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# Multi-spin Strings on $AdS_4 \times S^7$: Giant Magnon Solutions. Part II*

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**Abstract.** In this paper we study multi-spin magnon solutions of strings on $AdS_4 \times S^7$ background. We apply a reduction to Neumann-Rosochhatius integrable system and obtain the most general dispersion relations. By recently discovered duality and AdS/CFT dictionary, these dispersion relations are supposed to describe the anomalous dimensions of certain gauge theory operators.

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## 1 Introduction

Currently string/gauge theory duality, the idea of which is rooted in [1], is one of the hottest topics. One of the main tools in studying this duality is the so called AdS/CFT correspondence [2] conjectured by Maldacena. The subsequent developments made the subject a major research area and we witnessed many fascinating discoveries in the last decade. The conjectured correspondence provides a powerful tool for studies and future advances in the understanding of gauge theories at strong coupling.

Let us briefly sketch the recent advances in the AdS/CFT correspondence. The best studied example is based on superstring theory on $AdS_5 \times S^5$ background where the corresponding gauge theory is $\mathcal{N} = 4$ supersymmetric Yang-Mills. One of the predictions of the correspondence is the equivalence between the spectrum of free string theory on $AdS_5 \times S^5$ and the spectrum of anomalous dimensions of gauge invariant operators in the planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. In order to check this conjecture we need the full spectrum of free string theory on a curved space, such as $AdS_5 \times S^5$, which is

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still an unsolved problem. In certain limits, [3], allowing reliability of the semi-classical results on both sides, the techniques of integrable systems have become useful in studying the AdS/CFT correspondence in detail. Assuming that these theories are integrable, the dynamics should be encoded in an appropriate scattering matrix $S$. This can be interpreted from both sides of the correspondence as follows. On the string side, in the strong-coupling limit the $S$-matrix can be interpreted as describing the two-body scattering of elementary excitations on the worldsheet. When their worldsheet momenta become large, these excitations can be described as special types of solitonic solutions, or giant magnons, and the interpolating region is described by the dynamics of the so-called near-flat-space regime [6, 11, 14].

On the gauge theory side, the action of the dilatation operator on single-trace operators is the same as that of a Hamiltonian acting on the states of a certain spin chain [4]. This turns out to be of great advantage because one can diagonalize the matrix of anomalous dimensions by using the algebraic Bethe ansatz technique (see [5] for a nice review on the algebraic Bethe ansatz). Then one can look for the S-matrix determined by the integrable structures. The dispersionless scattering described in this approximation hopefully can be extended to the regions where, due to the nature of this duality, validity of the consideration on one side rules out the validity on the other side of the correspondence. However employing a certain spin chain as a mediator between the two sides of the correspondence seems very helpful. Indeed, in several papers the relation between strings, $N = 4$ SYM theory and spin chains was established, see for instance [27–30] and references therein. This idea opened the way for a remarkable interplay between spin chains, gauge theories, string theory[1] and integrability (the integrability of classical strings on $AdS_5 \times S^5$ was proven in [31]). Concerning the integrability properties of the superstring on $AdS_5 \times S^5$, one should point out several key developments leading to the current understanding. In [6] the authors have considered Bethe ansatz for quantum strings and introduced the dressing phase as well as the "symplectic" form of the charges appearing in the exponential.[2] The integrable structures were further studied and further important developments were found in [7–10].

All these advantages lead to the idea that it would be useful to look for string solutions governing various corners of the spin chain spectrum. The most studied cases were spin waves in the long-wave approximation corresponding to rotating and pulsating strings in certain limits, see for instance the reviews [24–26] and references therein. Another interesting case are the low lying spin chain states corresponding to the magnon excitations. One class of string solutions already presented in a number of papers is the string theory on pp-wave backgrounds. The latter, although interesting and important, describes point-like strings which are only part of the whole picture.

[1]For very nice reviews on the subject with a complete list of references see [24–26]
[2]See [6] for further details.

One of the important explicit examples of string solutions corresponding to gauge states of interest are the so-called "giant magnon" solutions. The relation between spin chain "magnon" states and specific rotating semiclassical string states on $R \times S^2$ was suggested by Hofman and Maldacena [11] (for an early attempt see also [12]). As we already pointed out, the investigation of various string solutions which have clear interpretation in terms of gauge theory operators is very important and this has motivated the authors of [15–18] to generalize the results to magnon bound states. The latter are dual to strings on $R \times S^3$ with two non-vanishing angular momenta. Certainly it is of great interest to generalize further these investigations to the cases of multi-spin magnon states. Such generalization was given in [18] where different giant magnon states with two and more spins were found, moreover the authors considered a string solutions moving on $AdS_3 \times S^1$, *i.e.*, having spins in both the AdS and the spherical part of the background (see also [19]). These classical string solutions were further generalized to include dynamics on the whole $S^5$ [20,21] and in fact a method to construct classical string solutions describing superposition of arbitrary scattering and bound states was found [22]. The semiclassical quantization of the giant magnon solution was performed in [23]. We note that all these developments were based on the integrability of the string theory [31] and reduction to various integrable modes started with [61, 62].

As we discussed above, one of the most important characteristics of the solutions of this type are the dispersion relations. To be more concrete, in the case of one spin giant magnon the latter is found in [11] to be

$$E - J = \frac{\sqrt{l}}{\pi}|\sin\frac{p}{2}|, \tag{1}$$

where $p$ is the magnon momentum which on the string side is interpreted as a difference in the angle $\phi$ (see [11] for details). In the multi-spin cases the $E - J$ relation was studied both on the string [16–18] and spin chain sides [15]. We will only note that a natural way to extend this analysis is to find giant magnon solutions in backgrounds which, *via* the AdS/CFT correspondence, are dual to less supersymmetric gauge theories [43], for instance, the gamma deformed $AdS_5 \times S^5$ background found by Lunin and Maldacena [44, 45]. Investigations in this direction, were pursued in [46, 47] (see [48]- [56] for interesting work on semiclassical string dual to marginally deformed $\mathcal{N} = 4$ SYM and [57] for a review).

An important part of the current understanding of the duality between gauge theories and strings (M-theory) is the worldvolume dynamics of the branes. Recently there has been a number of works focused on the understanding of the worldvolume dynamics of multiple M2-branes. The interest was inspired by Bagger, Lambert and Gustavsson investigations based on the structure of Lie 3-algebra.

The semi-classical strings have played, and still play, an important role in studying various aspects of $AdS_5/SYM_4$ correspondence. The development in this subject gives a strong hint about how the new emergent duality can be investigated. An important role in these studies plays the integrability. In these intensive studies many properties were uncovered and impressive results obtained, but still the understanding of this duality is far from completeness.

## 2 The $S^7$ Sigma Model, Reduction to NR System and Some Particular Solutions

One parametrization is as follows. First we define a seven-dim sphere. The $S^7$ can be parametrized with four complex coordinates

$$Z_1 = x_1 + ix_2 = r_1 e^{\phi_1}, \qquad Z_2 = x_3 + ix_4 = r_2 e^{\phi_2},$$
$$Z_3 = x_5 + ix_6 = r_3 e^{\phi_3}, \qquad Z_4 = x_7 + ix_8 = r_4 e^{\phi_4}, \tag{2}$$

satisfying

$$\sum_{k=1}^{4} Z_k \bar{Z}_k = 1 = \sum_{j=1}^{8} x_j^2 = \sum_{k=1}^{4} r_k^2, \tag{3}$$

where $i, j = 1, \ldots, 8$ and $k = 1, \ldots, 4$.

Then the lagrangian becomes

$$\mathcal{L} = \sum_{\alpha=\tau,\sigma} \left[ \partial_\alpha t \partial^\alpha t + \sum_{k=1}^{4} \partial_\alpha Z_k \partial^\alpha \bar{Z}_k \right] + \Lambda(Z_k \bar{Z}_k - 1). \tag{4}$$

### 2.1 Reduction of $R \times S^7$ to NR Integrable System

Motivated by the considerations made in the case of magnon string configurations [21], we find it useful to apply the generalized Neumann-Rosochatius(NR) ansatz [62]

$$Z_k = x_k(\xi) e^{i\omega_k \tau},\ \xi = \alpha\sigma + \beta\tau, \quad x_k(\xi + 2\pi\alpha) = x_k(\xi),\ k = 1, 2, 3, 4 \tag{5}$$

for the spherical part of the geometry and

$$Y_0 = t = \kappa\tau, \quad Y_l = 0, \quad l = 1, 2 \tag{6}$$

for the $AdS_4$ piece. Let us restrict our attention here to the spherical part, $S^7$. The embedding we choose is subject also to the constraint $\sum_k Z_k \bar{Z}_k = 1$, which is additional to the conformal ones. The latter are very important when we are dealing with the Polyakov string action, so let us briefly discuss it here.

***The conformal constraints.*** In general the Virasoro constraints can be written as

$$\sum_k \left[|\partial_\tau Z_k|^2 + |\partial_\sigma Z_k|^2\right] = \kappa^2, \tag{7}$$

where we have used that $t = \kappa\tau$ and the rest of the AdS part is turned off. Using that

$$\partial_\tau Z_k = (\beta x_k' + i\omega_k x_k)e^{i\omega_k\tau}, \quad \partial_\sigma Z_k = \alpha x_k' e^{i\omega_k\tau}; \quad {}' \equiv \frac{d}{d\xi}$$

one finds for (7)

$$\sum_k \left[(\beta x_k' + i\omega_k x_k)(\beta\bar{x}_k' - i\omega_k\bar{x}_k) + \alpha^2 x_k'\bar{x}_k'\right] = \kappa^2. \tag{8}$$

For notational simplicity we define

$$\Xi_k = i(x_k\bar{x}_k' - x_k'\bar{x}_k). \tag{9}$$

After some manipulations we write down the final expressions we will use

$$(\alpha^2 - \beta^2)\sum_k x_k'\bar{x}_k' + \sum_k \omega_k^2 x_k\bar{x}_k = \kappa^2, \tag{10}$$

$$-\frac{(\alpha^2 - \beta^2)}{2\beta}\sum_k \omega_k\Xi_k + \sum_k \omega_k^2 x_k\bar{x}_k = \kappa^2. \tag{11}$$

***The Lagrangian.*** The consistent reduction to the Neumann-Rosochatius interable system requires further specification of the parametrization (5). The parametrization we need is defined as

$$x_k(\xi) = r_k(\xi)e^{i\phi_k(\xi)}, \quad |x_k'|^2 = {r_k'}^2 + r_k^2{\phi_k'}^2, \quad \Xi_k = 2r_k^2\phi_k'. \tag{12}$$

In these variables the Lagrangian takes the following form:

$$\mathcal{L} = \sum_k \Big[(\alpha^2 - \beta^2){r_k'}^2 + (\alpha^2 - \beta^2)r_k^2\Big(\phi_k' - \frac{\beta\omega_k}{\alpha^2 - \beta^2}\Big)^2 - \frac{\alpha^2}{\alpha^2 - \beta^2}\omega_k^2 r_k^2\Big] + \Lambda\Big(\sum_k r_k^2 - 1\Big). \tag{13}$$

It is easy to eliminate the angular degrees of freedom $\phi_k$. First we can integrate the equations of motion for $\phi_k$ to get

$$\phi_k' = \frac{1}{\alpha^2 - \beta^2}\left(\frac{C_k}{r_k^2} + \beta\omega_k\right), \tag{14}$$

where $C_k$ are constants of motion. On other hand for $r_k$ we get

$$(\alpha^2-\beta^2)r_k'' - \frac{C_k^2}{\alpha^2-\beta^2}\frac{1}{r_k^3} + \frac{\alpha^2}{\alpha^2-\beta^2}\omega_k^2 r_k - \Lambda r_k = 0, \tag{15}$$

where we used the equations for $\phi_k$ (14). The net result of this considerations is that we reduced the essential degrees of freedom to $r_k$. Before to proceed further, one should note that the periodicity condition (5) translates to $r_k$ and $\phi_k$ as follows

$$r_k(\xi+2\pi\alpha) = r_k(\xi), \quad \phi_k(\xi+2\pi\alpha) = \phi_k(\xi) + 2\pi n_k. \tag{16}$$

The periodicity conditions (16) together with (14) impose on the parameters $\omega_k, C_k, \alpha$ and $\beta$ a non-trivial relation

$$\frac{C_k}{2\pi}\int\limits_0^{2\pi\alpha}\frac{d\xi}{r_k^2} = (\alpha^2-\beta^2)n_k - \alpha\beta\omega_k. \tag{17}$$

The next step is to obtain the effective theory for $r_k$. The equations of motion for $r_k$ can be obtained from the following effective Lagrangian:

$$\mathcal{L} = \sum_k\left[(\alpha^2-\beta^2){r_k'}^2 - \frac{C_k^2}{\alpha^2-\beta^2}\frac{1}{r_k^2} - \frac{\alpha^2}{\alpha^2-\beta^2}\omega_k^2 r_k^2\right] + \Lambda\Big(\sum_k r_k^2 - 1\Big), \tag{18}$$

which is nothing but the well known integrable Neumann-Rosochatius system [62]. One can easily find the corresponding on-shell Hamiltonian which is one of the classical constants of motion

$$H = \sum_a\left[(\alpha^2-\beta^2){r_k'}^2 + \frac{C_k^2}{\alpha^2-\beta^2}\frac{1}{r_k^2} + \frac{\alpha^2}{\alpha^2-\beta^2}\omega_k^2 r_k^2\right]. \tag{19}$$

The considerations from now on will be essentially based on the above reduction.

***More on the conformal constraints.*** As we argued above, the reduction to the string theory with desired properties requires certain ansatz for the string embedding coordinates. Using the definition $\Xi_k = 2r_k^2\phi_k'$ and the equations of motion for $\phi_k$ (14), the second constraint (11) one can find

$$\Xi_k = \frac{2}{\alpha^2-\beta^2}\left[C_k + \beta\omega_k r_k^2\right], \quad \Longrightarrow \quad \sum_k \omega_k C_k + \beta\kappa^2 = 0. \tag{20}$$

In more details, using that

$$|x_k'|^2 = r_k'^2 + \frac{1}{(\alpha^2-\beta^2)^2}\left[\frac{C_k^2}{r_k^2} + \beta^2\omega_k^2 r_k^2 + 2\beta C_k\omega_k\right] \tag{21}$$

one can rewrite the first Virasoro constraint (10) as

$$\sum_k \left[ (\alpha^2 - \beta^2) r_k'^{\,2} + \frac{C_k^2}{\alpha^2 - \beta^2} \frac{1}{r_k^2} + \frac{\alpha^2}{\alpha^2 - \beta^2} \omega_k^2 r_k^2 + \frac{2\beta C_k \omega_k}{\alpha^2 - \beta^2} \right] = \kappa^2. \quad (22)$$

Comparing with (19) we observe that the first three terms are exactly the on-shell Hamiltonian, *i.e.*

$$H + \sum_k \frac{2\beta C_k \omega_k}{\alpha^2 - \beta^2} = \kappa^2. \quad (23)$$

But according to the second constraint (20) we can rewrite the above equation as

$$H = \kappa^2 + \frac{2\beta^2}{\alpha^2 - \beta^2} \kappa^2 = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \kappa^2. \quad (24)$$

One can write then the final form of the Virasoro constraints as

$$H = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \kappa^2, \quad (25)$$

$$\sum_k \omega_k C_k + \beta \kappa^2 = 0. \quad (26)$$

We should also impose periodicity conditions on $r_a$ and $\phi_a$

$$r_k(\xi + 2\pi\alpha) = r_k(\xi), \quad \phi_k(\xi + 2\pi\alpha) = \phi_k(\xi) + 2\pi n_k, \; n_k - \text{integer}. \quad (27)$$

As we discuss above, first of all we must ensure the conformal symmetry, *i.e.* one must satisfy the Virasoro constraints. However, this is not the whole story. On the top of the Virasoro constraints we have additional, geometrical constraints. One set of these tells us that we are using the parametrization of the $S^7$ sphere. In other words we have

$$\sum_k r_k^2 = 1 \quad (28)$$

which gives the standard $S^7$ sphere constraints. Combining the solutions for $\phi_k'$

$$\phi_k' = \frac{1}{\alpha^2 - \beta^2} \left( \frac{C_k}{r_k^2} + \beta \omega_k \right) \quad (29)$$

with the Virasoro constraints

$$\sum_k \omega_k C_k + \beta \kappa^2 = 0, \quad (30)$$

we find all the conditions that have to be satisfied

$$\sum_k r_k^2 = 1, \quad \sum_k C_k \omega_k + \beta \kappa^2 = 0. \quad (31)$$

We mention also that

$$H = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \kappa^2. \quad (32)$$

***Conserved quantities.*** Since in our case the background metric does not depend on $t$ and $\varphi_a$, the conserved quantities are the string energy $E_s$ and four angular momenta $J_a$ corresponding to the isometry directions, and the latter are given by

$$E_s = -\int d\sigma \frac{\partial \mathcal{L}}{\partial(\partial_0 t)}, \quad J_a = \int d\sigma \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_a)}. \tag{33}$$

The explicit form of these is given by

$$E = \frac{\kappa\sqrt{\lambda}}{\alpha} \int d\xi, \tag{34}$$

$$J_a = \frac{\sqrt{\lambda}}{\alpha^2 - \beta^2} \int d\xi \left( \frac{\beta}{\alpha} C_a + \alpha \omega_a r_a^2 \right). \tag{35}$$

## 3 Multi-Spin Solutions

In this Section we will explore the power of integrable Neumann-Rosochatius system to obtain the most general multi-spin solutions following the techniques of [21] (see also [63]).

Since it is known for along time that NR system is integrable, one has to use the separation of variables to integrate the dynamical system and then to obtain the conserved quantities we are interested in. Let us start with mentioning the separation of variables in the general case.

***Separation of variables.*** Suppose that $\zeta_j(x)$ are the solutions of

$$u(\zeta) = 0, \quad u(\zeta) = \sum_{a=1}^{N} \frac{r_a^2}{\zeta - \omega_a^2}, \quad \sum_a r_a^2 = 1. \tag{36}$$

Considering the graph of $u(t)$ it is easy to see that

$$\omega_1^2 \le \zeta_1 \le \omega_2^2 \le \cdots \le \omega_{N-1}^2 \le \zeta_{N-1} \le \omega_N^2 \tag{37}$$

and we have a bijection of this domain $\mathcal{D}$ of the $\zeta_j$'s on the "quadrant" $r_k > 0\,\forall k$ of the sphere. In general the sphere appears as a $2^N$–fold covering of the domain $\mathcal{D}$, ramified on the edges of $\mathcal{D}$. For instance $r_1 = \sqrt{\zeta_1 - \omega_1^2}\,\varphi(\zeta_2, \cdots, \zeta_{N-1})$ so that $r_1$ changes its sign when we turn around the ramification point $\zeta_1 = \omega_1^2$ in the complex plane. By similar analytic continuations we can cover the whole sphere.

The inverse transformation can be obtained starting from

$$\sum_{a=1}^{N} \frac{r_k^2}{\zeta - \omega_a^2} = \frac{\prod_{j=1}^{N-1} (\zeta - \zeta_j)}{\prod_{a=1}^{N} (\zeta - \omega_a^2)}. \tag{38}$$

Comparing the poles and zeroes we get

$$r_a^2 = \frac{\prod_{j=1}^{N-1} (\omega_a^2 - \zeta_j)}{\prod_{b \neq a} (\omega_a^2 - \omega_b^2)}. \tag{39}$$

One can prove that $\{\zeta_j\}$ is an orthogonal system.

***Finding $\mathcal{L}$ in the new coordinates and Hamilton-Jacobi equation.*** To find the Hamilton-Jacobi equation in the new coordinates it is convenient to start from Lagrangian formalism because it is well suited to the change variables

$$\mathcal{L} = \frac{1}{2} \sum_a \dot{r_a}^2 - \mathcal{U} = \frac{1}{2} \sum_{jj'} g_{jj'} \dot{\zeta}_j \dot{\zeta}_{j'} - \mathcal{U}, \tag{40}$$

where the metric $g_{jj'}$ is given by

$$g_{jj'} = -\frac{1}{4} \delta_{jj'} \frac{\prod_{n \neq j} (\zeta_n - \zeta_j)}{\prod_k (\zeta_j - \omega_k^2)}. \tag{41}$$

The potential of the Neumann-Rosochatius integrable system is given by

$$\mathcal{U} = U_1 + U_2, \quad U_1 = \sum_a \omega_a^2 r_a^2, \quad U_2 = \sum_a \frac{C_a^2}{r_a^2}. \tag{42}$$

This potential can be obtained as a isospectral deformation of the Neumann integrable system, defined with $U_1$ only. Using partial fraction decomposition and the resolvent identity $(\mu_i - A)^{-1}(\mu_k - A)^{-1} = -\left((\mu_k - A)^{-1} - (\mu_i - A)^{-1}\right)/(\mu_k - \mu_i)$, one can find[1]

$$\mathcal{U} = \frac{1}{2} \sum_k \omega_k^2 r_k^2 + \frac{1}{2} \sum_k \frac{C_k^2}{r_k^2} = \frac{1}{2} \left( \sum_k \omega_k^2 - \sum_j \zeta_j \right) + \frac{1}{2} \sum_a C_a^2 \prod_{b \neq a} (\omega_a^2 - \omega_b^2) \sum_j \frac{1}{\prod_{n \neq j} (\zeta_n - \zeta_j)(\zeta_j - \omega_a^2)}. \tag{43}$$

The conjugate momentum to $\zeta_j$ is $p_j \equiv \partial \mathcal{L} / \partial \dot{\zeta}_j = g_{jj} \dot{\zeta}_j$ leading to the hamiltonian

$$H = \sum_j p_j \dot{\zeta}_j - \mathcal{L} = \frac{1}{2} \sum_j g^{jj} p_j^2 + \mathcal{U}, \tag{44}$$

where $g^{jj} = (g^{-1})_{jj} = 1/g_{jj}$. The Hamilton–Jacobi equation is a first–order non–linear partial differential equation obtained by substituting in $H$: $p_j \rightarrow$

[1] Here we omitted for a moment the factor $1/(1 - \beta^2)$.

$\partial S/\partial\zeta_j$. The action $S$ is function of the space coordinates $t_j$'s. For a fixed energy $E$ the Hamilton–Jacobi equation reads

$$-2\sum_j \frac{\prod_k(\zeta_j-\omega_k^2)}{\prod_{n\neq j}(\zeta_n-\zeta_j)}\left(\frac{\partial S}{\partial\zeta_j}\right)^2+\mathcal{U}-E=0. \tag{45}$$

The method of separation of variables consists in looking for a so-called complete solution of Eq. (45), *i.e.* depending on $(N-1)$ arbitrary constants, of the form

$$S(\zeta_1,\cdots,\zeta_{N-1})=S_1(\zeta_1)+\cdots+S_{N-1}(\zeta_{N-1}). \tag{46}$$

Let us remark that all $S_j$'s satisfy the same equation which we can write as

$$4\,\Delta(\zeta)\left(\frac{dS}{d\zeta}\right)^2+\sum_a C_a^2\prod_{b\neq a}\omega_{ba}^2(\zeta-\omega_a^2)^{-1}+\prod_{n=1}^{N-1}(\zeta-d_n)=0, \tag{47}$$

where $\Delta(\zeta)=\prod_k(\zeta-\omega_k^2)$ and $d_n$'s are $(N-1)$ independent constants. The energy is obtained in terms of the $d_n$'s as: $E=1/2\left(\sum_k\omega_k^2-\sum_n d_n\right)$.

We notice that the action itself is given by a similar integral

$$S(\zeta_1,\cdots,\zeta_{N-1})=\frac{1}{2}\sum_j\int_{P_0}^{P_j=(\zeta_j,s_j)}\Big\{\frac{d\zeta}{s}$$
$$\times\sum_a C_a^2\prod_{b\neq a}\omega_{ba}^2(\zeta-\omega_a^2)^{-1}+\prod_{n=1}^{N-1}(\zeta-d_n)\Big\}, \tag{48}$$

where

$$s^2+P(\zeta)=0,$$
$$P(\zeta)=\prod_{k=1}^{N}(\zeta-\omega_k^2)\left[\sum_a C_a^2\prod_{b\neq a}\omega_{ba}^2(\zeta-\omega_a^2)^{-1}+\prod_{n=1}^{N-1}(\zeta-d_n)\right]. \tag{49}$$

We note that the terms $\zeta^k$ for $k=1,\cdots,N-2$ in $\prod_n(\zeta-d_n)=\zeta^{N-1}+\sum_0^{N-2}c_k\,\zeta^k$ lead to Abelian integrals of first kind, while the term $\zeta^{N-1}$ leads to an integral of second kind, having a double pole at $\infty$. So the action may be seen as a multivalued meromorphic function on the Jacobian. More explicitly, one can schematically write

$$S=\frac{1}{2}\sum_{k=1}^{N-1}c_{k-1}\,\Omega_k+S_0, \tag{50}$$

where the $c_k$'s are linear functions of $F_k$'s[1] with coefficients polynomials in the $\omega_k^2$'s and $S_0$ is

$$S_0(P_1 + \cdots + P_{N-1}) = \frac{1}{2} \sum_j \int_{P_0}^{P_j} \frac{\zeta^{N-1}\, d\zeta}{s} \,. \tag{51}$$

As a function of the divisor $P_1 + \cdots + P_{N-1}$, $S_0$ has a simple pole on a variety of codimension 1 obtained when one of the $P_j$'s goes to $\infty$, which is well known to be the divisor of a theta function on the Jacobian torus. By restriction to real variables, $S$ becomes a real multivalued *analytic* function on the $(N-1)$-dimensional real Liouville torus.

$S^7$ ***case.*** In the case of string on $R \times S^7$ we have the coordinates on $S^7$ defined as

$$\sum_{a=1}^{4} \frac{r_a^2}{\zeta - \omega_a^2} = \frac{(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)}{\prod\limits_{a=1}^{4}(\zeta - \omega_a^2)}, \tag{52}$$

where $\zeta_i$ are the roots of the cubic equation obtained by taking the common denominator. Note that these are *unconstrained coordinates* and they are such that $\omega_4^2 < \zeta_3 < \omega_3^2 < \zeta_2 < \omega_2^2 < \zeta_1 < \omega_1^2$. The inverse map is

$$r_a^2 = \frac{(\zeta_1 - \omega_a^2)(\zeta_2 - \omega_a^2)(\zeta_3 - \omega_a^2)}{\prod\limits_{b \neq a}(\omega_b^2 - \omega_a^2)}. \tag{53}$$

After the separation of variables the Hamiltonians $\mathcal{W}$, $\mathcal{H} = \mathcal{W}_1 + \mathcal{W} + \mathcal{W}_3$ satisfy the same equation but with respect to different variables $\zeta_i$. Hence, we will denote these actions by $\mathcal{W}$ and distinguish them by explicitly writing the dependence on $\zeta_i$ (and relevant constants).

The EOM reduce to

$$\frac{\partial \mathcal{W}(\zeta_1, V, U, E)}{\partial V} + \frac{\partial \mathcal{W}(\zeta_2, V, U, E)}{\partial V} + \frac{\partial \mathcal{W}(\zeta_3, V, U, E)}{\partial V} = K, \tag{54}$$

$$\frac{\partial \mathcal{W}(\zeta_1, V, U, E)}{\partial U} + \frac{\partial \mathcal{W}(\zeta_2, V, U, E)}{\partial U} + \frac{\partial \mathcal{W}(\zeta_3, V, U, E)}{\partial U} = L, \tag{55}$$

$$\frac{\partial \mathcal{W}(\zeta_1, V, U, E)}{\partial E} + \frac{\partial \mathcal{W}(\zeta_2, V, U, E)}{\partial E} + \frac{\partial \mathcal{W}(\zeta_3, V, U, E)}{\partial E} = \xi, \tag{56}$$

where $K$ and $L$ are the new constants and $\xi$ is the "time" variable[2]. The first two equations, (54) and (55), serve to determine $\zeta_1$ as a function of $\zeta_2$, $\zeta_3$ and the constants $K, L$. The last equation gives the dependence on the proper time $\xi$.

[1]Here $F_k$ are the Uhlenbeck constants of motion $F_k = r_k^2 + \sum_{l \neq k} I_{kl}^2/(\omega_k^2 - \omega_l^2)$, $\sum_k F_k = 1$ where $I_{kl}^2 = (r_k r_l' - r_l r_k')^2 + C_k^2 r_l^2/r_k^2 + C_l^2 r_k^2/r_l^2$.

[2]Actually these constants can be absorbed into the integrals on the left hand side.

To write explicit expressions for (54-56), we use (45). The result is

$$\int\limits_{P_0}^{(\zeta_1,s_1)} \frac{d\zeta}{s} + \int\limits_{P_0}^{(\zeta_2,s_2)} \frac{d\zeta}{s} + \int\limits_{P_0}^{(\zeta_3,s_3)} \frac{d\zeta}{s} = K, \tag{57}$$

$$\int\limits_{P_0}^{(\zeta_1,s_1)} \frac{\zeta d\zeta}{s} + \int\limits_{P_0}^{(\zeta_2,s_2)} \frac{\zeta d\zeta}{s} + \int\limits_{P_0}^{(\zeta_3,s_3)} \frac{\zeta d\zeta}{s} = L, \tag{58}$$

$$\int\limits_{P_0}^{(\zeta_1,s_1)} \frac{\zeta^2 d\zeta}{s} + \int\limits_{P_0}^{(\zeta_2,s_2)} \frac{\zeta^2 d\zeta}{s} + \int\limits_{P_0}^{(\zeta_3,s_3)} \frac{\zeta^2 d\zeta}{s} = \frac{\xi}{(1-\beta^2)}, \tag{59}$$

where $s$ is related to a polynomial of degree seven, $P_7(\zeta)$, by $s^2 = P_7(\zeta)$.

## 4 Giant Magnon Solutions

We start with the general expression we found in the previous Section (57-59). First we note that the constants $K$ and $L$ can be absorbed in the integration and therefore we find

$$\int\limits_{\bar\zeta_1}^{\zeta_1} \frac{d\zeta}{s} + \int\limits_{\bar\zeta_2}^{\zeta_2} \frac{d\zeta}{s} + \int\limits_{\bar\zeta_3}^{\zeta_3} \frac{d\zeta}{s} = 0, \tag{60}$$

$$\int\limits_{\bar\zeta_1}^{\zeta_1} \frac{\zeta d\zeta}{s} + \int\limits_{\bar\zeta_2}^{\zeta_2} \frac{\zeta d\zeta}{s} + \int\limits_{\bar\zeta_3}^{\zeta_3} \frac{\zeta d\zeta}{s} = 0, \tag{61}$$

$$\int\limits_{\bar\zeta_1}^{\zeta_1} \frac{\zeta^2 d\zeta}{s} + \int\limits_{\bar\zeta_2}^{\zeta_2} \frac{\zeta^2 d\zeta}{s} + \int\limits_{\bar\zeta_3}^{\zeta_3} \frac{\zeta^2 d\zeta}{s} = -2\frac{\xi}{(1-\beta^2)}. \tag{62}$$

We can consider the shape of the generic string solutions but here we will be interested in the class of solutions describing strings with one infinite momentum. Such solutions arise when $\zeta_i$ reach their extremal values $\omega_2^2,\ \omega_3^2$ and $\omega_4^2$. These conditions entails vanishing of the following constants $C_2 = C_3 = C_4 = 0$. To ensure this, we must have double zeroes of the polynomial $P_7(\zeta)$ at these values. Having the explicit form of this polynomial one can adjust appropriately the integration constants. To this end, one can write the polynomial $P_7(\zeta)$ as

$$P_7(\zeta) = (\zeta - \omega_2^2)(\zeta - \omega_3^2)(\zeta - \omega_4^2) \times \left[ C_1^2 \prod_{b\neq 1}(\omega_b^2 - \omega_1^2) + (\zeta - \omega_1^2)(\zeta - d_1)(\zeta - d_2)(\zeta - d_3) \right] = (\zeta - \omega_2^2)(\zeta - \omega_3^2)(\zeta - \omega_4^2)\, P_4(\zeta), \quad (63)$$

where

$$\begin{aligned} P_4(\zeta) =& C_1^2 \prod_{b\neq 1}(\omega_b^2 - \omega_1^2) + (\zeta - \omega_1^2)(\zeta - d_1)(\zeta - d_2)(\zeta - d_3) \\ =& \zeta^4 - (d_1 + d_2 + d_3 + \omega_1^2)\zeta^3 \\ &+ \left[d_1 d_2 + d_1 d_3 + d_2 d_3 + \omega_1^2(d_1 + d_2 + d_3)\right] \zeta^2 \\ &- \left[d_1 d_2 d_3 + \omega_1^2(d_1 d_2 + d_1 d_3 + d_2 d_3)\right] \zeta \\ &+ \omega_1^2 d_1 d_2 d_3 + C_1^2 \prod_{b\neq 1}(\omega_b^2 - \omega_1^2). \end{aligned} \quad (64)$$

The polynomial $P_4(\zeta)$ has zeroes at $\omega_2^2,\ \omega_3^2$ and $\omega_4^2$ when the parameters satisfy the following algebraic system

$$d_1 + d_2 + d_3 + \omega_1^2 = \omega_2^2 + \omega_3^2 + \omega_4^2 + \bar{\zeta}_1 \quad (65)$$

$$\omega_1^2(d_1 + d_2 + d_3) + d_1 d_2 + d_1 d_3 + d_2 d_3 = \omega_2^2\omega_3^2 + \omega_2^2\omega_4^2 + \omega_3^2\omega_4^2 + \bar{\zeta}_1(\omega_2^2 + \omega_3^2 + \omega_4^2) \quad (66)$$

$$\omega_1^2(d_1 d_2 + d_1 d_3 + d_2 d_3) + d_1 d_2 d_3 = \omega_2^2\omega_3^2\omega_4^2 + \bar{\zeta}_1(\omega_2^2\omega_3^2 + \omega_2^2\omega_4^2 + \omega_3^2\omega_4^2) \quad (67)$$

$$\omega_1^2 d_1 d_2 d_3 + C_1^2 \prod_{b\neq 1}(\omega_b^2 - \omega_1^2) = \bar{\zeta}_1\omega_2^2\omega_3^2\omega_4^2. \quad (68)$$

From here we find that

$$\bar{\zeta}_1 = \omega_1^2 - C_1^2, \quad (69)$$

and

$$d_1 + d_2 + d_3 = \omega_2^2 + \omega_3^2 + \omega_4^2 - C_1^2. \quad (70)$$

From the explicit form of the energy in terms of the parameters $\omega_a^2$ and $d_n$ one finds

$$\omega_1^2 + C_1^2 - E = 0. \quad (71)$$

On the other hand $E = \dfrac{1+\beta^2}{1-\beta^2}\,\kappa^2$, ( without loss of generality we set $\alpha = 1$) and hence, the following equality must hold

$$(\omega_1^2 + C_1^2 + \kappa^2)\beta^2 = \omega_1^2 + C_1^2 - \kappa^2. \quad (72)$$

As in the case of $S^5$, there are two solutions to this equation and the magnon solution corresponds to the choice $\kappa^2 = \omega_1^2 - C_1^2$, and then $\beta^2 = \frac{C_1^2}{\omega_1^2}$ .

Now the integrals we have to solve are

$$\int\limits_{\bar{\zeta}_1}^{\zeta_1} \frac{d\zeta}{(\zeta-\omega_2^2)(\zeta-\omega_3^2)(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int\limits_{\bar{\zeta}_2}^{\zeta_2} \frac{d\zeta}{(\zeta-\omega_2^2)(\zeta-\omega_3^2)(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int\limits_{\bar{\zeta}_3}^{\zeta_3} \frac{d\zeta}{(\zeta-\omega_2^2)(\zeta-\omega_3^2)(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} = 0 \tag{73}$$

$$\int\limits_{\bar{\zeta}_1}^{\zeta_1} \frac{\zeta d\zeta}{(\zeta-\omega_2^2)(\zeta-\omega_3^2)(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int\limits_{\bar{\zeta}_2}^{\zeta_2} \frac{\zeta d\zeta}{(\zeta-\omega_2^2)(\zeta-\omega_3^2)(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int\limits_{\bar{\zeta}_3}^{\zeta_3} \frac{\zeta d\zeta}{(\zeta-\omega_2^2)(\zeta-\omega_3^2)(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} = 0 \tag{74}$$

$$\int\limits_{\bar{\zeta}_1}^{\zeta_1} \frac{\zeta^2 d\zeta}{(\zeta-\omega_2^2)(\zeta-\omega_3^2)(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int\limits_{\bar{\zeta}_2}^{\zeta_2} \frac{\zeta^2 d\zeta}{(\zeta-\omega_2^2)(\zeta-\omega_3^2)(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int\limits_{\bar{\zeta}_3}^{\zeta_3} \frac{\zeta^2 d\zeta}{(\zeta-\omega_2^2)(\zeta-\omega_3^2)(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} = -2\frac{\xi}{(1-\beta^2)} \tag{75}$$

Now we combine appropriately (60-62) to find

$$\int_{\bar{\zeta}_1}^{\zeta_1} \frac{d\zeta}{(\zeta-\omega_2^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int_{\bar{\zeta}_2}^{\zeta_2} \frac{d\zeta}{(\zeta-\omega_2^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int_{\bar{\zeta}_3}^{\zeta_3} \frac{d\zeta}{(\zeta-\omega_2^2)\sqrt{\bar{\zeta}_1-\zeta}} = -2\frac{\xi}{(1-\beta^2)} \quad (76)$$

$$\int_{\bar{\zeta}_1}^{\zeta_1} \frac{d\zeta}{(\zeta-\omega_3^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int_{\bar{\zeta}_2}^{\zeta_2} \frac{d\zeta}{(\zeta-\omega_3^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int_{\bar{\zeta}_3}^{\zeta_3} \frac{d\zeta}{(\zeta-\omega_3^2)\sqrt{\bar{\zeta}_1-\zeta}} = -2\frac{\xi}{(1-\beta^2)} \quad (77)$$

$$\int_{\bar{\zeta}_1}^{\zeta_1} \frac{d\zeta}{(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int_{\bar{\zeta}_2}^{\zeta_2} \frac{d\zeta}{(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} + \int_{\bar{\zeta}_3}^{\zeta_3} \frac{d\zeta}{(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} = -2\frac{\xi}{(1-\beta^2)}. \quad (78)$$

The evaluation of the integrals strongly depends on where the lower limit of integration lies. In our case we have $\omega_2^2 < \bar{\zeta}_2 < \omega_3^2$ and $\omega_3^2 < \bar{\zeta}_3 < \omega_4^2$.

The integrals on the rhs of (76-78) we are dealing with can be solve as follows. First of all, we have

$$\int_{\zeta_1}^{\bar{\zeta}_1} \frac{d\zeta}{(\zeta-\omega_j^2)\sqrt{\bar{\zeta}_1-\zeta}} = \frac{2}{\sqrt{\bar{\zeta}_1-\omega_j^2}} \operatorname{arctanh}\frac{\sqrt{\bar{\zeta}_1-\zeta_1}}{\sqrt{\bar{\zeta}_1-\omega_j^2}}, \quad (79)$$

for $j = 2, 3, 4$.

Next, we have

$$\int_{\zeta_i}^{\bar{\zeta}_i} \frac{d\zeta}{(\zeta-\omega_2^2)\sqrt{\bar{\zeta}_1-\zeta}} = \frac{2}{\sqrt{\bar{\zeta}_1-\omega_2^2}} \left[\operatorname{arctanh}\frac{\sqrt{\bar{\zeta}_1-\omega_2^2}}{\sqrt{\bar{\zeta}_1-\zeta_i}} - \operatorname{arctanh}\frac{\sqrt{\bar{\zeta}_1-\omega_2^2}}{\sqrt{\bar{\zeta}_1-\bar{\zeta}_i}}\right], \quad (80)$$

for $i = 2, 3$.

The other three integrals we need are

$$\int_{\zeta_2}^{\bar{\zeta}_2} \frac{d\zeta}{(\zeta-\omega_3^2)\sqrt{\bar{\zeta}_1-\zeta}} = \frac{2}{\sqrt{\bar{\zeta}_1-\omega_3^2}}\left[\operatorname{arctanh}\frac{\sqrt{\bar{\zeta}_1-\zeta_2}}{\sqrt{\bar{\zeta}_1-\omega_3^2}} - \operatorname{arctanh}\frac{\sqrt{\bar{\zeta}_1-\bar{\zeta}_2}}{\sqrt{\bar{\zeta}_1-\omega_3^2}}\right], \quad (81)$$

$$\int_{\zeta_3}^{\bar{\zeta}_3} \frac{d\zeta}{(\zeta-\omega_3^2)\sqrt{\bar{\zeta}_1-\zeta}} = \frac{2}{\sqrt{\bar{\zeta}_1-\omega_3^2}}\left[\operatorname{arctanh}\frac{\sqrt{\bar{\zeta}_1-\omega_3^2}}{\sqrt{\bar{\zeta}_1-\zeta_3}} - \operatorname{arctanh}\frac{\sqrt{\bar{\zeta}_1-\omega_3^2}}{\sqrt{\bar{\zeta}_1-\bar{\zeta}_3}}\right], \quad (82)$$

$$\int_{\zeta_i}^{\bar{\zeta}_i} \frac{d\zeta}{(\zeta-\omega_4^2)\sqrt{\bar{\zeta}_1-\zeta}} = \frac{2}{\sqrt{\bar{\zeta}_1-\omega_4^2}}\left[\operatorname{arctanh}\frac{\sqrt{\bar{\zeta}_1-\zeta_i}}{\sqrt{\bar{\zeta}_1-\omega_4^2}} - \operatorname{arctanh}\frac{\sqrt{\bar{\zeta}_1-\bar{\zeta}_i}}{\sqrt{\bar{\zeta}_1-\omega_4^2}}\right], \quad (83)$$

for $i = 2, 3$.

To write the solutions of the system (76-78) in a compact form we define

$$\sqrt{\bar{\zeta}_1-\zeta_i} = s_i, \quad i = 1,\, 2,\, 3, \quad (84)$$

$$\sqrt{\bar{\zeta}_1-\omega_j^2} = b_j, \quad j = 2,\, 3,\, 4, \quad (85)$$

$$\sqrt{\bar{\zeta}_1-\bar{\zeta}_i} = a_i, \quad i = 1,\, 2,\, 3, \quad (\text{with } a_1 = 0\,). \quad (86)$$

The system we have to solve then can be written as

$$\frac{b_2^2(s_1+s_2+s_3)+s_1s_2s_3}{b_2^2+s_1s_2+s_1s_3+s_2s_3} = b_2A_2(\xi), \quad (87)$$

$$\frac{b_3^2(s_1+s_2+s_3)+s_1s_2s_3}{b_3^2+s_1s_2+s_1s_3+s_2s_3} = b_3A_3(\xi), \quad (88)$$

$$\frac{b_4^2(s_1+s_2+s_3)+s_1s_2s_3}{b_4^2+s_1s_2+s_1s_3+s_2s_3} = b_4A_4(\xi), \quad (89)$$

or

$$b_k^2(s_1+s_2+s_3)+s_1s_2s_3=b_kA_k(\xi)(b_k^2+s_1s_2+s_1s_3+s_2s_3),\ \ k=2,3,4. \quad (90)$$

In the above expressions we denoted by $A_j(\xi)$ and $B_j$ the following expressions:

$$A_2(\xi)=\tanh\left[-\frac{b_2\xi}{1-\beta^2}+B_2\right],\quad B_2=\operatorname{arctanh}\left[\frac{b_2(a_2+a_3)}{b_2^2+a_2a_3}\right], \quad (91)$$

$$A_3(\xi)=\coth\left[-\frac{b_3\xi}{1-\beta^2}+B_3\right],\quad B_3=\operatorname{arctanh}\left[\frac{b_3^2+a_2a_3}{b_3(a_2+a_3)}\right], \quad (92)$$

$$A_4(\xi)=\tanh\left[-\frac{b_4\xi}{1-\beta^2}+B_4\right],\quad B_4=\operatorname{arctanh}\left[\frac{b_4(a_2+a_3)}{b_4^2+a_2a_3}\right]. \quad (93)$$

In terms of these notations the solutions can be written in the form

$$\begin{aligned} r_a^2 &= \frac{(\zeta_1-\omega_a^2)(\zeta_2-\omega_a^2)(\zeta_3-\omega_a^2)}{\prod\limits_{b\neq a}(\omega_b^2-\omega_a^2)} \\ &= \frac{(b_a^2-s_1^2)(b_a^2-s_2^2)(b_a^2-s_3^2)}{\prod\limits_{b\neq a}(\omega_b^2-\omega_a^2)} \\ &= \frac{b_a^2[b_a^2+s_1s_2+s_1s_3+s_2s_3]^2-[b_a^2(s_1+s_2+s_3)+s_1s_2s_3]^2}{\prod\limits_{b\neq a}(\omega_b^2-\omega_a^2)} \\ a &= 2,\ 3,\ 4, \end{aligned} \quad (94)$$

or, using (90) we get

$$r_k^2=\frac{b_k^2\left(1-A_k^2(\xi)\right)\left(b_k^2+s_1s_2+s_1s_3+s_2s_3\right)^2}{\prod\limits_{b\neq k}(\omega_b^2-\omega_k^2)},\ \ k=2,\ 3,\ 4, \quad (95)$$

where

$$\begin{aligned} &s_1s_2+s_1s_3+s_2s_3 \\ &\quad=\frac{b_2^3(b_4^2-b_3^2)A_2(\xi)+b_3^3(b_2^2-b_4^2)A_3(\xi)+b_4^3(b_3^2-b_2^2)A_4(\xi)}{b_2(b_3^2-b_4^2)A_2(\xi)+b_3(b_4^2-b_2^2)A_3(\xi)+b_4(b_2^2-b_3^2)A_4(\xi)}. \end{aligned} \quad (96)$$

The final form of the multi-spin giant magnon solutions to our reduced integrable system can be written as

$$\begin{aligned} r_2^2(\xi) &= \frac{b_2^2(\omega_2^2-\omega_3^2)(\omega_2^2-\omega_4^2)}{(\omega_1^2-\omega_2^2)} \\ &\times\frac{[1-A_2^2(\xi)][b_3A_3(\xi)-b_4A_4(\xi)]^2}{[-b_2(\omega_3^2-\omega_4^2)A_2(\xi)+b_3(\omega_2^2-\omega_4^2)A_3(\xi)-b_4(\omega_2^2-\omega_3^2)A_4(\xi)]^2}, \end{aligned} \quad (97)$$

$$r_3^2(\xi) = \frac{b_3^2(\omega_3^2 - \omega_4^2)(\omega_2^2 - \omega_3^2)}{(\omega_1^2 - \omega_3^2)} \times \frac{[A_3^2(\xi) - 1][b_2 A_2(\xi) - b_4 A_4(\xi)]^2}{[-b_2(\omega_3^2 - \omega_4^2)A_2(\xi) + b_3(\omega_2^2 - \omega_4^2)A_3(\xi) - b_4(\omega_2^2 - \omega_3^2)A_4(\xi)]^2}, \quad (98)$$

$$r_4^2(\xi) = \frac{b_4^2(\omega_2^2 - \omega_4^2)(\omega_3^2 - \omega_4^2)}{(\omega_1^2 - \omega_4^2)} \times \frac{[1 - A_4^2(\xi)][b_2 A_2(\xi) - b_3 A_3(\xi)]^2}{[-b_2(\omega_3^2 - \omega_4^2)A_2(\xi) + b_3(\omega_2^2 - \omega_4^2)A_3(\xi) - b_4(\omega_2^2 - \omega_3^2)A_4(\xi)]^2}, \quad (99)$$

and

$$r_1^2(\xi) = 1 - \sum_{k=2}^{4} r_k^2(\xi). \quad (100)$$

## 5 Conclusions

In this paper we find and study multi-spin magnon solutions of strings on $AdS_4 \times S^7$ background. These solutions are very important for understanding the detailed mechanism of AdS/CFT correspondence. They correspond to long gauge theory operators, as discussed in the Introduction. The exact maps between dispersion relation of string solutions and the anomalous dimensions of the gauge theory operators is in the core of AdS/CFT correspondence and to establish the latter it should be extended to wider energy scales. Moreover, such studies give strong arguments for conjecturing the exact form of the scattering matrix governing the correspondence. The latter is based on integrable structures on both sides of the correspondence. In view to this, we approach the problem of finding certain type of solutions, magnon solutions, using a reduction to the Neumann-Rosochatius integrable system. It turns out that all magnon type solutions can be obtained from this system using a certain choice of the parameters describing the families of solutions, which we have demonstrated to be true for the class of solutions we are interested in.

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