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Complexification of Three Potential Model

S. Meyur[1], **S. Debnath**[2]

[1]Tansuk Rai Ganapat Rai Khemka High School, 23, Rabindra Sarani, Liluah, Howrah-711204, India

[2]Department of Mathematics, Jadavpur University, Kolkata-700032, India

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Abstract. The new kind of $\mathcal{PT}$ and non $\mathcal{PT}$-symmetric complex potentials are constructed from a group theoretical viewpoint of sl(2,$\mathbb{C}$) potential algebras. The real eigenvalues and corresponding regular eigenfunctions are also obtained. The results are compared with ones obtained before.

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1 Introduction

$\mathcal{PT}$-symmetric quantum mechanics has generated much interest in recent years [1-10]. A few years ago, Bender and others [1-2,4,9] have looked at several complex potentials with $\mathcal{PT}$-symmetry and have shown that the energy eigenvalues are real when $\mathcal{PT}$-symmetry is unbroken, whereas they come in complex conjugate pairs when $\mathcal{PT}$-symmetry is spontaneously broken. Recently Mostafazadeh [11] in his very noteworthy work has introduced the concept of pseudo-Hermiticity and he has pointed out that all the $\mathcal{PT}$-symmetric Hamiltonians regarded so far are actually $\mathcal{P}$-pseudo Hermitian, namely $\mathcal{P}H\mathcal{P}^{-1} = H^\dagger$. Again, it is claimed that generally, it is the η-pseudo Hermiticity, *i.e.* $\eta H \eta^{-1} = H^\dagger$ [10] and not the $\mathcal{PT}$-symmetry, of a Hamiltonian which is the necessary condition for its real spectrum.

Bagchi and Quesne [11,12] have discussed the Lie algebra for hyperbolic potential. In this paper we have illustrated the Lie algebra for deformed type hyperbolic Scarf-II , Pöschl-Teller potential and Morse potential. We have shown that results are consistent with [12].

The paper is organized as follows. In Section 2, we present a brief discussion of sl(2,$\mathbb{C}$) potential algebra and its realization. In Section 3, we obtain general results for complex potential associated with sl(2,$\mathbb{C}$) potential algebra. In Sections 4, 5, 6 we discuss respectively the solutions of Scarf-II, Pöschl-Teller and Morse potential. Section 7 is kept for conclusions and discussions.

2 sl(2,$\mathbb{C}$) Potential Algebras

The most general differential realization of the sl(2,$\mathbb{C}$) algebra is [12]

$$J_0 = -i\frac{\partial}{\partial\phi}, J_\pm = e^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} + \left(i\frac{\partial}{\partial\phi} \mp \frac{1}{2}\right) f(x) + g(x)\right], \qquad (1)$$

where $o \le \phi < 2\pi$, $x \in \mathbb{R}$ and the two functions $f(x), g(x) \in \mathbb{C}$ satisfy

$$\frac{df}{dx} = 1 - f^2, \qquad \frac{dg}{dx} = -fg \qquad (2)$$

and the generators are connected by

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0\,. \qquad (3)$$

The Casimir operator corresponding to the above generators is

$$J^2 = -J_\pm J_\mp + J_0^2 \mp J_0\,. \qquad (4)$$

Using (1) and (4) one can obtain

$$J^2 = \frac{\partial^2}{\partial x^2} - \left(\frac{\partial^2}{\partial\phi^2} + \frac{1}{4}\right)\frac{df}{dx} + 2i\frac{\partial}{\partial\phi}\frac{dg}{dx} - g^2 - \frac{1}{4}\,. \qquad (5)$$

For bound states, the unitary irreducible representations of the type D_j^+, spanned by states $|jm\rangle$, $j > 0$ and $m = j + n$, $n = 0, 1, 2, \ldots$ such that

$$J_0|jm\rangle = m|jn\rangle, \qquad m = j, j+1, \ldots \qquad (6)$$

$$J^2|jm\rangle = j(j-1)|jn\rangle, \qquad m = j, j+1, \ldots \qquad (7)$$

In the realization (1), the states are given by

$$|jm >= \psi_{jm}(x,\phi) = \psi_{jm}\frac{e^{im\phi}}{\sqrt{2\pi}}\,, \qquad (8)$$

where $\psi_{jm}(x) = \psi_n^{(m)}(x)$, satisfying the Schrödinger equation

$$-{\psi_n^{(m)}}'' + V_m\psi_n^{(m)} = -E_n^{(m)}\psi_n^{(m)}\,. \qquad (9)$$

The potential $V_m(x)$ is represented by

$$V_m(x) = \left(\frac{1}{4} - m^2\right)\frac{df}{dx} + 2m\frac{dg}{dx} + g^2 \qquad (10)$$

and the energy eigenvalues are given by

$$E_n^{(m)} = -\left(m - n - \frac{1}{2}\right)^2\,. \qquad (11)$$

The potential $V_m(x)$, $m = j, j+1, j+2, \ldots$ supporting the same eigenvalues $-\left(m-n-\frac{1}{2}\right)^2$ produce a potential algebra. Solving the differential equation $J_-\psi_0^{(m)}(x) = 0$, the eigenfunctions $\psi_0^{(m)}(x)$ are easily obtained. The remaining eigenfunctions are obtained by successive application of J_+ on $\psi_0^{(m)}(x)$. For bound states ($\psi_n^{(m)}(\pm\infty) \rightarrow 0$), n is restricted to the range $n = 0, 1, 2, \ldots, n_{\max} < m - \frac{1}{2}$.

3 General Results

The most general solutions of the equation (2) are

$$\begin{aligned} f(x) &= \tanh_q(x-c-i\sigma), \\ g(x) &= (d_1+id_2)\,\mathrm{sech}_q(x-c-i\sigma) \end{aligned} \qquad \text{when} \quad f^2 < 1 \tag{12}$$

$$\begin{aligned} f(x) &= \coth_q(x-c-i\sigma), \\ g(x) &= (d_1+id_2)\,\mathrm{cosech}_q(x-c-i\sigma) \end{aligned} \qquad \text{when} \quad f^2 > 1 \tag{13}$$

$$f(x) = \lambda,\ g(x) = (d_1+id_2)e^{-\lambda(x-c-i\sigma)} \qquad \text{when} \quad f^2 = 1\,, \tag{14}$$

where

$$q(>0), \quad c, d_1, d_2(\neq 0) \in R, \quad \lambda = \pm 1, \quad -\frac{\pi}{4} \leq \sigma \leq \frac{\pi}{4}.$$

From (10) and (12) we have the nonsingular Scarf-II[SF] potential [13], given by

$$\begin{aligned} V_m^{SF}(x) =& \Big[(d_1+id_2)^2 + \Big(\frac{1}{4}-m^2\Big)q\Big]\mathrm{sech}_q^2(x-c-i\sigma) \\ &- 2m(d_1+id_2)\mathrm{sech}_q(x-c-i\sigma)\tanh_q(x-c-i\sigma) \\ =& \frac{2}{\big(\cosh_{q^2}(2x-2c)+q\cos 2\sigma\big)^2}\Big\{\Big[d_1^2-d_2^2+\Big(\frac{1}{4}-m^2\Big)q\Big] \\ &\times(\cosh_{q^2}(2x-2c)\cos 2\sigma+q) - 2d_1d_2\sinh_{q^2}(2x-2c)\sin 2\sigma \\ &- 2m\Big[d_1\sinh_{q^2}(x-c)\cos\sigma\,\big(\cosh_{q^2}(2x-2c)-q\cos 2\sigma+2q\big) \\ &- d_2\cosh_{q^2}(x-c)\sin\sigma\,\big(\cosh_{q^2}(2x-2c)-q\cos 2\sigma-2q\big)\Big]\Big\} \\ &+ \frac{2i}{(\cosh_{q^2}(2x-2c)+q\cos 2\sigma)^2}\Big\{\Big[d_1^2-d_2^2+\Big(\frac{1}{4}-m^2\Big)q\Big] \\ &\times\sinh_{q^2}(2x-2c)\sin 2\sigma + 2d_1d_2(\cosh_{q^2}(2x-2c)\cos 2\sigma+q) \\ &- 2m\Big[d_1\cosh_{q^2}(x-c)\sin\sigma(\cosh_{q^2}(2x-2c)-q\cos 2\sigma-2q) \\ &+ d_2\sinh_{q^2}(x-c)\cos\sigma(\cosh_{q^2}(2x-2c)-q\cos 2\sigma+2q)\Big]\Big\}. \end{aligned} \tag{15}$$

From (10) and (13) we have the nonsingular Pöschl-Teller[PTL] potential [13], given by

$$\begin{aligned}
V_m^{PTL}(x) =&\Big[(d_1+id_2)^2-\Big(\frac{1}{4}-m^2\Big)q\Big]\mathrm{cosech}_q^2(x-c-i\sigma)\\
&-2m(d_1+id_2)\mathrm{cosech}_q(x-c-i\sigma)\mathrm{coth}_q(x-c-i\sigma)\\
=&\frac{2}{(\cosh_{q^2}(2x-2c)-q\cos 2\sigma)^2}\Big\{\Big[d_1^2-d_2^2-\Big(\frac{1}{4}-m^2\Big)q\Big]\\
&\times(\cosh_{q^2}(2x-2c)\cos 2\sigma-q)-2d_1d_2\sinh_{q^2}(2x-2c)\sin 2\sigma\\
&-2m\Big[d_1\cosh_{q^2}(x-c)\cos\sigma(\cosh_{q^2}(2x-2c)+q\cos 2\sigma-2q)\\
&-d_2\sinh_{q^2}(x-c)\sin\sigma(\cosh_{q^2}(2x-2c)+q\cos 2\sigma+2q)\Big]\Big\}\\
&+\frac{2i}{(\cosh_{q^2}(2x-2c)-q\cos 2\sigma)^2}\Big\{\Big[d_1^2-d_2^2-\Big(\frac{1}{4}-m^2\Big)q\Big]\\
&\times\sinh_{q^2}(2x-2c)\sin 2\sigma+2d_1d_2(\cosh_{q^2}(2x-2c)\cos 2\sigma-q)\\
&-2m\Big[d_1\sinh_{q^2}(x-c)\sin\sigma(\cosh_{q^2}(2x-2c)+q\cos 2\sigma+2q)\\
&+d_2\cosh_{q^2}(x-c)\cos\sigma(\cosh_{q^2}(2x-2c)+q\cos 2\sigma-2q)\Big]\Big\}
\end{aligned}\tag{16}$$

and from (10) and (14) we have the nonsingular Morse potential [14], given by

$$\begin{aligned}
V_m^{MP}(x) &= (d_1+id_2)^2e^{-2\lambda x}-2m(d_1+id_2)e^{-\lambda x}\\
&=(d_1^2-d_2^2)e^{-2\lambda x}-2\lambda m d_1e^{-\lambda x}+id_2(2d_1e^{-2\lambda x}\mp 2me^{-\lambda x}),
\end{aligned}\tag{17}$$

where (17) corresponds to $\lambda = 1$ and we use the relations

$$\cosh_q(x+i\sigma)\,\cosh_q(x-i\sigma)=\frac{1}{2}[\cosh_{q^2}2x+q\cos 2\sigma]\,,\tag{18}$$

$$\sinh_q(x+i\sigma)\,\sinh_q(x-i\sigma)=\frac{1}{2}[\cosh_{q^2}2x-q\cos 2\sigma]\,.\tag{19}$$

For $q = 1$ the equations (15) and (16) coincide with [12]. The above potentials give a quite complete generalization of sl(2,$\mathbb{C}$) algebra corresponding to the representation (1). In order to obtain the regular wave function $\psi_0^{(m)}(x)$ that solves the differential equation $J_-\psi_{mm}(x,\phi)=0$, we have

$$\psi_0^{(m)}(x)\propto(\mathrm{sech}_q x')^{(m-\frac{1}{2})}\,\exp\left[\frac{(d_1+id_2)}{\sqrt{q}}\arctan\left(\frac{1}{\sqrt{q}}\sinh_q x'\right)\right]\tag{20}$$

$$\psi_0^{(m)}(x)\propto\left[\sinh_{\sqrt{q}}\left(\frac{x'}{2}\right)\right]^{(-m+\frac{1}{2}+\frac{d_1+id_2}{\sqrt{q}})}\left[\cosh_{\sqrt{q}}\left(\frac{x'}{2}\right)\right]^{(-m+\frac{1}{2}-\frac{d_1+id_2}{\sqrt{q}})}\tag{21}$$

$$\psi_0^{(m)}(x) \propto \exp\left[-\left(m-\frac{1}{2}\right)x' - (d_1+id_2)e^{-x'}\right], \tag{22}$$

where $x' = x - c - i\sigma$. Equations (20) and (21) are regular when $m > \frac{1}{2}$ and (22) is regular when $m > \frac{1}{2}$ and $d_2 > 0$.

4 Complexification of Scarf-II Potential

The most general form of Scarf-II potential [13] is

$$V(x) = -V_1 \mathrm{sech}_q^2 x - iV_2 \mathrm{sech}_q x \tanh_q x, \qquad V_1 > 0, \quad V_2 \neq 0\,. \tag{23}$$

The potential (23) is $\mathcal{PT}$-symmetric under $\mathcal{P}: x \to \log q - x$ $\mathcal{T}: i \to -i$ and η-pseudo Hermitian under $\eta x \eta^{-1} = x + i\pi$. Now for $c = \sigma = 0$, comparing (23) with (15) we have

$$d_1^2 - d_2^2 + \left(\frac{1}{4} - m^2\right) q = -V_1\,, \tag{24}$$

$$d_1 d_2 = 0\,, \tag{25}$$

$$m d_1 = 0\,, \tag{26}$$

$$2 m d_2 = V_2\,. \tag{27}$$

Using regularity condition $m > \frac{1}{2}$, from (26) we have $d_1 = 0$ and from (27) we have

$$m = \frac{V_2}{2d_2}\,. \tag{28}$$

From (24) and (28) we have

$$d_2^2 = \frac{1}{4}\left[\sqrt{V_1 + \frac{q}{4} + \sqrt{q}V_2} \pm \sqrt{V_1 + \frac{q}{4} - \sqrt{q}V_2}\right]^2 \tag{29}$$

provided

$$|V_2| \leq \frac{1}{\sqrt{q}}\left(V_1 + \frac{q}{4}\right).$$

However, the proper choice of the sign in the expression for d_2 is extremely important. Due to the fact that $m > \frac{1}{2}$, the sign of d_2 is the same as the sign of V_2 in the equation (28),

$$\text{when } d_2 > 0,\ V_2 > 0, \quad d_2 = \frac{1}{2}\left[\sqrt{V_1 + \frac{q}{4} + \sqrt{q}V_2} \pm \sqrt{V_1 + \frac{q}{4} - \sqrt{q}V_2}\right]$$

$$\text{when } d_2 < 0,\ V_2 < 0, \quad d_2 = -\frac{1}{2}\left[\sqrt{V_1 + \frac{q}{4} + \sqrt{q}V_2} \pm \sqrt{V_1 + \frac{q}{4} - \sqrt{q}V_2}\right].$$

In both cases

$$m=\frac{1}{2}\left[\sqrt{V_1+\frac{q}{4}+\sqrt{q}V_2}\pm\sqrt{V_1+\frac{q}{4}-\sqrt{q}V_2}\right], \tag{30}$$

where

$$|V_2|\le\frac{1}{\sqrt{q}}\left(V_1+\frac{q}{4}\right)\text{ and }\sqrt{V_1+\frac{q}{4}+\sqrt{q}V_2}\pm\sqrt{V_1+\frac{q}{4}-\sqrt{q}V_2}>1$$

From (11) and (29) we have two energy levels

$$E_n=-\left[\frac{1}{2}\left(\sqrt{V_1+\frac{q}{4}+\sqrt{q}V_2}\pm\sqrt{V_1+\frac{q}{4}-\sqrt{q}V_2}\right)-n-\frac{1}{2}\right]^2, \tag{31}$$

where

$$n<0,1,2,\ldots,\left(\frac{1}{2}\left(\sqrt{V_1+\frac{q}{4}+\sqrt{q}V_2}\pm\sqrt{V_1+\frac{q}{4}-\sqrt{q}V_2}\right)-\frac{1}{2}\right).$$

Let us take the potential parameter as $V_1=(B^2-A(A+\sqrt{q}))$, $V_2=-(B/\sqrt{q})\left(2A+\sqrt{q}\right)$, the potential (23) is invariant under the transformation $A+\sqrt{q}/2\longleftrightarrow B$. Two values of m are $A/\sqrt{q}+1/2$, and $B/\sqrt{q}$ and the two energy levels are

$$E_n^{\left(\frac{1}{\sqrt{q}}\left(A+\frac{\sqrt{q}}{2}\right)\right)}=-\left(\frac{A}{\sqrt{q}}-n\right)^2,\ n=0,1,2,3,\ldots,n_{\max}<\frac{A}{\sqrt{q}} \tag{32}$$

$$E_n^{\left(\frac{B}{\sqrt{q}}\right)}=-\left(\frac{B}{\sqrt{q}}-n-\frac{1}{2}\right)^2,\ n=0,1,2,3,\ldots,n_{\max}<\left(\frac{B}{\sqrt{q}}-\frac{1}{2}\right). \tag{33}$$

Now for the special choice $B=\sqrt{q}$, $A+\sqrt{q}/2=-\lambda\sqrt{q}$, $(\lambda<0)$, the energies (32) obtained from the first sl(2,$\mathbb{C}$) algebra become $E_n^{(-\lambda)}=-\left(\lambda+n+\frac{1}{2}\right)^2$, while the second sl(2,$\mathbb{C}$) algebra leads to a single energy level $E_0^{(1)}=-\frac{1}{4}$.

5 Complexification of Pöschl-Teller Potential

The generalized Pöschl-Teller potential is usually given in the form [13]

$$\begin{aligned}V(x)&=V_1\mathrm{cosech}_q^2(x-c-i\sigma)\\&\quad-V_2\mathrm{cosech}_q(x-c-i\sigma)\coth_q(x-c-i\sigma),\quad V_1>-\frac{q}{4},\ V_2\neq 0.\end{aligned} \tag{34}$$

The potential (34) is $\mathcal{PT}$-symmetric under $\mathcal{P}:x\to\log q-x+2c\ \mathcal{T}:i\to -i$. Comparing equation (34) with equation (16) we have

$$d_1^2 - d_2^2 - \left(\frac{1}{4} - m^2\right) q = -V_1 \,, \tag{35}$$

$$d_1 d_2 = 0 \,, \tag{36}$$

$$2md_1 = V_2 \,, \tag{37}$$

$$md_2 = 0 \,. \tag{38}$$

Using same technique in the previous section we have

$$m = \frac{1}{2}\left[\sqrt{V_1 + \frac{q}{4} + \sqrt{q}V_2} \pm \sqrt{V_1 + \frac{q}{4} - \sqrt{q}V_2}\right] \tag{39}$$

provided

$$|V_2| \leq \frac{1}{\sqrt{q}}\left(V_1 + \frac{q}{4}\right).$$

From (11) and (39) we have two energy levels

$$E_n = -\left[\frac{1}{2}\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}V_2} \pm \sqrt{V_1 + \frac{q}{4} - \sqrt{q}V_2}\right) - n - \frac{1}{2}\right]^2 \tag{40}$$

$$n < 0, 1, 2, \ldots, \left[\frac{1}{2}\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}V_2} \pm \sqrt{V_1 + \frac{q}{4} - \sqrt{q}V_2}\right) - \frac{1}{2}\right].$$

6 Complexification of Morse Potential

The generalized Morse potential is usually given by [14]

$$V(x) = V_1 e^{-2x} - V_2 e^{-x}, \qquad V_1, V_2 \in \mathbb{C}\,. \tag{41}$$

The potential (41) is non-$\mathcal{PT}$-symmetric under $\mathcal{P} : x \to -x$ $\mathcal{T} : i \to -i$ but it is η-pseudo Hermitian[14]. Comparing equation (41) with equation (17) we have

$$(d_1 + id_2)^2 = V_1, \quad 2m(d_1 + id_2) = V_2, \quad c = \sigma = 0. \tag{42}$$

Solving the equations given in (42) we have $m = \dfrac{V_2}{2\sqrt{V_1}}$, known as real effective parameter [14]. The energy eigenvalues are

$$E_n = -\left(\frac{V_2}{2\sqrt{V_1}} - n - \frac{1}{2}\right)^2, \qquad n < 0, 1, 2, \ldots, \left(\frac{V_2}{2\sqrt{V_1}} - \frac{1}{2}\right), \tag{43}$$

which coincides with [14]. Now as $m > \dfrac{1}{2}$, we have $\dfrac{V_2}{2\sqrt{V_1}} - \dfrac{1}{2} > 0$ which confirms that (17) has only real eigenvalues corresponds to $n < 0, 1, 2, \ldots, \left(\dfrac{V_2}{2\sqrt{V_1}} - \dfrac{1}{2}\right)$ and hence there is no symmetry breaking range.

7 Conclusion

In this paper, the bound state eigenvalues of Scarf-II, Pöschl-Teller and Morse potential have been solved by sl(2,$\mathbb{C}$) potential algebra. For the case of Scarf-II and Pöschl-Teller potential, we have shown that symmetry breaking occurs when $|V_2| > \frac{1}{\sqrt{q}}\left(V_1 + \frac{q}{4}\right)$. We have also shown that for the Morse potential there is no symmetry breaking range.

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