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U_q(so(1,2)) Representations at Roots of Unity and Their Realizations on the Finite Circle

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Abstract. We consider some infinitesimally unitarizable $U_q(so(1,2))$ highest weight modules and provide realizations of them on $\mathcal{L}^2(\mathbb{Z}/(N-1)\mathbb{Z})$, the finite dimensional Hilbert space of square integrable functions on the finite circle. Actions of the $U_q(so(1,2))$ generators on basis functions are given. In particular the Cartan generator is given as an explicit function of the adjacency operator. We also show how these representations at roots of unity go over into discrete series representations of $so(1,2)$ as N goes to infinity and q goes to one. We conclude with an application of these results to the Casimir effect for a massless scalar field on the two-dimensional Einstein universe, and make some comments about possible generalizations of these results to higher dimensions.

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1 Some Representations of U_q(so(1,2))

The q-deformation $U_q(so(3,\mathbb{C}))$ may be defined as the associative algebra over $\mathbb{C}$ with generators H, $X^{\pm}$ and relations [1,2]:

$$[H, X^{\pm}] = \pm 2X^{\pm}\,, \tag{1}$$

$$[X^+, X^-] = [H]_q\,. \tag{2}$$

Let I be the unit element in $U_q(so(3,\mathbb{C}))$. The Casimir element of $U_q(so(3,\mathbb{C}))$ is

$$\begin{aligned}\Delta_q &= X^+X^- + \left(\left[\frac{1}{2}(H - I)\right]_q\right)^2 - \frac{1}{4} \\ &= X^-X^+ + \left(\left[\frac{1}{2}(H + I)\right]_q\right)^2 - \frac{1}{4}\,. \end{aligned} \tag{3}$$

The real form $U_q(so(2,1))$ of $U_q(so(3,\mathbb{C}))$ is defined as follows. The generators of $U_q(so(2,1))$ are given by the following expressions:

$$\mathbf{L}_{32} = -\frac{i}{2}(X^+ - X^-), \quad \mathbf{L}_{13} = \frac{1}{2}(X^+ + X^-), \quad \mathbf{L}_{21} = \frac{i}{2}\,H\,. \tag{4}$$

Thus

$$X^{\pm} \;=\; \mathbf{L}_{13} \pm i\mathbf{L}_{32}\,. \tag{5}$$

The operators $i\mathbf{L}_{12}$, $i\mathbf{L}_{13}$, $i\mathbf{L}_{32}$ are preserved under the following antilinear anti-involution ω of $U_q(so(3,\mathbb{C}))$

$$\omega(H) = H\,, \qquad \omega(X^{\pm}) = -X^{\mp}\,. \tag{6}$$

For the coproduct on $U_q(so(3,\mathbb{C}))$ we take [1,2]

$$\Delta(H) \;=\; H\otimes I \;+\; I\otimes H\,, \qquad \Delta(X^{\pm}) \;=\; X^{\pm}\otimes q^{\frac{H}{4}} \;+\; q^{-\frac{H}{4}}\otimes X^{\pm}\,. \tag{7}$$

For $q^M = 1$ ($M\in\mathbb{Z}$, $M>2$), let $q=\exp(2\pi i/m)$ and set $M=m$ for m odd, and set $M=m/2$ for m even. Define $\sigma = \frac{1}{2}(d-1) - \frac{1}{2}M$ ($d=1,2,\ldots,M$) and let $V_d =$ linear span of the $|s_3>$ ($s_3 = \sigma, \sigma-1, \ldots, \sigma-(d-1)$). The action $d\pi^{\sigma}$ of the basic generators H and $X^{\pm}$ on V_d is given by [3]

$$d\pi^{\sigma}(H)|s_3> = -2s_3|s_3>\,, \qquad d\pi^{\sigma}(X^{\pm})|s_3> = [-\sigma\pm s_3]_q|s_3\pm 1>\;. \tag{8}$$

These finite dimensional highest weight modules are all infinitesimally unitary [4].

Let $\mathcal{H}_{\epsilon}$ be the complex Hilbert space with orthonormal basis consisting of the set of vectors: ($\epsilon = 0, \frac{1}{2}$)

$$\{|m>|m \;=\; n \;+\; \epsilon, \quad n \;=\; 0, \pm1, \pm2, \cdots\}. \tag{9}$$

For $\sigma\in\mathbb{C}$ the following formulae define a representation $d\pi^{\sigma,\epsilon}$ of $U_q(so(1,2))$ on $\mathcal{H}_{\epsilon}$:

$$d\pi^{\sigma,\epsilon}(H)|m> \;\;=\;\; 2m|m> \tag{10}$$

$$d\pi^{\sigma,\epsilon}(X^{\pm})|m> \;\;=\;\; [-\sigma\pm m]_q|m\pm 1>\;. \tag{11}$$

Indeed, these formulae specify the action of the generators H and $X^{\pm}$ in the representation $d\pi^{\sigma,\epsilon}$ of $U_q(so(1,2))$, which is defined on the dense subspace ($\ell.s.$ means linear span)

$$\mathcal{D}^{\sigma,\epsilon} \;=\; \ell.s.\{|m>|m \;=\; n \;+\; \epsilon\,, \quad n \;=\; 0, \pm1, \pm2, \ldots\}\subset\mathcal{H}_{\epsilon}\,. \tag{12}$$

It is easy to check that the commutation relations (1) and (2) are satisfied, and for the action of Casimir element in the representation $d\pi^{\sigma,\epsilon}$ one finds

$$d\pi^{\sigma,\epsilon}(\Delta_q) \;=\; \left\{\left[\sigma \;+\; \frac{1}{2}\right]_q^2 \;-\; \frac{1}{4}\right\}\times I\,, \tag{13}$$

where I is the identity operator on $\mathcal{H}_{\epsilon}$.

The representation $d\pi^{\sigma,\epsilon}$ is irreducible if and only if $2\sigma\neq 2\epsilon(\mod 2)$ [5]. If $2\sigma \;=\; 2\epsilon(\mod 2)$, then $d\pi^{\sigma,\epsilon}$ is indecomposable or (fully) reducible, and we now discuss the composition series for these cases. We let $\sigma = \ell$ with $2\ell = 2\epsilon (\mod 2)$ and $\ell \geq 0$. We define

$$\mathcal{W}^{\ell,\epsilon} = \ell.s.\{|m> | -\ell \leq m \leq \ell\}, \tag{14}$$

$$V_+^{\ell,\epsilon} = \ell.s.\{|m> | -\ell \leq m\}, \tag{15}$$

$$V_-^{\ell,\epsilon} = \ell.s.\{|m> |m \leq \ell\}. \tag{16}$$

$\mathcal{W}^{\ell,\epsilon}$ is a finite dimensional invariant subspace of $\mathcal{D}^{\ell,\epsilon}$ on which $U_q(so(1,2))$ acts irreducibly. $U_q(so(1,2))$ acts irreducibly via the quotient action of $d\pi^{\ell,\epsilon}$ on $V_+^{\ell,\epsilon}/\mathcal{W}^{\ell,\epsilon}$ and $V_-^{\ell,\epsilon}/\mathcal{W}^{\ell,\epsilon}$. $V_+^{\ell,\epsilon}$ and $V_-^{\ell,\epsilon}$ are the essential extensions[1] of $\mathcal{W}^{\ell,\epsilon}$ given in Eqs. (15) and (16), respectively.

For $q = 1$ the $d\pi^{\sigma,\epsilon}$ acting on $V_+^{\ell,\epsilon}/\mathcal{W}^{\ell,\epsilon}$ and $V_-^{\ell,\epsilon}/\mathcal{W}^{\ell,\epsilon}$are the well known (unitarizable) holomorphic and anti-holomorphic discrete series representations of $so(1,2)$. We denote the irreducible representations of $U_q(so(1,2))$ on $V_+^{\ell,\epsilon}$ and $V_-^{\ell,\epsilon}$ by $d\pi_+^{\ell+1,\epsilon}$ and $d\pi_-^{-\ell-1,\epsilon}$, respectively. For the case $\ell = -\frac{1}{2}$ all of this which is stated applies except that the finite dimensional subspace $\mathcal{W}^{\ell,\epsilon}$ is trivial, *i.e.* it is the zero vector.

2 Models of the Representations

2.1 Geometrical description of discrete series representations of so(1,2)

We now present a geometrical construction [6, 7, 9–12], which is valid for all $SO(p,q)$ groups. We restrict the discussion to the relevant non-trivial lowest dimensional case which is used for the description of certain discrete series representations of $so(1,2)$. This description is important for us, since, as we shall see in Section 2.2, the model for roots of unity representations $d\pi^\sigma$ for $\sigma = -1$ introduced above is just a "discretization" of this geometric construction for these $so(1,2)$ discrete series. We shall make use of the discrete model (for the roots of unity representations) in the physical applications given in Section 3.

Let X denote the real vector space of dimension four on which a symmetric nonsingular bilinear form $B(\,\cdot\,,\,\cdot\,)$ having signature $(2,2)$ is defined. Let $SO(X)$ be the special (pseudo-)orthogonal group of X. One has the isomorphism

$$SO(X) \simeq SO(2,2). \tag{17}$$

If $G = SO_0(2,2)$ denotes the identity component of $SO(2,2)$ then one has the covering map

$$\pi : SO_0(2,2)^\sim \longrightarrow SO_0(2,2) \tag{18}$$

[1]Let $(d\pi(U_q(\mathcal{A}),\ W)$ be a representation of an algebra $\mathcal{A}$ on a vector space W. An *essential extension* of the representation $(d\pi(\mathcal{A}),\ W)$ is a representation of $\mathcal{A}$ on a vector space V, which has W as an invariant subspace, but which has no complementary invariant subspace, and for which the quotient representation on V/W has no nontrivial invariant subspace.

defining a representation on X of $SO_0(2,2)^\sim$, the simply connected covering group of $SO_0(2,2)$.

Now let $C \subset X$ be the 3-dimensional cone defined by

$$C = \{x \in X | Q(x) = 0\}, \tag{19}$$

where $Q(x) = (1/2)B(x,x)$ is the quadratic form associated with $B(\ \cdot\ ,\ \cdot\ )$. Let X^* and C^* be the sets of nonzero elements in X and C, respectively, and let $Proj(X)$ be the real projective variety of all one-dimensional subspaces in X. Then we have the map

$$X^* \longrightarrow Proj(X) \qquad x \longrightarrow \overline{x}\,. \tag{20}$$

Let $\mathbf{M} \subset Proj(X)$ be the image of C^* under (20). $\mathbf{M}$ is a 2-dimensional subvariety of $Proj(X)$, and the action of $SO_0(2,2)^\sim$ on X induces an action of $SO_0(2,2)^\sim$ on $Proj(X)$. Since obviously C is stable under the action of $SO_0(2,2)^\sim$, $\mathbf{M}$ must be stable under the action of $SO_0(2,2)^\sim$ on $Proj(X)$. It is easy to see that $SO_0(2,2)$ is transitive on C^*, and hence $SO_0(2,2)$ (and $SO_0(2,2)^\sim$) is transitive on $\mathbf{M}$. $\mathbf{M}$ is naturally diffeomorphic to $(S^1 \times S^1)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ action is the product of antipodal maps on the S^1's of the first and second factors. We denote $S^1 \times S^1$ by $\bar{\mathbf{M}}$. The two-dimensional space time $\tilde{\mathbf{M}}$, which is the simply connected covering space of $\mathbf{M}$ is topologically $\mathbb{R} \times \mathbb{R}$, and it is the two-dimensional Einstein universe.

The line bundle $L^s(\mathbf{M})$ over $\mathbf{M}$ associated with the character

$$\lambda \to |\lambda|^{-s} \tag{21}$$

of $\mathbb{R}^*$ is the bundle whose fibre over $\overline{x}$ is the set of all pairs

$$(\lambda x, |\lambda|^s) \in C \times \mathbb{C}\,, \tag{22}$$

($s \in \mathbb{C}$). Denote by $\Gamma^s(\mathbf{M})$ the space of smooth sections of $L^s(\mathbf{M})$. There is a unique isomorphism between $\Gamma^s(\mathbf{M})$ and the space of smooth functions $f : C^* \longrightarrow \mathbb{C}$ which satisfy the homogeneity condition

$$f(\lambda x) = |\lambda|^{-s} f(x). \tag{23}$$

$\Gamma^s(\mathbf{M})$ is an $SO_0(2,2)^\sim$ module with respect to the representation π_s defined by

$$(\pi_s(g)\ f)(x) = f(g^{-1}x) \tag{24}$$

($f \in \Gamma^s(\mathbf{M})$, $g \in SO_0(2,2)$, $x \in C^*$, and $g^{-1}\ x$ denotes the action of g^{-1} on $x \in C^*$). We denote the associated representation of the Lie algebra $so(2,2)$ by $d\pi_s$. Let $C^s_K(\bar{\mathbf{M}})$ be the space of all K finite elements of $C^\infty(\bar{\mathbf{M}})$ for which

$$\phi(w) = (-1)^s \phi(-w) \tag{25}$$

($w \in \bar{\mathbf{M}}$). ($K \simeq SO(2) \times SO(2)$.) $C_K^s(\bar{\mathbf{M}})$ is an $so(2,2)$ module with the representation on $C_K^s(\bar{\mathbf{M}})$ being $d\pi_s$.

If we let $v = (u_{-1}, u_0) \in S^1$ and $u = (u_1, u_2) \in S^1$, then a point in $\overline{\mathbf{M}}$ is determined by

$$u_{-1}^2 + u_0^2 = u_1^2 + u_2^2 = 1. \tag{26}$$

We may introduce spherical coordinates on $\bar{\mathbf{M}}$ as follows:

$$u_{-1} = \cos\tau, \qquad u_0 = \sin\tau, \qquad u_1 = \sin\rho, \qquad u_2 = \cos\rho \tag{27}$$

with the ranges of the angular parameters being

$$0 < \tau < 2\pi, \qquad 0 < \rho < 2\pi\,.$$

A basis for $C_K^0(\bar{\mathbf{M}})$ is given by the $K-$finite functions

$$\phi_{n,m}(\tau,\rho) = e^{i(n\tau + m\rho)} \tag{28}$$

with $n, m \;=\; 0, \pm 1, \pm 2, \pm 3, \ldots$, and with $(-1)^{(n+m)} = 1$. We let H_K ($\subset C_K^0(\bar{\mathbf{M}})$) be the linear span of the set of all $\phi_{n,m}(\tau,\rho)$ for which

$$\Big(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\rho^2}\Big)\phi_{n,m}(\tau,\rho) \;=\; 0\,, \tag{29}$$

which leads to the spectral equation

$$n^2 - m^2 = 0\,. \tag{30}$$

H_K is an $so(2,2)$ (and therefore $so(1,2)$) invariant subspace of $C_K^0(\bar{\mathbf{M}})$, and we have the decomposition

$$H_K = (H_K^+ \oplus H_K^-) \oplus H_0 \oplus (H_K^{+'} \oplus H_K^{-'}), \tag{31}$$

where H_K^+ and H_K^- are the spaces spanned by the basis elements (28) for which $n \;=\; m$ $(m > 0)$ and $n \;=\; -m$ $(m < 0)$, respectively, and $H_K^{+'}$ and $H_K^{-'}$ are the spaces spanned by the basis elements (28) for which $-n \;=\; m$ $(m > 0)$ and $-n \;=\; -m$ $(m < 0)$, respectively, and, finally H_0 is the one-dimensional space spanned by the constant function, which is given by (28) with $n = m = 0$.

In Ref. [13], we have explicitly proved that the various factors of the decomposition, Eq. (31), for the various cases given above are, except for the one-dimensional representation on H_0, the essential extensions of the holomorphic discrete series representations $d\pi_+^{1,\epsilon=0}$ and $d\pi_-^{-1,\epsilon=0}$ of $U_q(so(1,2))$ (with $q = 1$) described in Section 1. For example, for case of $n = m > 0$, the vector $|m> \in \mathcal{H}_\epsilon$ in Section 1 corresponds to the function $\phi_{m,m}(\tau,\rho)$, *i.e.*

$$\phi_{m,m}(\tau,\rho) = e^{im(\tau+\rho)} = (|\tau,\rho>, |m>) \tag{32}$$

where $(\ \cdot\ ,\ \cdot\ )$ is, for $m \neq 0$, the inner product for the space $V_+^{0,0}$ defined in Section 1.

The Einstein energy is just the generator of rotations of the S^1 on $\bar{\mathbf{M}}$, and it is

$$E = i\frac{\partial}{\partial \tau}\,. \tag{33}$$

The completion of each one of the four infinite dimensional factors in (31) with respect to their discrete series norms are representation spaces for irreducible, positive and negative energy representations of $SO_0(2,2)$ which describe massless particles (and antiparticles) on $\bar{\mathbf{M}}$, where positive (resp. negative) energy means

$$(\psi, E\psi) > 0 \ \ (\text{resp.}\ \ (\psi, E\psi) < 0) \tag{34}$$

for ψ an arbitrary vector in the representation space. The $H_K^{\pm}$ are positive energy representations and the $H_K^{\pm'}$ are negative energy representations of $SO_0(2,2)$.

2.2 Geometrical description of the representations at roots of unity

The model for realizing representations of $U_q(so(1,2))$ at roots of unity consists of replacing the spatial part S^1 of the two-dimensional Einstein universe, $\bar{\mathbf{M}}$, by a lattice of N points symmetrically placed along S^1. We call this lattice of points the discrete circle.

To obtain a finite approximation on the discrete circle of a massless scalar field in $\bar{\mathbf{M}}$ we consider N particles each of mass m occupying positions at the N lattice sites of the discrete circle introduced above, and assume motion of the lattice points produced by a linear restoring force (Hooke's law) between the nearest neighbors. This is an exactly solvable model for which the linear wave equation in $\bar{\mathbf{M}}$ is obtained upon taking the limit $N \to \infty$ (see Ref. [13]).

Let $t = \frac{R}{c}\tau$ and let $\phi_i(t)$ be the angular displacement of the i-th particle of mass m from its equilibrium position. Then, it is an easy exercise in elementary physics to show that the equation of motion for the ith mass point is

$$m\ddot{\phi}_i - k\phi_{i+1} \ + \ 2k\phi_i - k\phi_{i-1} = 0 \qquad (i = 1, 2, \ldots, N)\,. \tag{35}$$

Define the adjacency operator A by

$$A\phi_i := \phi_{i+1} + \phi_{i-1}\,, \tag{36}$$

and also define the operators δ_q and Δ_q by

$$\delta_q := (N-1)^2(A - 2\,I)\,, \tag{37}$$

and

$$\Delta_q := \frac{\delta_q}{2(N-1)^2}\,. \tag{38}$$

Then we can rewrite (35) as the equation

$$m\ddot{\phi}_i(t) = k(A - 2\,I)\phi_i(t) = \frac{k}{(N-1)^2}\delta_q\phi_i(t)\,. \tag{39}$$

We now proceed to solve these N differential-difference equations for the N unknowns $\phi_i(t)$. We use separation of variables [14] and we let

$$\phi_i(t) = e^{i\omega\ t} f(i)\,. \tag{40}$$

Then substitution of (40) into (39) gives the eigenvalue problem

$$\frac{-m(N-1)^2\omega^2}{k} f(j) = \delta_q f(j)\,. \tag{41}$$

which is equivalent to the following system of N linear equations in the N unknowns $f(i)$:

$$\begin{aligned}
&-m\omega^2 f(1) + 2kf(1) - kf(2) - kf(N-1) = 0\\
&-m\omega^2 f(2) + 2kf(2) - kf(3) - kf(1) = 0\\
&-m\omega^2 f(3) + 2kf(3) - kf(4) - kf(2) = 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&-m\omega^2 f(N-1) + kf(N-1) - kf(N) - kf(N-2) = 0\\
&-m\omega^2 f(N) + 2kf(N) - kf_1 - kf(N-1) = 0\,.
\end{aligned} \tag{42}$$

The allowed frequencies (or eigenvalues) are:

$$\omega_a = 2\sqrt{\frac{k}{m}}\sin\big(\frac{a\pi}{N-1}\big) \qquad (a = 0, 1, 2, \ldots, N-2). \tag{43}$$

For each a the eigenvector is given by

$$f_a(j) = e^{ij\frac{2\pi a}{(N-1)}}\,. \tag{44}$$

It is shown in Ref. [13] that in the limit $N \to \infty$ the system of equations, Eqs (42), goes over into the wave equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial s^2} = 0, \tag{45}$$

where $\phi = \phi(t,s) = \lim_{N\to\infty}\phi(t,s_i)$ with $\phi(t,s_i) = \phi_i(t)$, and for the wave speed we have

$$c \ = \ \lim_{N\to\infty}\frac{2\pi R}{N-1}\sqrt{\frac{k}{m}}\,. \tag{46}$$

Notice that for $\phi_a(t,s_i) := e^{i\omega_a t} f_a(i)$ we have with $n^2 = m^2 = a^2$ and $n > 0$, $m < 0$:

$$\lim_{N\to\infty} \phi_a(t,s_i) = e^{i(n\tau - m\rho)}, \tag{47}$$

where the $e^{i(n\tau-m\rho)}$ are the $K-$finite function introduced in (28). Also we have as $N \to \infty$

$$\omega_a \sim 2\sqrt{\frac{k}{m}}\frac{a\pi}{N-1} \sim \frac{c}{R}a\,. \tag{48}$$

We verify the orthogonality and completeness relations for the $f_a(j) \in \mathcal{L}^2(\mathbb{Z}/(\mathrm{N}-1)\mathbb{Z})$:

$$\sum_{j=1}^{N-1} \overline{f_a(j)} f_b(j) = \delta_a(b) := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}, \tag{49}$$

and

$$\sum_{a=1}^{N-1} \overline{f_a(j)} f_a(j') = \delta_j(j')\,. \tag{50}$$

Now we realize on $\mathcal{L}^2(\mathbb{Z}/(\mathrm{N}-1)\mathbb{Z})$ the finite dimensional representation $d\pi^\sigma$ ($\sigma = -1$) of $U_q(so(1,2))$ which is defined in Section 1. Let $q = e^{\frac{2\pi i}{N-1}}$ be an $(N-1)$th root of unity, then we have

$$\frac{(N-1)}{\pi}\sin(\frac{\pi a}{N-1}) = [a]_q \cdot \frac{\sin(\frac{\pi}{N-1})}{(\frac{\pi}{N-1})}\,. \tag{51}$$

Let $M = N - 1 \in \mathbb{Z}$. With $s_3 = -a$ we let $|s_3>$ in Section 1 (c.f. Eq. (8)) correspond to the function $\phi_a(t,s_i) = e^{i\omega_a} f_a(i)$. Thus

$$s_3 = -1,-2,-3,\ldots,-(N-1) \quad\Longleftrightarrow\quad a = 1,2,3,\ldots,N-1\,. \tag{52}$$

We define

$$H := \frac{2(N-1)}{\pi}\arcsin\left(\sqrt{-\Delta_q}\right), \tag{53}$$

where $\sqrt{-\Delta_q}$ is the unique positive square root of the Hermitian operator $-\Delta_q$ and by $\arcsin$ we mean the principal value. Further define operators X^+ and X^- on $\mathcal{L}^2(\mathbb{Z}/(\mathrm{N}-1)\mathbb{Z})$ by their following action on the basis $|s_3>$:

$$X^\pm|s_3> = [-\sigma \pm s_3]_q|s_3 \pm 1>\ . \tag{54}$$

Then with these definitions it is trivial to verify that Eqs. (10) and (11) are satisfied. In particular

$$H|s_3> = -2s_3|s_3>\ . \tag{55}$$

We have thus provided a realization of $d\pi^\sigma$ on $\mathcal{L}^2(\mathbb{Z}/(\mathrm{N}-1)\mathbb{Z})$. With this realization it is easy to see that as $N \to \infty$ this representation of $U_q(so(1,2))$

goes over into the realization of the holomorphic discrete series representation $d\pi_+^{+1,0}$ of $so(1,2)$ given in Section 1. The details are given in Ref. [13]. We may give a similar construction for the lowest weight representation $d\pi^{\sigma'}$ (c.f. [3, 4]) and to show that it goes over into the corresponding realization of the antiholomorphic discrete series representation $d\pi_-^{+1,0}$ of $so(1,2)$ described in Section 1.

3 Applications to the Casimir Effect

We start with a real massless scalar field $\phi(t,s)$ on $\overline{\mathbf{M}}$ and consider the interval which is an arc of the circle S^1 (the spatial part of the Einstein universe) satisfying $0 \leq s \leq L$. We shall impose Dirichlet boundary conditions on this region

$$\phi(t,0) = \phi(t,L) = 0\,. \tag{56}$$

We assume that the circumference C of the circle is commensurate with L, *i.e.* $C = ML$ and that M is an odd whole number.

The standard quantization of the field ϕ is performed by means of the expansion

$$\phi(t,s) = \sum_{n=0}^{\infty} \Big\{ a_n \phi_n^-(t,s) + a_n^\dagger \phi_n^+(t,s) \Big\}, \tag{57}$$

where now $\phi(t,s)$ is an operator valued distribution which for fixed $(t,s) \in \overline{\mathbf{M}}$ acts on the Fock space as a densely defined self-adjoint operator. The quantities a_n and $a_n^\dagger$ satisfy the commutation relations

$$[a_n, a_m^\dagger] = \delta_{n,m} \tag{58}$$

and

$$[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0\,. \tag{59}$$

The energy operator is given by

$$H = \sum_{n=0}^{\infty} \hbar\omega_n \Big[a_n^\dagger a_n + \frac{1}{2} \Big] \cdot I, \tag{60}$$

where I is the identity operator on Fock space. H is formally self adjoint. The vacuum state is defined by the equation

$$a_n|n >= 0\,. \tag{61}$$

Thus the total vacuum or zero point energy is

$$E = \frac{1}{2}\hbar \sum_{n=0}^{\infty} \omega_n\,. \tag{62}$$

Now the equation

$$S^1 = [0, L] \cup (S^1 \backslash [0, L]) \tag{63}$$

leads to the following interpretation: the total zero point energy is the sum of two parts. The first part is E', which is the energy associated with the interval $[0, L]$. The second part, E'', is the energy associated with $S^1 \backslash [0, L]$, the complement of $[0, L]$ in S^1. Thus the Casimir energy is

$$E_{tot} = E' + E'' \,, \tag{64}$$

where each of the terms E' and E'' are given by divergent sums similar to Eq. (62).

In the presence of the boundary conditions the fields on each side of the boundaries decouple from each other, and we can think of the space H_K splitting into the direct sum of two identical copies of itself

$$H'_K \oplus H''_K \,, \tag{65}$$

and similarly there are two independent Fock spaces with quantized fields $\phi'(t, s)$ and $\phi''(t, s)$ satisfying boundary conditions:

$$\phi'(t, 0) = \phi'(t, L) = 0 \tag{66}$$

$$\phi''(t, 0) = \phi''(t, C - L) = 0 \,. \tag{67}$$

The boundary conditions provide constraints on the fields $\phi'(t, s)$ and $\phi''(t, s)$, so they are not completely independent of one another, *i.e.* both fields $\phi'(t, s)$ and $\phi''(t, s)$ must go to the zero operator continuously in the operator topology as $s \to 0$, and as $s \to L$ or $s \to C - L$ for $\phi'(t, s)$ or $\phi''(t, s)$ respectively.

Our regularization scheme uses the approximation of massless scalar fields on $\overline{\mathbf{M}}$ by representation spaces of $U_q(so(1, 2))$ for q a root of unity that was described in the previous section. Let N be an even positive integer and q an $(N - 1)$-th root of unity, *i.e.* $q = \exp[2\pi i/(N - 1)]$. For the discrete circle and we have

$$s_j \in [0, L], \quad s_j = R\theta_j, \quad \theta_j = \frac{2\pi j}{N - 1} \qquad (j = 1, 2, \ldots, N - 1). \tag{68}$$

We have

$$L = \frac{C}{M} = \frac{2\pi R}{M} \Rightarrow \frac{2\pi R\tilde{j}}{N - 1} = \frac{2\pi R}{M} \Rightarrow M\tilde{j} = N - 1 \tag{69}$$

so $N - 1$ is a multiple of M and

$$L = R\theta_{\tilde{j}} = \frac{2\pi R\tilde{j}}{N - 1} \,. \tag{70}$$

The boundary condition at zero implies that $\phi'(t,x)$ is an odd function of x, *i.e.* [13, 15]

$$\phi'(t,x) = e^{i\omega_n\, t}\sin(\frac{2\pi n x}{2\pi R})\,, \tag{71}$$

and the boundary condition at the other end on the interval $[0,L]$ gives [13, 15]

$$\phi'(t,L) = 0 \Rightarrow \sin\left(\frac{2\pi n L}{C}\right) = 0 \tag{72}$$

so that n must be a multiple of M/2.

On the finite circle equations such as (62) become finite sums (where ∞ becomes replaced by $N-1$) and using the above results about the allowed integer values we can perform the finite sums to obtain a closed form for our regularization of the total zero point energy, Eq. (64): [13]

$$E^{reg}_{tot} = \frac{(N-1)\hbar c}{2\pi R}\left\{\cot\left(\frac{\pi M}{4(N-1)}\right) + \cot\left(\frac{\pi M}{4(N-1)(M-1)}\right)\right\} \tag{73}$$

In order to extract the finite part of Eq. (73) in the limit $N \to \infty$ we now use the following expansion of $\cot x$, valid for real x, such that $x^2 < \pi^2$:

$$\cot x = \frac{1}{x} + 2\sum_{p=1}^{\infty}(-1)^p\frac{B_{2p}}{(2p)!}(2x)^{2p-1}, \tag{74}$$

where B_{2p} are Bernoulli numbers. We obtain

$$E^{reg}_{tot} = \frac{4(N-1)^2\hbar c}{2\pi^2 R} - \frac{\hbar c M B_2}{4\,R}\left(1 + \frac{1}{M-1}\right) + o(\frac{1}{N-1})\,, \tag{75}$$

where $o(\frac{1}{N-1})$ means terms with power $\frac{1}{N-1}$ or smaller. Now we use $M = \frac{2\pi R}{L}$ and $B_2 = \frac{1}{6}$ and obtain for the term involving L and not depending upon N

$$E(L) = -\frac{\pi\hbar c}{12L}\left(1 + \frac{L}{2\pi R - L}\right). \tag{76}$$

This agrees with the known results [15].

4 Conclusions and Further Developments

It would be interesting to develop an analogous treatment for our regularization scheme in the higher dimensional cases based on higher dimensional quantum symmetries. This probably will require detailed and explicit knowledge about the q deformations of massless representations of $SO_0(2,n-1)^\sim$. For the physically important case of $n = 4$ this is what we have done in Refs. [16, 17]: namely, an explicit constructions of the q deformations of $U_q(so(2,3))$ massless representations for generic $|q| = 1$ and for q a root of unity. We also mention the possibility of a role for the matrix quantum group $SO_q(r,s)$ dual to $U_q(so(r,s))$ and noncommutative space-time in our regularization scheme for the Casimir effect. For this purpose it would be interesting to have an explicit construction of the induced representations of $SO_q(2,3)$ (work with V.K. Dobrev, *in progress*).

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