu^2 y_\mu^i = y_\mu^i\rho_\nu^2 \cdot \begin{cases} q^{-2} & \nu < \mu \\ 1, & \nu \geq \mu \end{cases} \\ &\rho_\mu^2\xi^i = \xi^i\rho_\mu^2, && \partial_i\rho_\mu^2 = \rho_\mu^2\partial_i. \\ &y_\mu^i y_\nu^j = q\hat{\mathsf{R}}^{ij}_{hk}y_\nu^h y_\mu^k && \nu < \mu, \\ &\mathsf{P}_{A\,hk}^{\,ij}y_\mu^h y_\mu^k = 0. \end{aligned} \tag{30}$$

($\mu, \nu = 0, 1, ..., n$, and we have set $x^i \equiv y_0^i$).

The last relation states that for any fixed ν the 4 coordinates y_ν^i generate a copy of $\mathcal{A}$. The last but one states that the various copies of $\mathcal{A}$ are *braided* [19] w.r.t. each other (this is necessary for the $SO_q(4)$ covariance of the overall algebra).

The obvious consequence of the nontrivial commutation relations (30) is that in a complete theory **the instanton moduli space must be a noncommutative manifold**.

At least for low n, we have been able to go to a 'regular' (*i.e.* analytic in $z_\mu^i := x^i - v_\mu^i$) gauge potential A by a 'singular gauge transformation' (as in the $q = 1$ case [15, 23, 27]).

References

[1] M.F. Atiyah (1979) *Geometry on Yang-Mills Fields*, Lezioni Fermiane, Scuola Normale Superiore di Pisa.

[2] M.F. Atiyah, N.J. Hitchin, V. G. Drinfel'd, and Yu.I. Manin (1978) *Phys. Lett.* **A65** 185.

[3] A.A. Belavin, A.M. Polyakov, A.S. Schwarz, and Yu.S. Tyupkin (1975) *Phys. Lett.* **B59** 85.

[4] U. Carow-Watamura, M. Schlieker, S. Watamura (1991) *Z. Phys.* **C49** 439.

[5] A. Connes (1994) *Noncommutative Geometry* (Academic Press).

[6] L.D. Faddeev, N.Y. Reshetikhin, and L. Takhtadjan (1989) *Algebra and Analysis* **1** 178 (in Russian); English translation: *Leningrad Math. J.* **1** 193 (1990).

[7] J.M. Gracia-Bonda, J.C. Varilly, and H. Figueroa (2001) *Elements of Noncommutative Geometry* (Birkhäuser, Boston).

[8] G. Fiore (1993) *Int. J. Mod. Phys.* **A8** 4679-4729.

[9] G. Fiore (1994) *q-Euclidean Covariant Quantum Mechanics on $\mathbb{R}_q^N$: Isotropic Harmonic Oscillator and Free particle*, PhD Thesis, SISSA-ISAS.

[10] G. Fiore (1994) *J. Phys. A: Math Gen.* **27** 3795.

[11] G. Fiore (2004) *J. Phys. A: Math. Gen.* **37** 9175-9193.

[12] G. Fiore, " On the hermiticity of q-differential operators and forms on the quantum Euclidean spaces $\mathbb{R}_q^N$", math.QA/0403463.

[13] G. Fiore, "From q-quaternions to q-deformed $su(2)$ instantons", Preprint 05-54 Dip. Matematica e Applicazioni, Università di Napoli, and DSF/43-2005.

[14] G. Fiore, "On q-quaternions", Preprint 05-55 Dip. Matematica e Applicazioni, Università di Napoli, and DSF/44-2005.

[15] J.J. Giambiaggi and K.D. Rothe (1977) *Nucl. Phys.* **B129** 111-124; S. Sciuto (1979) *Riv. Nuovo Cim.* **2** 16.

[16] B. Jurco, L. Möller, S. Schraml, P. Schupp, and J. Wess (2001) *Eur. Phys. J.* **C21** 383-388; hep-th/0104153.

[17] G. Landi (1997) *An Introduction to Noncommutative Spaces and Their Geometries*, (Springer-Verlag, Berlin).

[18] S. Majid (1994) *J. Math. Phys.* **35** 5025-5034.

[19] See for instance: S. Majid (1995) *Foundations of Quantum Groups* (Cambridge Univ. Press).

[20] N. Nekrasov and A. Schwarz (1998) *Commun. Math. Phys.* **198** 689-703.

[21] O. Ogievetsky (1992) *Lett. Math. Phys.* **24** 245.

[22] O. Ogievetsky and B. Zumino (1992) *Lett. Math. Phys.* **25** 121-130.

[23] D.I. Olive, S. Sciuto and R.J. Crewther (1979) *Riv. Nuovo Cim.* **2** 1-117.

[24] N. Seiberg and E. Witten (1999) *JHEP* **9909** 032.

[25] H. Steinacker (1996) *J. Math. Phys.* **37** 4738.

[26] G. 't Hooft (1976) *Phys. Rev.* **D14** 3432-3450.

[27] F. Wilczeck (1977) in: *Quark confinement and field theory*, eds. D. Stump and D. Weingarten (John Wiley and Sons, New York); E. Corrigan and D.B. Fairlie (1977) *Phys. Lett.* **B67** 69; R. Jackiw, C. Nohl, and C. Rebbi (1977) *Phys. Rev.* **D15** 1642.

[28] S.L. Woronowicz (1987) *Commun. Math. Phys.* **111** 613-665.

[29] S.L. Woronowicz (1987) *Publ. RIMS, Kyoto Univ.* **23** 117.

[30] S.L. Woronowicz (1989) *Commun. Math. Phys.* **122** 125.