

# Non-Hermitian Hamiltonian with Gauge-Like Transformation

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**Abstract.** The non-Hermitian Hamiltonians are discussed for the case of Wood-Saxon and Morse potential. Possible generalization of our approach is outlined.

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## 1 Introduction

$\mathcal{PT}$ -symmetric quantum systems have generated much interest in recent years [1]. Bender and Boettcher in their noteworthy papers [1,2] have claimed that the real energy eigenvalue of Hamiltonian is due to  $\mathcal{PT}$ -symmetry. The linear operator  $\mathcal{P}$  (for parity) and another anti-linear operator  $\mathcal{T}$  (for time-reversal) act on the fundamental observables according to

$$\mathcal{P}x\mathcal{P}^{-1} = -x; \mathcal{P}p\mathcal{P}^{-1} = \mathcal{T}p\mathcal{T}^{-1} \quad (1)$$

and  $\mathcal{T}i\mathcal{T}^{-1} = iI$  remembering that

$$x^\dagger = x, i^\dagger = -i, p^* = -p, p^\dagger = p. \quad (2)$$

In the explicit form the condition of the  $\mathcal{PT}$ -symmetry for the potential  $V(x)$  reads

$$V^*(x) = V(x). \quad (3)$$

In particular, the spectrum of the Hamiltonian is real if  $\mathcal{PT}$ -symmetry is not spontaneously broken and hence, the property of exactness guarantees the real energies [2–7].

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Different techniques have been applied in the study of  $\mathcal{PT}$ -invariant potentials such as variational methods [8], numerical approaches [9], Fourier analysis [10], semi-classical estimates [11], quantum field theory [12] and Lie group theoretical approaches [10–14]. In the present paper, our aim is to solve the general model of non-Hermitian Hamiltonians *via* the gauge-like transformation, using Nikiforov-Uvarov method, for which the corresponding eigenvalue problem can be solved exactly.

The paper is organized as follows. We discuss the general model Hamiltonian and gauge-like transformation in Section 2. In Section 3, we give a brief review of NU method. In Section 4 and 5 we consider the Woods-Saxon potential and Morse model and obtain its eigenvalue and eigenfunctions for the corresponding Hamiltonian by NU method and also discuss its  $\mathcal{PT}$  and non  $\mathcal{PT}$ -symmetric solutions respectively. Section 8 is kept for conclusions and discussions.

### 2 Gauge-Like Transformation on Non-Hermitian Hamiltonian

Let the Hamiltonian be of the form

$$H_{alpha} = [p + i\alpha g(x)]^2 + V(x), \quad (2m = 1 = \hbar). \quad (4)$$

By applying gauge-like transformation [14]

$$e^{f(x)}[p + i\alpha g(x)]e^{-f(x)} = p - i\alpha g(x),$$

where  $f(x) = -2\alpha g(x) \int g(x)dx$ . Again

$$\begin{aligned} e^{f(x)}[p + i\alpha g(x)]^2 e^{-f(x)} &= e^{f(x)}[p + i\alpha g(x)][p + i\alpha g(x)]e^{-f(x)} \\ &= e^{f(x)}[p + i\alpha g(x)]e^{-f(x)} e^{f(x)}[p + i\alpha g(x)]e^{-f(x)} \\ &= [p - i\alpha g(x)][p - i\alpha g(x)] \\ &= [p - i\alpha g(x)]^2. \end{aligned}$$

Similarly  $e^{f(x)}[p + i\alpha g(x)]^n e^{-f(x)} = [p - i\alpha g(x)]^n$ . We shall now discuss a Hamiltonian of the type

$$H_{\alpha} = [p + i\alpha x^m]^2 + V(x), \quad (5)$$

where  $m$  is some positive integer. The eigenvalue equation is

$$H_{\alpha}\phi_n(x) = E_n\phi_n(x). \quad (6)$$

Applying a suitable transformation  $H_{\alpha}$  becomes

$$e^{f(x)/2}H_{\alpha}e^{-f(x)/2} = H_T = p^2 + V(x), \quad (7)$$

where  $f(x) = -\frac{2\alpha}{m+1}x^{m+1}$  and  $\phi_n$  becomes

$$\phi_n(x) = \exp\left[\frac{\alpha}{m+1}x^{m+1}\right]\psi_n(x), \quad (8)$$

where  $\psi_n(x)$  are eigenfunctions of transformed Hamiltonian  $H_T$ . The orthonormality condition becomes

$$\int_{-\infty}^{\infty} \exp\left[-\frac{\alpha x^{m+1}}{m+1}\right] \phi_{mq}(x)\phi_{nq}(x)dx = \delta_{m,n}. \quad (9)$$

### 3 Nikiforov-Uvarov Method

The differential equations whose solutions are the special functions can be solved by using NU method which has been developed by Nikiforov and Uvarov [15]. In this method, for a given  $V(x)$ , the one-dimensional Schrödinger equation is reduced to an equation which is

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \quad (10)$$

where  $\sigma(s)$  and  $\tilde{\sigma}(s)$  are polynomials, at most of second degree, and  $\tilde{\tau}(s)$  is a polynomial, at most of first degree. The wave function is constructed as a multiple of two independent parts

$$\psi(s) = \phi(s)y(s) \quad (11)$$

and (10) becomes

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0, \quad (12)$$

where

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \quad (13)$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s). \quad (14)$$

$\lambda$  is defined as

$$\lambda_n + n\tau' + \frac{n(n-1)}{2}\sigma'' = 0, \quad n = 0, 1, 2, \dots \quad (15)$$

determine  $\pi(s)$  and  $\lambda$  by defining

$$k = \lambda - \pi'(s) \quad (16)$$

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and the linear polynomial  $\pi(s)$  becomes

$$\pi(s) = \left( \frac{\sigma' - \tilde{\tau}}{2} \right) \pm \sqrt{\left( \frac{\sigma' - \tilde{\tau}}{2} \right)^2 - \tilde{\sigma} + k\sigma}. \quad (17)$$

Since  $\pi(s)$  has to be a polynomial of degree at most one in (17) the expression under the square root must be the square of a polynomial of first degree. This is possible only when its discriminant is zero. The differential Equation (10) is a hypergeometric type, whose polynomial solutions are given by Rodrigues relation as [16]

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)], \quad (18)$$

where  $C_n$  is normalization constant and the weight function satisfies the relation as

$$\frac{d}{ds} [\sigma(s)\rho(s)] = \tau(s)\rho(s). \quad (19)$$

#### 4 Woods-Saxon Potential

Let  $V(x)$  be a Woods-Saxon potential [17]

$$V(x) = -V_1 \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}} + V_2 \frac{e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2}. \quad (20)$$

The Schrödinger equation for the transformed Hamiltonian  $H_T$  becomes

$$\frac{d^2\psi}{dx^2} + \left[ E + V_1 \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}} - V_2 \frac{e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2} \right] \psi = 0, \quad (21)$$

where  $\hbar = 2m = 1$ . Setting the following notations

$$\varepsilon = -\frac{E}{\alpha^2}, \quad \beta = \frac{V_1}{q\alpha^2} (> 0), \quad \gamma = \frac{V_2}{q^2\alpha^2} (> 0), \quad \text{and } s = -qe^{-\alpha x} \quad (22)$$

with  $\varepsilon > 0$  ( $E < 0$ ) for bound states, Equation (21) becomes

$$\frac{d^2\psi}{ds^2} + \frac{1-s}{s-s^2} \frac{d\psi}{ds} + \frac{1}{(s-s^2)^2} [-(\varepsilon + \gamma - \beta)s^2 + (2\varepsilon - \beta)s - \varepsilon] \psi = 0. \quad (23)$$

After the comparison of Equation (23) with Equation (10), we have

$$\begin{aligned} \tilde{\tau}(s) &= 1 - s, \quad \sigma(s) = s - s^2, \\ \tilde{\sigma}(s) &= -(\varepsilon + \gamma - \beta)s^2 + (2\varepsilon - \beta)s - \varepsilon. \end{aligned} \quad (24)$$

Substituting these polynomials into (17), we have

$$\begin{aligned}\pi(s) &= -\frac{s}{2} \pm \frac{1}{2} [(2\sqrt{\varepsilon} + \sqrt{1+4\alpha t})s - 2\sqrt{\varepsilon}] \text{ if } k = -\beta - \sqrt{\varepsilon(1+4\gamma)} \\ \pi(s) &= -\frac{s}{2} \pm \frac{1}{2} [(2\sqrt{\varepsilon} - \sqrt{1+4\alpha t})s - 2\sqrt{\varepsilon}] \text{ if } k = -\beta + \sqrt{\varepsilon(1+4\gamma)}.\end{aligned}\quad (25)$$

After proper choice of the polynomials  $\pi(s)$  and  $k$ , we can write the function  $\tau(s)$  as follows:

$$\tau(s) = 1 - [2 + 2\sqrt{\varepsilon} + \sqrt{1+4\gamma}]s + 2\sqrt{\varepsilon} \quad (26)$$

and its negative derivative becomes

$$\tau'(s) = - [2 + 2\sqrt{\varepsilon} + \sqrt{1+4\gamma}].$$

Therefore from Equation (15) and (16) we have

$$\lambda = \lambda_n = n [2 + 2\sqrt{\varepsilon} + \sqrt{1+4\gamma}] + n(n-1)$$

and

$$\lambda = k + \pi'(s) = -\beta - \sqrt{\varepsilon(1+4\gamma)} - \frac{1}{2} - \frac{1}{2} [2\sqrt{\varepsilon} + \sqrt{1+4\gamma}]. \quad (27)$$

With the comparison of  $\lambda$ , we obtain the energy eigenvalues as follows:

$$E_n = -\alpha^2 \left[ \frac{2n+1}{2} - \frac{n(n+1) - \frac{V_1}{q\alpha^2}}{2n+1 + \sqrt{1 + \frac{4V_2}{q^2\alpha^2}}} \right]^2, \quad n \geq 0, \quad q \geq 1. \quad (28)$$

For  $\hbar = 2m = 1$ , Equation (28) is consistent with [17]. Using Equations (11), (13), (18) and (19), the wave functions can be written as

$$\psi_n(s) = N_n s^{\sqrt{2\varepsilon}} (1-s)^{\frac{1}{2}(1+\sqrt{1+4\gamma})} P_n^{(2\sqrt{\varepsilon}, \sqrt{1+4\gamma})}(1-2s), \quad (29)$$

where  $N_n$  is normalization constant and  $P_n^{(a,b)}(x)$  is Jacobi Polynomial [16].

#### 4.1 $\mathcal{PT}$ -Symmetric Woods-Saxon Case

Now let us take the potential parameters as  $\alpha \rightarrow i\alpha$  and  $V_1, V_2 \in \mathbb{R}$ , then (4) becomes

$$\begin{aligned}H_\alpha &= [p - \alpha g(x)]^2 + V_1 \frac{\cos 2\alpha x + q - i \sin 2\alpha x}{1 + q^2 + 2q \cos 2\alpha x} \\ &+ V_2 \frac{(q^2 + 2q \cos 2\alpha x + \cos 4\alpha x) - i(2q \sin 2\alpha x + \sin 4\alpha x)}{(1 + q^2 + 2q \cos 2\alpha x)^2}.\end{aligned}\quad (30)$$

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Then  $(\mathcal{PT})V(x)(\mathcal{PT})^{-1} = V(x)$ . Let us remember that

$$\begin{aligned}\mathcal{P}x\mathcal{P}^{-1} &= -x; \quad \mathcal{P}p\mathcal{P}^{-1} = \mathcal{T}p\mathcal{T}^{-1}; \quad \mathcal{T}i\mathcal{T}^{-1} = iI; \\ x^\dagger &= x; \quad i^\dagger = -i; \quad p^* = -p; \quad p^\dagger = p.\end{aligned}$$

Also  $H_\alpha$  satisfies

- (i)  $H_\alpha$  is non-Hermitian.
- (ii)  $H_\alpha$  is  $\mathcal{PT}$ -symmetric if  $m$  is even.
- (iii)  $H_\alpha$  is  $\mathcal{P}$ -pseudo-Hermitian if  $m$  is odd.

The real energy eigenvalues are given by

$$E_n = \alpha^2 \left[ \frac{2n+1}{2} - \frac{n(n+1) - \frac{V_1}{q\alpha^2}}{2n+1 + \sqrt{1 + \frac{4V_2}{q^2\alpha^2}}} \right]^2, \quad n \geq 0, \quad q \geq 1. \quad (31)$$

Thus, by choosing the parameter  $\alpha$  as purely imaginary, we found the energy eigenvalues obtained for non-Hermitian and  $\mathcal{PT}$ -symmetric Woods-Saxon potential are not similar to Equation (28). A positive energy spectra is obtained if and only if

$$n < \sqrt{\frac{V_1}{q\alpha^2} - \frac{V_2}{q^2\alpha^2}} - \frac{1}{2} \sqrt{1 - \frac{4V_2}{q^2\alpha^2}} - \frac{1}{2}, \quad (32)$$

since the energy eigenvalues of Woods-Saxon potential are negative. The eigenvalues are always positive real when  $V_2 = 0$  and then condition for  $n$  is  $n < \frac{1}{2} \sqrt{\frac{V_1}{q\alpha^2}} - 1$ , but can be complex for  $V_2 > \frac{q^2\alpha^2}{4}$ . Since these one-dimensional non-Hermitian Hamiltonians are invariant under  $\mathcal{PT}$ -transformation, they possessed real spectra. Thus, their real spectral properties may be linked with their  $\mathcal{PT}$ -symmetry. The eigenfunctions of the Hamiltonian (30) are

$$\begin{aligned}\phi_n(x) &= C_n \exp \left[ -i\alpha x \left( 2\sqrt{\varepsilon} - \frac{x^m}{m+1} \right) \right] (1 + qe^{-i\alpha x})^{\frac{1}{2}} (1 + \sqrt{1 - \frac{4V_2}{q^2\alpha^2}}) \times \\ &\quad \times P_n^{(2\sqrt{\varepsilon}, \sqrt{1 - \frac{4V_2}{q^2\alpha^2}})} (1 + 2qe^{-i\alpha x}). \quad (33)\end{aligned}$$

#### 4.2 Non $\mathcal{PT}$ -Symmetric Woods-Saxon Case

Next let us take  $V_1 \rightarrow iV_1$ ,  $V_2 \in \mathbb{R}$  and  $\alpha \rightarrow i\alpha$ , then (4) takes the form

$$\begin{aligned}H_{i\alpha} &= [p - \alpha g(x)]^2 - V_1 \frac{\sin 2\alpha x + i(\cos 2\alpha x + q)}{1 + q^2 + 2q \cos 2\alpha x} \\ &\quad + V_2 \frac{(q^2 + 2q \cos 2\alpha x + \cos 4\alpha x) - i(2q \sin 2\alpha x + \sin 4\alpha x)^2}{(1 + q^2 + 2q \cos 2\alpha x)}. \quad (34)\end{aligned}$$

Such a potential is non- $\mathcal{PT}$ -symmetric. The energy eigenvalues are given by

$$E_n = \alpha^2 \left[ \frac{2n+1}{2} - \frac{n(n+1) + \frac{iV_1}{q\alpha^2}}{2n+1 + \sqrt{1 - \frac{4V_2}{q^2\alpha^2}}} \right]^2, \quad n \geq 0, q \geq 1, \quad (35)$$

simplification of which gives

$$E_n = \frac{\alpha^2}{16} \left( 2n+1 + \sqrt{1 - \frac{4V_2}{q^2\alpha^2}} \right)^2 + \frac{1}{\alpha^4 q^2} \frac{v_2^2 - q^2 V_1^2}{\left( 2n+1 + \sqrt{1 - \frac{4V_2}{q^2\alpha^2}} \right)^2} + \frac{V_2}{2q^2} - i \left[ \frac{2V_1 V_2}{q^3 \alpha^2 \left( 2n+1 + \sqrt{1 - \frac{4V_2}{q^2\alpha^2}} \right)^2} + \frac{V_1}{2q} \right]. \quad (36)$$

Visible energies are obtained when

$$n < \frac{1}{q\alpha} (q^2 V_1^2 - V_2^2)^{1/4} - \frac{1}{2} \sqrt{1 - \frac{4V_2}{q^2\alpha^2}} - \frac{1}{2}. \quad (37)$$

If  $q \rightarrow iq$  then from (35) we have real energy

$$E_n = \alpha^2 \left[ \frac{2n+1}{2} - \frac{n(n+1) + \frac{V_1}{q\alpha^2}}{2n+1 + \sqrt{1 + \frac{4V_2}{q^2\alpha^2}}} \right]^2, \quad n \geq 0, q \geq 1 \quad (38)$$

$$\text{and } n < \sqrt{\frac{V_1}{q\alpha^2} - \frac{V_2}{q^2\alpha^2}} - \frac{1}{2} \sqrt{1 + \frac{4V_2}{q^2\alpha^2}} - \frac{1}{2}.$$

## 5 Morse Potential

Let us take  $V(x)$  be Morse potential of the form

$$V(x) = V_1 e^{-2\alpha x} - V_2 e^{-\alpha x}. \quad (39)$$

The Schrödinger equation for the transformed Hamiltonian  $H_T$  becomes

$$\frac{d^2\psi}{dx^2} + [E - V_1 e^{-2\alpha x} + V_2 e^{-\alpha x}] \psi = 0, \quad (40)$$

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where  $\hbar = 2m = 1$ . Setting the following notations:

$$\varepsilon = -\frac{E}{\alpha^2}, \quad \beta = \frac{1}{\alpha}, \quad \gamma = \frac{V_2}{\alpha^2\sqrt{V_1}}, \quad \text{and } s = \sqrt{V_1}e^{-\alpha x}$$

with  $\varepsilon > 0$  ( $E < 0$ ) for bound states, Equation (40) becomes

$$\frac{d^2\psi}{ds^2} + \frac{1}{s} \frac{d\psi}{ds} - \frac{1}{s^2} [\beta^2 s^2 - \gamma s + \varepsilon] \psi = 0. \quad (41)$$

Comparing the Equation (41) with Equation (10), we have

$$\begin{aligned} \tilde{\tau}(s) &= 1, \quad \sigma(s) = s, \\ \tilde{\sigma}(s) &= \beta^2 s^2 - \gamma s + \varepsilon. \end{aligned} \quad (42)$$

Using (41) and (17), we have

$$\begin{aligned} \pi(s) &= \pm(\beta s + 2\sqrt{\varepsilon}) & \text{if } k &= \gamma + 2\beta\sqrt{\varepsilon} \\ \pi(s) &= \pm(\beta s - 2\sqrt{\varepsilon}) & \text{if } k &= \gamma - 2\beta\sqrt{\varepsilon}. \end{aligned} \quad (43)$$

After proper choice of the polynomial  $\pi(s)$  and  $k$ , we can write the function  $\tau(s)$  which has a negative derivative as follows:

$$\tau(s) = 1 + 2\sqrt{\varepsilon} - 2\beta s \quad (44)$$

and then its negative derivatives become

$$\tau'(s) = -2\beta.$$

Therefore from Equation (15) and (16) we have

$$\lambda = \lambda_n = 2n\beta \quad (45)$$

and

$$\lambda = k - \beta. \quad (46)$$

With the comparison of  $\lambda$  in Equations (45) and (46), we obtain the energy eigenvalues as follows:

$$E_n = -\frac{\alpha^2}{4} \left( 2n + 1 - \frac{V_2}{\alpha\sqrt{V_1}} \right)^2, \quad (n = 0, 1, 2, \dots). \quad (47)$$

From (13), (14), (18) and (19), we have the wave functions as

$$\psi_n(s) = N_n s^{\sqrt{\varepsilon}} e^{-\beta s} L_n^{(2\sqrt{\varepsilon})}(2\beta s), \quad (48)$$

where  $N_n$  is normalization constant and  $L_n^{(2\mu)}(x)$  is associated Laguerre Polynomials [16].

### 5.1 $\mathcal{PT}$ -Symmetric Non-Hermitian Morse Potential

We are going to consider different forms of generalized Morse potential, viz at least one of the parameters is purely imaginary. When  $\alpha \rightarrow i\alpha$  and  $V_1, V_2 \in \mathbb{R}$ , the Hamiltonian (4) becomes

$$H_\alpha = [p - \alpha g(x)]^2 + V_1 \cos 2\alpha x - V_2 \cos \alpha x - i(V_1 \sin 2\alpha x - V_2 \sin \alpha x). \quad (49)$$

Then  $V(x)$  satisfies the relation  $(\mathcal{PT})V(x)(\mathcal{PT})^{-1} = V(x)$  and  $H_\alpha$  satisfies

- (i)  $H_\alpha$  is non-Hermitian.
- (ii)  $H_\alpha$  is  $\mathcal{PT}$ -symmetric if  $m$  is even.
- (iii)  $H_\alpha$  is  $\mathcal{P}$ -pseudo-Hermitian if  $m$  is odd.

The energy eigenvalues and eigenfunctions are given by

$$E_n = \frac{\alpha^2}{4} \left( 2n + 1 + i \frac{V_2}{\alpha \sqrt{V_1}} \right)^2, \quad n = 0, 1, 2, \dots \quad (50)$$

$$\phi_n(x) = C_n \exp \left[ -i\alpha x \left( \sqrt{\varepsilon} - \frac{x^m}{m+1} \right) - \beta \sqrt{V_1} e^{-i\alpha x} \right] L_n^{(2\sqrt{\varepsilon})} (2\beta \sqrt{V_1} e^{-i\alpha x}). \quad (51)$$

### 5.2 Non $\mathcal{PT}$ -Symmetric Non-Hermitian Morse Potential

When  $V_1, V_2 \in \mathbb{C}$  and  $\alpha \in \mathbb{R}$ , then  $V(x)$  is non- $\mathcal{PT}$ -symmetric. It is of interest to consider the two special cases:

**Case-I:**  $V_1 = (A + iB)^2$ ,  $V_2 = (2C + 1)(A + iB)$ ,  $\alpha = 1$  and  $A, B, C \in \mathbb{R}$ . In this case, Equation (38) takes the form

$$V(x) = (A + iB)^2 e^{-2x} - (2C + 1)(A + iB) e^{-x} \quad (52)$$

$$E_n = -(n - C)^2, \quad n = 0, 1, 2, \dots \quad (53)$$

which is completely real and independent from the potential parameters  $A, B$  and consistent with Ref. [18]. Since  $\varepsilon > 0$ , using Equations (45) and (46) we have  $n < C = \frac{V_2}{2\sqrt{V_1}} - \frac{1}{2}$ .

**Case-II:**  $V_1 \in \mathbb{R}$ ,  $V_2 = (A + iB)$ . The energies are

$$E_n = \frac{\alpha^2}{4} \left( 2n + 1 - \frac{A + iB}{\alpha \sqrt{V_1}} \right)^2, \quad n = 0, 1, 2, \dots \quad (54)$$

There are real energies only when

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- (i)  $\text{Im}(V_2) = 0, V_1 > 0$  and  $\alpha \in \mathbb{R}$
- (ii)  $\text{Im}(V_2) = 0, V_1 < 0$  and  $\alpha \in i\mathbb{R}$
- (iii)  $\text{Re}(V_2) = 0, V_1 < 0$  and  $\alpha \in \mathbb{R}$
- (iv)  $\text{Re}(V_2) = 0, V_1 > 0$  and  $\alpha \in i\mathbb{R}$

## 6 Conclusions

We have discussed the eigenvalues and eigenfunctions of non-Hermitian Hamiltonian for Woods-Saxon and Morse potential and the orthogonality condition. We have also discussed the pseudo-Hermiticity of this Hamiltonians. Finally, we have emphasized that the method discussed here can be generated for other potentials.

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