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Higher Dimensional Vertex Algebras and Rational Conformal Field Theory Models

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Abstract. The notion of *global conformal invariance* (GCI) in Minkowski space allows to prove rationality of correlation functions and to extend the concept of vertex algebra to any number D of space-time dimensions. The case of even D, which includes a conformal stress-energy tensor with a rational 3-point function, is of particular interest. Recent progress, reviewed in the talk, includes a full account of Wightman positivity at the 4-point level for D=4, and a study of modular properties of thermal expectation values of the conformal energy operator.

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1 Introduction

Invariance of Wightman functions in quantum field theory under finite conformal transformations in Minkowski space has far reaching implications: local fields commute for non-light-like separations and have, as a result, rational correlation functions. The theory is reformulated in a complex realization of compactified Minkowski space, in which the forward tube is mapped inside the unit ball, and yields a higher dimensional extension of the notion of a vertex algebra. Thermal correlators are shown to be elliptic functions of the conformal time variable. Energy mean values in a Gibbs state for free massless fields are expressed in terms of modular forms. For large compactification radii the energy density approaches the Minkowski space thermodynamic limit, reproducing the Stefan–Boltzmann law.

The talk is based on work of Nikolay M. Nikolov, Karl-Henning Rehren, Yassen S. Stanev and the authors [1–8]. The present outline is chiefly meant as a general introduction to the subject.

1.1 Why should one care at all for conformally invariant quantum field theory models?

Every once in a while since the discovery (by Cunningham and Bateman in 1910) of the conformal invariance of (vacuum) Maxwell's electrodynamics the conformal group is attracting the attention of mathematical physicists by its elusive beauty. The "real world" is certainly not conformally invariant: the presence of discrete positive masses of atoms and elementary particles signals violation of even the weaker *dilation symmetry*. One is entitled to ask: does the invariance of free massless equations provide enough rationale to care for conformally invariant quantum field theory (QFT)? Let me cite two reasons why the study of conformal models may still be of interest.

The first is a negative one: the relativistic quantum theory of the real world has proven too difficult for us. The first attempts to formulate QFT date over three quarter of a century ago, but in spite of vigorous efforts by theoretical – and mathematical – physicists (as we now distinguish between the two brands) we still have no mathematically established interacting QFT in four (or higher) space-time dimensions. This justifies the study of not entirely realistic QFT models – either in low space-time dimensions or having a higher symmetry (or both).

A positive argument affirms that conformal invariance may be a meaningful approximation to a realistic QFT at very short distances (or large energies and transferred momenta) when particle masses can be neglected. A QFT with a (ultraviolet stable) renormalization group fixed point has to be dilation invariant at that point. On the other hand, a dilation invariant QFT with a stress-energy tensor is expected to be (under reasonable assumptions) also conformally invariant. (A recently discussed counterexample [9] – which violates those assumptions – also violates Wightman positivity and thus looks rather pathological.) Progress in physics often needs idealizations: without neglecting friction Galileo could not have discovered the law of inertia.

Two-dimensional (2D) conformal field theory (CFT) not only provides a rich family of soluble QFT models and thereby of universality classes of 2D critical phenomena (and string vacua), [10], it also gives rise to a new fruitful mathematical concept, the notion of a *(chiral) vertex algebra* which naturally incorporates an important class of infinite dimensional Lie algebras, [11–15]. Recently, a (not fully understood) intriguing relation was discovered between Zhu's vertex algebra approach [13] to rational CFT and Haag's [16] von Neumann algebra framework applied to the classification of chiral CFT models [17–20]. It is all the more interesting that the concept of a vertex algebra – and the associated modular properties of thermal energy mean values – appear to admit a higher dimensional generalization, [1, 6, 7].

The paper is organized as follows. In the rest of this introduction (Subsection 1.2) some background material and early developments (of [1, 2, 21]), are sketched, including a complex realization of compactified Minkowski space which prepares the ground for a higher dimensional extension of the notion of vertex algebra [6]. Section 2 reviews the construction of 4-point functions of scalar fields [3,4] (providing some new formulae for the $d = 3$ case). Section 3 outlines the explicit construction of conformal partial wave expansions and its application to the study of Wightman positivity [8]. The final Section 4 gives a bird's-eye view of equilibrium states in a GCI QFT, including the appearance of elliptic thermal correlation functions and an application of modular invariance to the Gibbs energy-mean-value.

1.2 Background and early results

The concept of conformal invariance in Minkowski space involves a subtlety absent in the Euclidean formulation of the theory. While the spinorial Euclidean conformal group, $Spin(5, 1)$, is simply connected and so is the (one-point) conformal compactification, $\mathbb{S}^4$, of the Euclidean 4-space, the corresponding Minkowski space group, $\mathcal{C} = Spin(4, 2) = SU(2, 2)$, and compactified space-time, $\bar{M} = \mathbb{S}^3 \times \mathbb{S}^1/\mathbb{Z}_2$, are not: in fact, each has an infinite sheeted universal cover. It follows that infinitesimal conformal transformations of Euclidean Green's functions can be integrated and invariance under such transformations is thus equivalent to a global invariance property (see Section 1.1 of [7]). This implies, in turn, that the corresponding Minkowski space QFT can be continued to the (simply connected) universal cover of $\bar{M}$, the cylinder space $\tilde{M} = \mathbb{S}^3 \times \mathbb{R}$, and is invariant under the universal cover $\tilde{\mathcal{C}}$ of the conformal group [21]. The projection of this conformal group action on (compactified) Minkowski space itself is, in general, multivalued. The condition of global conformal invariance (GCI) in Minkowski space introduced in [2] is, therefore, stronger as it allows to continue the Wightman functions to invariant distributions on $\bar{M}$. It turns out that this rather natural condition has far reaching implications. Combined with locality GCI on M yields the *Huygens principle* – the vanishing of (observable, Bose) field commutators for non-light-like separations. Together with Wightman axioms [22] this implies rationality of correlation functions (Theorem 3.1 of Ref. [2]).(It has been noted long ago that such a condition can be realized in the context of (generalized) free fields – see, *e.g.* [23].)

Mentioning *observable* local fields is not accidental. One cannot expect that, *e.g.* gauge dependent quantities satisfy even (the weaker) infinitesimal conformal invariance. It is therefore important to state which fields are assumed to be observable. The stress energy tensor T is an obvious candidate for such a role [24, 25]. The general form of the 3-point function of T [26] implies that T can only be assumed GCI in even dimensional space time – in line with the validity domain of the classical Huygens principle. For the sake of definiteness (and to stay closer to reality) we shall restrict attention in what follows to the $D = 4$

case. We have proposed recently a model in which the observable algebra is generated by a GCI Lagrangian density $\mathcal{L}$ of scale dimension $D(=4)$ [4] (and contains an infinite ladder of conserved tensor fields starting with T).

Rationality of Wightman functions signals the possibility of an algebraic formulation of the theory – in the spirit of chiral vertex algebras (or "meromorphic CFT" in the terminology of the physicist oriented early survey [12]). This is made easier by introducing appropriate complex parametrization of compactified Minkowski space and of the *future tube* – the analyticity domain of vector valued functions of the form $\phi(z)|0\rangle$ for any local field ϕ (the counterpart of the unit circle and the unit disk in the 1D chiral case). To describe it we perform a complex conformal transformation from the Cartesian Minkowski space coordinates $x = (x^0, \mathbf{x})$ to the complex 4-vector $z = (\mathbf{z}, z_4) = z(x)$ [1,2]

$$\mathbf{z} = \frac{\mathbf{x}}{\omega(x)}, \quad z_4 = \frac{1-x^2}{2\omega(x)}, \quad 2\omega(x) = 1 + x^2 - 2ix^0, \tag{1}$$

$$z^2 = \mathbf{z}^2 + z_4^2 = \frac{\bar{\omega}}{\omega}, \quad x^2 = \mathbf{x}^2 - (x^0)^2 (= \frac{1+z^2-2z_4}{1+z^2+2z_4}). \tag{2}$$

In the z coordinates the image T_+ of the future tube is the connected component of the complement of compactified Minkowski space

$$\bar{M} = \{z \in \mathbb{C}^4\,;\ z = \frac{\bar{z}}{\bar{z}^2} = e^{2\pi i\zeta} u\, \zeta \in \mathbb{R}\,,\ u \in \mathbb{R}^4, z^2 = \sum_{\alpha=1}^{4} (z_\alpha)^2 = e^{4\pi i\zeta}\} \tag{3}$$

($\bar{M} = \mathbb{S}^3 \times \mathbb{S}^1/\mathbb{Z}_2$) containing the origin

$$T_+ = (z \in \mathbb{C}^4; |z^2| < 1, |z|^2 = \sum_{i=1}^{4} |z_i|^2 < \frac{1+|z^2|^2}{2}). \tag{4}$$

Fields $\phi(z)$ are then defined as formal power series of the form

$$\phi(z) = \sum_{n\in\mathbb{Z}} \sum_{m\geq 0} (z^2)^n \phi_{nm}(z)\,, \tag{5}$$

$\phi_{nm}(z)$ being an operator valued polynomial in z that is homogeneous of degree m and harmonic. The Huygens principle admits an algebraic formulation in terms of such formal power series. If ϕ is a GCI irreducible spin-tensor of dimension d and $SU(2) \times SU(2)$ weight $(j_1, j_2; 2j_{1,2} = 0, 1, ...)$, then the strong locality condition reads

$$(z_{12}^2)^n (\phi(z_1)\phi^*(z_2) - (-1)^{2j_1+2j_2} \phi^*(z_2)\phi(z_1)) = 0, \tag{6}$$

$$for\, n \geq d + j_1 + j_2 (\in \mathbb{N}),\ z_{12} = z_1 - z_2. \tag{7}$$

We assume that the field algebra is spanned by *conformal* fields, transforming homogeneously under infinitesimal conformal transformations. In particular, under commutation with the *conformal Hamiltonian* H, the generator of the centre of the maximal compact subgroup $U(1) \times Spin(4)$ of $\mathcal{C}$ (whose significance has been emphasized by Segal [27]), a field ϕ of scale dimension d satisfies

$$[H,\, \phi(z)] = (z\frac{\partial}{\partial z} + d)\phi(z)\,. \tag{8}$$

H is related to the Minkowski space energy P^0 by $2H = P^0 + wP^0w^{-1}$, where w is the *Weyl inversion* – the proper conformal transformation that changes the sign of z (cf. [27]); it follows that the conformal energy H is positive whenever the Minkowski space one is. Energy positivity implies analyticity of the vector-valued function $\phi(z)|0\rangle$ in T_+, and hence the vanishing of $\phi_{nm}|0\rangle$ for negative n. Covariance of local fields under (complex) translations T_α allows to formulate the *state-field correspondence* as follows: to each finite-energy state v corresponds a unique local field (or *vertex operator*) $Y(v,z)$), such that

$$Y(v,0)|0\rangle = v\,,\; [T_\alpha,\, Y(v,z)] = \frac{\partial}{\partial z}Y(v,z). \tag{9}$$

2 General Rational 4-Point Functions of Scalar Fields

The fact that GCI implies rationality of correlation functions provides a powerful tool for explicit construction of Wightman functions.

Given the conformally invariant 2-point function of a scalar field of dimension d,

$$\langle 12\rangle (= \langle 12\rangle_d) = N(z_{12}^2)^{-d}\,, \quad N = N(d) > 0, \tag{10}$$

we can write its most general rational conformal 4-point function as

$$\langle 1234\rangle \;=\; \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle +\; p(z_{ij}^2)F(s,t), \tag{11}$$

where s and t are the conformally invariant cross ratios

$$s = \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2},\; t = \frac{z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2}; \tag{12}$$

the prefactor $p(\rho_{ij})$ is a monomial in its six arguments and their inverses homogeneous of degree $-d$ in $\rho_{ij} = \rho_{ji}$ for each fixed i and $j = 1,...,4, j \neq i$; the invariant amplitude F is a polynomial in s, t, s^{-1}, t^{-1} of degree restricted by Wightman positivity. For the cases $d = 2, 3, 4$ of interest the truncated 4-point function $p.F$, obeying locality and conformal invariance, depends on $2d - 3$ parameters. Using the analysis of [3–5, 8] we can choose the prefactor p and the amplitude F in these three cases as

$$d = 2 : p(\rho_{ij}) = (\rho_{12}\rho_{23}\rho_{34}\rho_{14})^{-1},\; F(s,t) = c(1+s+t);$$
$$d = 3 : p(\rho_{ij}) = (\Pi_{i<j}\rho_{ij})^{-1},\; F(s,t) = \sum_{i=0,1} a_i \tilde{J}_i \;+ b;$$
$$d = 4 : p(\rho_{ij}) = (\rho_{12}\rho_{23}\rho_{34}\rho_{14})^{-2},$$
$$stF(s,t) = \sum_{i=0}^{2} a_i J_i(s,t) \;+ st(bD(s,t) + b'Q(s,t))\,. \tag{13}$$

Here $J_i(s,t)$ are polynomials of overall degree 5 in their arguments (given in [5,8]), $D(s,t)$ is a second degree polynomial (given in (22) below), $Q(s,t) = s+t+st$;

$$\tilde{J}_0 \;=\; s + t + \frac{t+1}{s} + \frac{s+1}{t}\;,$$
$$\tilde{J}_1 \;=\; \frac{(1-t)(1-t^2)}{st} - t^{-1} - t - s(1+t^{-1}) + \frac{s^2}{t}\;, \tag{14}$$

are the symmetrized twist two contributions to the 4-point function which are computed as follows. We organize, following [4], the *operator product expansion* (OPE) of two hermitian scalar fields of (integer) dimension d in terms of (mutually orthogonal) bilocal fields $V_\kappa(z_1, z_2)$ of twist 2κ:

$$\phi(z_1)\phi(z_2) = \langle 12\rangle + \sum_{\kappa>0}(z_{12}^2)^{\kappa-d} V_\kappa(z_1,z_2). \tag{15}$$

The first of them, V_1 can be expanded in an infinite series of even-rank conserved symmetric traceless tensors and, as a consequence, is harmonic in each argument (see Proposition 2.1 of [4]). This allows to compute the 4-point function of two V_1 as a finite linear combination of a standard basis of GCI solutions:

$$z_{13}^2 z_{24}^2\langle 0|V_1(z_1,z_2)V_1(z_3,z_4)|0\rangle = \sum_{\nu=0}^{d-1} a_\nu j_\nu(s,t), j_0 = 1 + t^{-1}, ... \tag{16}$$

(We shall display the general form of j_ν in Section 3, below.) Then $\tilde{J}_\nu$ and J_ν are crossing symmetric expressions satisfying

$$\frac{s}{t}\tilde{J}_\nu(s,t) - j_\nu(s,t) = O(s),\; t^{-3}J_\nu(s,t) - j_\nu(s,t) = O(s) \tag{17}$$

(see [5] Section 5.2).

Here is also an example of a mixed 4-point function of a pair of scalar fields $t(z)$ and $W(z)$ of dimensions 2 and 3, respectively

$$\langle 0|t(z_1)W(z_2)W(z_3)t(z_4)|0\rangle - \frac{N}{(z_{14}^2)^2(z_{23}^2)^3} = \frac{c(1+s)+c't}{z_{12}^2 z_{13}^2 z_{23}^2 z_{24}^2 z_{34}^2 t}\,. \tag{18}$$

The 1D reduction of Eqs. (13) for $d = 2$ and $d = 3$ and (18) is an extension of the 1985 Zamolodchikov's W_3 model (for a review see [28]). This model would be recovered if there were a single field of dimension four in the $1D$ restriction (which would then necessarily coincide with the normal square of $t(z)$ and yield $a_0 = c$, $6a_1 + b = \frac{16c(2-c)}{5c+22}$, $N = \frac{c^2}{6}$, $c' = 2c$). This is not the case, however, since the 4D stress-energy tensor will also appear as a chiral field of dimension $d = 4$ in the 1D restriction.

In the simplest non-trivial case, $d = 2$ the truncated 4-point function involves a single parameter c which can be defined by the invariant under rescaling of the basic field: $8\langle 12\rangle\langle 23\rangle\langle 13\rangle = c(\langle 123\rangle)^2$. The following result was established in [3].

Theorem 2.1. *The scalar GCI field $t(z)$ of dimension $d = 2$ with 2-point function given by (10) with $N = \frac{c}{2}$ obeys an OPE of the form (15) with a single singular term $V_1 \equiv V$ in the sum,*

$$t(z_1)t(z_2) = \langle 12\rangle + \frac{V(z_1, z_2)}{z_{12}^2} + :t(z_1)t(z_2):, \; V(z, z) = 2t(z). \qquad (19)$$

Wightman positivity implies that V generates a unitary vacuum representation of a central extension of the infinite dimensional real symplectic Lie algebra with positive integral central charge c. As a corollary t can be presented as (half) the sum of normal squares of c free commuting massless scalar fields.

The *proof* involves two essential steps. First, one finds the $2n$-point functions of V as sums of 1-loop graphs (see Section 2 of [3] and Appendix A.1 of [4]) and derives on this basis the commutation relations of the (extended) symplectic algebra for the V's. Secondly, one obtains (Section 5.1 of [3]) an analogue of Kac's determinant formula [29] for this infinite dimensional Lie algebra.

Remark 2.1. In a 1982 paper [30](which contains an early proof of the fact that the Huygens principle implies rationality) Baumann has proven that all massless scalar fields with a trivial S-matrix are Wick polynomials of free fields. It is, in fact, clear that the state-field correspondence and the presence of (zero-mass) asymptotically complete set of particle states implies that the field algebra is generated by free local fields.[1] Thus a CFT (even if it only obeys infinitesimal conformal invariance) can never have a (nontrivial) scattering theory. One should therefore use a more subtle criterion to distinguish nontrivial conformal models which are known to exist (at least in two space-time dimensions).

We view the case $d = 4$ [4, 6] corresponding to a (gauge invariant) Lagrangian density, whose dimension is expected to be protected, as the most promising one for providing a non-trivial GCI model. As discussed in [6] (see, in particular, Eq. (1.10) there, at the end of the Introduction) the systematic study of this theory also requires the knowledge of a system of scalar fields of lower dimensions $(d = 2, 3)$.

[1] I owe this remark to Nikolay Nikolov.

3 Conformal Partial Wave Expansion: a Tool for Verifying Wightman Positivity

A powerful tool in studying Wightman positivity is provided by operator product expansions [31, 32] that yield *conformal partial wave expansions* [33]. The positive definite 4-point function of a pair of hermitian scalar fields A and B of dimensions d and $d+\delta$, respectively, admits a conformal partial wave expansion of the form

$$\begin{aligned}\langle 0|A(z_1)B(z_2)B(z_3)A(z_4)|0\rangle &= \sum_{\kappa L}\langle 0|A(z_1)B(z_2)\Pi_{\kappa L}B(z_3)A(z_4)|0\rangle \\ &= [(z_{12})^2(z_{34})^2]^{-d}[(z_{23})^2]^{-\delta}\sum_{\kappa L}B_{\kappa L}\beta^{\delta}_{\kappa L}(s,t) \quad (20)\end{aligned}$$

where $\Pi_{\kappa L}$ is the projection on the positive energy irreducible representation (IR) of the conformal group of $u(1)\times su(2)\times su(2)$ weight $(2\kappa+L, \frac{1}{2}L, \frac{1}{2}L)$; the *conformal partial waves* $\beta_{\kappa L}$ are universal functions, only depending on the above IR. Following [34] they can be expressed in terms of hypergeometric functions by using a higher dimensional analogue u, v of the 2D chiral coordinates:

$$\begin{aligned}(u-v)\beta_{\kappa L} &= uv(G^{\delta}_{\kappa+L-\frac{\delta}{2}}(u)G^{\delta}_{\kappa-1-\frac{\delta}{2}}(v) - (u\leftrightarrow v)) \\ G^{\delta}_{\nu}(z) &= z^{\nu}F(\nu,\nu;2\nu+\delta;z), \quad (21)\end{aligned}$$

where u and v are related to s and t (12) by

$$s = uv,\ t = (1-u)(1-v),\ (u-v)^2 = (1-t)^2-2s(1+t)+t^2 =: D(s,t). \quad (22)$$

The full dynamical information is carried by the coefficients $B_{\kappa L}$. Wightman positivity for the 4-point function is equivalent to the requirement that all these coefficients are non-negative. The computation of $B_{\kappa L}$ [8] uses the expression for $(u-v)F(s,t)$ (13) as a sum of products of non-negative powers of u or $\frac{u}{1-u}$ and similar monomials in v. For the twist two contributions we have, in particular,

$$(u-v)j_{\nu}(s,t) = f_{\nu+1}(u) - f_{\nu+1}(v),\ f_{\nu}(u) = \frac{u^{\nu}}{(1-u)^{\nu}} - (-1)^{\nu}u^{\nu}. \quad (23)$$

To compare the expressions (13) and (18) with the expansion (20) one uses (special cases of) the identity

$$\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n(\beta)_n}{n!(\gamma+n-1)_n}z^nF(n+\alpha,n+\beta;2n+\gamma;z) = 1. \quad (24)$$

In the most interesting case in which the basic scalar field can be interpreted as a (gauge invariant) Lagrangian $\mathcal{L}(z)$ of $d=4$ (see the last equation (13)) we find for the twist two partial waves

$$\frac{1}{2}B_{1L} = a_0 + L(L+1)a_1 + \frac{1}{4}(L-1)L(L+1)(L+2)a_2. \qquad (25)$$

The (more complicated) expressions of all higher twist amplitudes are written down in (Section 2 of) [6] where it is established that Wightman positivity is satisfied for the closure of a non-empty open set in the five-dimensional parameter space describing the 4-point function. One finds, in particular, that $a_\nu, \nu = 0, 1, 2$ and b' should be non-negative, while if $b' = 0$, then

$$-3a_1 \le b \le \frac{1}{3}(2a_0 + a_1). \qquad (26)$$

As discussed in [6] it is of particular interest to consider the case in which the operator product expansion of $\mathcal{L}(z_1)\mathcal{L}(z_2)$ involves no scalar field of dimension 2 or 4, so that both a_0 and b' vanish and Eq. (26) further simplifies.

4 Thermal States. Modular Properties of Energy Mean Values

The conformal Hamiltonian H satisfying (8) has a discrete (integer or half-integer) spectrum in the vacuum space $\mathcal{V}$ of a GCI theory which is assumed finitely degenerate and such that there exist a partition function $Z(\tau)$ (defined as a trace over the Boltzmann weights in $\mathcal{V}$) and *thermal mean values* $\langle A\rangle_q$ for any product A of local GCI fields

$$Z(\tau)\langle A\rangle_q = tr(Aq^H),\; Z(\tau) = tr(q^H),\; q = e^{2\pi i\tau},\; Im\tau > 0\,(|q| < 1), \quad (27)$$

an assumption verified for (generalized) free fields [7]. Here τ is interpreted as the (complexified) inverse temperature; more precisely,

$$Im\tau = \frac{1}{kT}\,. \qquad (28)$$

In order to reveal the properties of thermal correlation functions it is advantageous to use (real variable)*compact picture fields* $\phi(\zeta, u)$ related to the corresponding analytic (z-picture) vertex operators $\phi(z)$ (of dimension d) by

$$\phi(\zeta, u) = e^{2\pi i d\zeta}\phi(z)\; for\; z = e^{2\pi i\zeta}\,. \qquad (29)$$

A compact picture field is (anti)periodic in ζ depending on its spin

$$\phi(\zeta + 1, u) = (-1)^{2j_1+2j_2}\phi(\zeta, u). \qquad (30)$$

The *Kubo-Martin-Schwinger boundary condition* [35] which says, *e.g.* for the 2-point function of ϕ,

$$w_q(\zeta_1 - \zeta_2 - \tau; u_1, u_2) \equiv \langle\phi(\zeta_1, u_1)\phi^*(\zeta_2, u_2)\rangle_q = \langle\phi^*(\zeta_2, u_2)\phi(\zeta_1, u_1)\rangle_q \qquad (31)$$

implies that the function $w_q(\zeta, u_1, u_2)$ has a second period, $\tau + 1$, related to the inverse temperature, on top of the period 1 (or 2, for Fermi fields). For a scalar field w_q depends on u_1, u_2 through their scalar product, $u_1 u_2 = cos(2\pi\alpha)$. In particular, for the free massless field φ we find

$$w_q(\zeta, \tau; \alpha) = \sum_{n \in \mathbb{Z}} w_0(\zeta + n\tau, \alpha), \tag{32}$$

where the vacuum 2-point function of φ is given by

$$\begin{aligned} w_0(\zeta, \alpha) \;&=\; \frac{-1}{4sin(\pi(\zeta+\alpha))sin(\pi(\zeta-\alpha))} \\ &=\; \frac{1}{4sin 2\pi\alpha}(cot(\pi(\zeta+\alpha)) - cot(\pi(\zeta-\alpha)))\,. \end{aligned} \tag{33}$$

On the other hand, for a suitable choice of the vacuum energy E_0, the thermal mean value in the theory of a free massless scalar field is given by the unique weight-two (normalized) modular form:

$$\langle H + E_0 \rangle_q = G_4(\tau) = -\frac{B_4}{8} + \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, E_0 = -\frac{B_4}{8} = \frac{1}{240} \tag{34}$$

(B_n being the Bernoulli numbers). Note that, restoring the energy units, $H + E_0 \mapsto \hbar\nu(H + E_0)$, it gives for $q = exp(-\frac{\hbar\nu}{kT})$ Planck's black-body energy distribution of the harmonic oscillator (with energy eigenvalues $n\hbar\nu$). Invariance under the modular inversion, $\tau^{-4} G_4(\frac{-1}{\tau}) = G_4(\tau)$, allows to compute the high temperature expansion of $\langle H \rangle_q$ in terms of its low temperature (small q) behavior. If we replace the unit 3-sphere in (3) by a sphere of radius R, substituting $z(x)$ by $Rz(\frac{x}{2R})$, and identifying the frequency ν in the definition of the resulting Hamiltonian by $\frac{c}{R}$, then the high temperature asymptotics reproduces the Stefan-Boltzmann law for the energy density in the infinite volume limit (Section 5.1 of [7]). This result is, in fact, valid under more general conditions (see Appendix A of [36]).

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