

Aspects of Purely Transmitting Defects in Integrable Field Theories*

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Abstract. Some classical and quantum aspects of integrable defects are reviewed with particular emphasis on the behaviour of solitons in the sine-Gordon model.

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1 Introduction

The purpose of this paper is to review recent work exploring the properties of ‘integrable defects’ in 1+1 dimensional field theories. At first sight, the idea of an integrable defect appears preposterous because the simplest type of defect – of δ -function type, where a field is continuous across the defect but its first spatial derivative is discontinuous - appears to violate the property of integrability. This particular setup has been explored numerically (see, for example [1]) and, while there are various interesting effects associated with defects of this type, the conclusion appears to be that one needs to look elsewhere for an example that preserves integrability. In the quantum domain, integrable models with defects were considered even earlier [2] [3] and, assuming a reasonable-looking set of relations, which should be satisfied by the scattering matrix, reflection and transmission matrices, integrability appeared to be incompatible (in almost all cases) with having both reflection and transmission. There are alternative suggestions for the relations expressing the compatibility of reflection and transmission with the scattering matrix [4] but they will not be pursued here for the following reason: the types of defect considered will turn out classically to be purely transmitting and it does not appear to be necessary within this context to consider reflection and transmission together.

A common ‘defect’ in the domain of continuum mechanics is a ‘bore’ or ‘shock’ in which a field variable (the fluid velocity) has a discontinuity (in effect modelling rapid variations over a small region of space), yet other quantities, such

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as momentum density, are continuous. A shock is in a way more dramatic than a δ -impurity, since there is a discontinuity in the field itself (perhaps it should be regarded as a δ' -defect), yet it turns out to have a counterpart in the integrable domain. Describing this type of defect or shock, first from a Lagrangian view point, then exploring its classical and quantum properties, is the purpose of this review. The main references, as far as this talk is concerned, are the articles [5]- [10] where many of the ideas have been introduced. There have also been subsequent developments along similar lines and these may be found in [11] and [12], and, as will be seen in the quantum domain, the article [3] contains much of interest for the sine-Gordon model.

2 The Setup

The simplest situation has two scalar fields, $u(x, t)$, $x < x_0$ and $v(x, t)$, $x > x_0$, with a Lagrangian density given formally by

$$\mathcal{L} = \theta(x_0 - x)\mathcal{L}_u + \theta(x - x_0)\mathcal{L}_v + \delta(x - x_0)\mathcal{B}(u, v). \quad (1)$$

The first two terms are the bulk Lagrangian densities for the fields u and v respectively, while the third term provides the sewing conditions. In principle, this term depends on u , v and their various derivatives evaluated at x_0 , but the interesting question is how to choose \mathcal{B} so that the resulting system remains integrable [5].

If u and v are free fields, there are many ways to choose \mathcal{B} . For example,

$$\mathcal{B}(u, v) = -\frac{1}{2}\sigma uv + \frac{1}{2}(u_x + v_x)(u - v), \quad (2)$$

with standard free-field choices for the bulk Lagrangians, with σ a free constant parameter, leads to the following set of field equations and sewing conditions:

$$\begin{aligned} (\partial^2 + m^2)u &= 0, & x < x_0, \\ (\partial^2 + m^2)v &= 0, & x > x_0, \\ u &= v, & x = x_0, \\ v_x - u_x &= \sigma u, & x = x_0, \end{aligned} \quad (3)$$

implying the fields are continuous with a discontinuity in the derivative controlled by the parameter σ . This is an example of a δ -impurity. Typically, the sewing conditions at $x = x_0$ lead to reflection and transmission and, sometimes (for $\sigma < 0$) a bound state. However, if the fields on either side have nonlinear but integrable interactions (e.g. each is a sine-Gordon field), the δ -impurity destroys the integrability, as mentioned before [1].

If both u and v are described in the bulk by integrable nonlinear wave equations, for example they are both sine-Gordon fields, a suitable choice of Lagrangian

would be to take

$$\begin{aligned}\mathcal{B}(u, v) &= \frac{1}{2}(vu_t - uv_t) + \mathcal{D}(u, v), \\ \mathcal{D}(u, v) &= -2\left(\sigma \cos \frac{u+v}{2} + \frac{1}{\sigma} \cos \frac{u-v}{2}\right)\end{aligned}\quad (4)$$

leading to the set of equations

$$\begin{aligned}\partial^2 u &= -\sin u, & x < x_0, \\ \partial^2 v &= -\sin v, & x > x_0 \\ u_x &= v_t - \sigma \sin \frac{u+v}{2} - \frac{1}{\sigma} \sin \frac{u-v}{2}, & x = x_0 \\ v_x &= u_t + \sigma \sin \frac{u+v}{2} - \frac{1}{\sigma} \sin \frac{u-v}{2}, & x = x_0.\end{aligned}\quad (5)$$

This setup is not at all the same as the δ -impurity because, typically, $u(x_0, t) - v(x_0, t) \neq 0$, implying a discontinuity in the fields at x_0 . Clearly, the equations (5) describe a ‘defect’ although the ‘physical’ details of the defect are hidden in the sewing conditions at x_0 . Note also that the sewing conditions are strongly reminiscent of a Bäcklund transformation, and would be a Bäcklund transformation if they were not ‘frozen’ at $x = x_0$ (see, for example, [13]). The fact the spatial derivatives are evaluated at a specific location implies that one cannot eliminate u in favour of v , or vice-versa, the usual trick associated with Bäcklund transformations. The setup was not supposed to be obvious and was originally determined by examining the spin ± 3 conserved charges within the sine-Gordon model and demanding the energy-like combination was preserved [5]. That the setup is integrable is indicated strongly in [5] and [6] by constructing Lax pairs using techniques similar to those described in [14] applicable to field theories restricted to a half-line.

Since the sewing conditions (5) are local, it is clear there could be many defects, with parameters σ_i , located at x_i along the x -axis.

As an exercise, it is worth looking at the linear approximation to the sine-Gordon equations and defect sewing conditions (5), and investigating what happens to a plane wave as it approaches the defect. Perhaps surprisingly, there turns out to be no reflected component and the transmitted wave collects an additional phase; there is no bound state for any value of σ .

2.1 Energy and Momentum

Although the first analysis of this problem concentrated on preserving the energy-like combination of spin ± 3 charges, it was natural to go back and consider energy and momentum, which are combinations of spin ± 1 charges. Despite the conditions (5) generally implying a discontinuity in the fields at the defect it might still be the case that the defect could exchange both energy and

momentum with the defect itself. If so, the defect would indeed be an integrable analogue of a shock.

Clearly, time translation invariance is not violated by the defect and therefore there is a conserved energy, which will include a contribution from the defect itself. On the other hand, space translation is violated by a defect located at a specific point and therefore momentum would not be expected to be conserved, even allowing for a contribution from the defect. This aspect was also treated in [5] and investigated there in a general context using the quantity $\mathcal{D}(u, v)$ appearing in (4). Surprisingly, including a suitably chosen defect contribution does lead to a conserved momentum (though this would not be possible for a δ -impurity).

The momentum carried by the fields on either side of the defect is given by

$$P = \int_{-\infty}^{x_0} dx u_x u_t + \int_{x_0}^{\infty} dx v_x v_t \quad (6)$$

and it is necessary to check the extent to which this fails to be conserved. Using the defect conditions coming from (4),

$$u_x = v_t - \frac{\partial \mathcal{D}}{\partial u}, \quad v_x = u_t + \frac{\partial \mathcal{D}}{\partial v}, \quad \text{at } x = x_0, \quad (7)$$

one finds

$$\dot{P} = \left[-v_t \frac{\partial \mathcal{D}}{\partial u} - u_t \frac{\partial \mathcal{D}}{\partial v} - V(u) + V(v) + \frac{1}{2} \left(\frac{\partial \mathcal{D}}{\partial u} \right)^2 - \frac{1}{2} \left(\frac{\partial \mathcal{D}}{\partial v} \right)^2 \right]_{x_0}. \quad (8)$$

In this expression, the fields on either side of the defect have been allowed to have more general potentials (and a further generalization would have a different potential for each field). Clearly, (8) is not generally a total time-derivative of a functional of the two fields evaluated at x_0 although that is what it is required to be in order to be able to construct a momentum contribution located at the defect. However, it will be provided the first two terms are a total time derivative and the other terms exactly cancel. In other words, the quantity \mathcal{D} and the bulk potentials must satisfy

$$\frac{\partial^2 \mathcal{D}}{\partial u^2} = \frac{\partial^2 \mathcal{D}}{\partial v^2}, \quad \frac{1}{2} \left(\frac{\partial \mathcal{D}}{\partial u} \right)^2 - \frac{1}{2} \left(\frac{\partial \mathcal{D}}{\partial v} \right)^2 = V(u) - V(v). \quad (9)$$

This set of conditions is satisfied by the sine-Gordon defect function (5). However, there are other solutions too, including cases with several scalar fields [5,6]. An intriguing feature is that the requirements of integrability, as expressed by analysing higher spin charges or by constructing suitable Lax pairs, are the same as the requirements necessary for a modified conserved momentum, at least for the sine-Gordon model. There is some evidence that not all the higher spin conserved charges are preservable for more general affine Toda field theories (see [6] where an example of an even spin charge is analysed in detail).

2.2 Classical Scattering of Solitons

As mentioned previously, it is easy to verify that the free-field limit of the sine-Gordon defect setup, provided by

$$\mathcal{D}(u, v) \rightarrow \frac{\sigma}{4}(u+v)^2 + \frac{1}{4\sigma}(u-v)^2, \quad (10)$$

together with quadratic limits of the bulk potentials, leads to conditions describing a purely transmitting defect. And, in any case it is straightforward to check directly that (10) satisfies the above conditions (9). Given this, it is natural to ask what happens to solitons in the full nonlinear sine-Gordon model as they encounter a defect. (For a treatise on solitons, see for example [15].)

A soliton travelling in the positive x direction (rapidity θ) in the absence of any defect is given by the expression

$$e^{iu/2} = \frac{1 + iE}{1 - iE}, \quad (11)$$

where

$$E = e^{ax+bt+c}, \quad a = \cosh \theta, \quad b = -\sinh \theta, \quad \text{with } e^c \text{ real.} \quad (12)$$

The expression (11) provides a real solution to the sine-Gordon equation that smoothly interpolates between $u(t, -\infty) = 0$ and $u(t, \infty) = 2\pi$. If on the right hand side of (11) E is replaced by $-E$ the resulting solution smoothly interpolates between $u(t, -\infty) = 2\pi$ and $u(t, \infty) = 0$, and is called an anti-soliton. (Note that shifting u everywhere by an integer multiple of 2π is an invariance of the sine-Gordon equation and the soliton and anti-soliton could be regarded alternatively as interpolating adjacent ‘constant solutions’, or ‘vacua’.)

If there is a defect then it turns out that a soliton meeting the defect will generally pass right through (that is, it is purely transmitted) and the solutions on either side of the defect will be given by

$$e^{iu/2} = \frac{1 + iE}{1 - iE}, \quad x < x_0; \quad e^{iv/2} = \frac{1 + izE}{1 - izE}, \quad x > x_0, \quad (13)$$

where z is determined by the sewing conditions. Trying to allow more general solutions (for example, to include a reflected and, therefore, additional soliton at late times) does not satisfy the defect sewing conditions. The defect conditions (5) satisfied requires the parameter z to be

$$z = \frac{e^{-\theta} + \sigma}{e^{-\theta} - \sigma} \equiv \coth \left(\frac{\eta - \theta}{2} \right), \quad (14)$$

where $\sigma = e^{-\eta}$.

One of the remarkable properties of sine-Gordon solitons is that in the absence of any defect two solitons originally ordered with the slower ahead of the faster will, over time, reorder to a situation in which the faster is eventually ahead. This property has been known since the 1960s (see, for example, [15]). This scattering process is registered by a positive ‘delay’, given by z^2 if a soliton of rapidity θ passes another of rapidity η .

However, when a soliton encounters a defect the quantity z given by (14) may change sign, implying a soliton might, depending on the sign of $\theta - \eta$, convert to an anti-soliton, be delayed, or even absorbed. In the latter case, the defect would gain a unit of topological charge in addition to storing the energy and momentum of the soliton; in the former, the defect gains (or loses) two units of topological charge, as measured by the difference $v(t, x_0) - u(t, x_0)$. In other words, a discontinuity at the location of the defect would be created in these cases. Because the defect potential has period 4π , all evenly charged defects have identical energy–momentum, as do all oddly charged defects.

A fascinating possibility associated with this type of defect (if it could be realized in practice) would be the capacity to control solitons. For example, in [7] it was pointed out how solitons together with a defect might be used to mimic a Toffoli universal logic gate. This idea is theoretically viable using two basic ingredients in addition to the above features. First, it should be agreed that a soliton represents the bit ‘1’ and an anti-soliton represents ‘0’; second, there needs to be a feedback mechanism that increases the defect parameter if a soliton passes, but not if an anti-soliton passes. The Toffoli gate manipulates three incoming bits in such a way that the third bit is flipped providing the first two bits were set to ‘1’ but not otherwise. The Toffoli gate is universal because it can be used in combinations to create all other gates. Of course, this cannot be other than a cute notion unless a physical situation is found where the conditions of the integrable defect are met.

Several defects affect progressing solitons independently and several solitons approaching a defect (inevitably possessing different rapidities) are affected independently, with at most one of the components being absorbed (and this fact was already alluded to in the previous paragraph). Notice, too, that the situation is not time-reversal invariant owing to the presence of explicit time derivatives in eqs(5). Also, one might imagine that starting with an odd charged defect energy–momentum conservation would permit a single soliton to emerge. However, within the classical picture this cannot happen because there is nothing to determine the origin of time for the emerging soliton (that is, nothing determines the time at which the decay would occur). Associated with this is an interesting question for quantum mechanics. In the quantised theory one might expect to calculate, starting with a suitably prepared defect carrying energy and momentum, a probability for its decay at any specified time. In a sense, this is what happens but to see how one needs to explore the properties of the transmission matrix within the sine-Gordon quantum field theory (see the next Section).

As a final comment, for which there is no space for details, it is worth pointing out that it is possible to generalise the setup to allow for moving defects [8]. In the classical picture a defect located at $x = p(t)$ satisfies $\ddot{p} = 0$. In other words it may move with constant speed or remain at rest; effectively, the defect experiences no forces as a consequence of its interaction with the fields on either side of it. Two defects moving with constant but differing speeds will change places eventually, or ‘scatter’, if the slower defect is ahead to start with. However, a soliton encountering the pair will be influenced in a manner independent of the ordering of the defects. In a way, it is natural that the defect experiences no force: if it did there would need to be an explanation of its mass. On the other hand, in the quantum domain the mass could be generated by quantum effects and the scattering of two defects might turn out to be interesting. So far, although there is a candidate for the S-matrix to describe the scattering of two defects [8] it is not yet clear that it can satisfy all the additional requirements of unitarity, crossing, and so on.

3 Quantum Picture

The sine-Gordon quantum field theory has been well-studied for many years and much is known about the bulk theory and the theory confined to a half-line; rather less is known about the theory confined to an interval. As far as this talk is concerned the essential ingredient needed from the bulk theory is Zamolodchikov’s soliton-soliton S-matrix, which will be used in the following form (for example, see the review [16]):

$$S_{kl}^{mn}(\Theta) = \rho(\Theta) \begin{pmatrix} a(\Theta) & 0 & 0 & 0 \\ 0 & c(\Theta) & b(\Theta) & 0 \\ 0 & b(\Theta) & c(\Theta) & 0 \\ 0 & 0 & 0 & a(\Theta) \end{pmatrix}, \quad (15)$$

where k, l label the incoming particles and m, n label the outgoing particles in a two-body scattering process, with the particles labelled k, n having rapidity θ_1 , and the particles labelled l, m having rapidity θ_2 . The various pieces of the matrix are defined by

$$a(\Theta) = \frac{qx_2}{x_1} - \frac{x_1}{qx_2}, \quad b(\Theta) = \frac{x_1}{x_2} - \frac{x_2}{x_1}, \quad c(\Theta) = q - \frac{1}{q}, \quad (16)$$

with

$$\Theta = \theta_1 - \theta_2, \quad q = e^{i\pi\gamma}, \quad x_p = e^{\gamma\theta_p}. \quad (17)$$

In this notation the crossing property of the S-matrix is represented by

$$S_{kl}^{mn}(i\pi - \Theta) = S_{k-m}^{-l n}(\Theta), \quad (18)$$

with the diagonal elements $S_{+-}^{+-}(\Theta)$ and $S_{-+}^{-+}(\Theta)$ crossing into themselves. The overall factor $\rho(\Theta)$ will be needed later and is given by

$$\rho(\Theta) = \frac{\Gamma(1 + i\gamma\Theta/\pi)\Gamma(1 - \gamma - i\gamma\Theta/\pi)}{2\pi i} \prod_{k=1}^{\infty} R_k(\Theta)R_k(i\pi - \Theta), \quad (19)$$

where

$$R_k(\Theta) = \frac{\Gamma(2k\gamma + i\gamma\Theta/\pi)\Gamma(1 + 2k\gamma + i\gamma\Theta/\pi)}{\Gamma((2k + 1)\gamma + i\gamma\Theta/\pi)\Gamma(1 + (2k - 1)\gamma + i\gamma\Theta/\pi)}. \quad (20)$$

Note, the conventions adopted by Konik and LeClair [3] have been used. Therefore, in particular, the coupling γ in terms of the Lagrangian coupling β is defined by

$$\frac{1}{\gamma} = \frac{\beta^2}{8\pi - \beta^2}. \quad (21)$$

Where $\hbar = 1$ and the conventions are those associated with the bulk Lagrangian

$$\mathcal{L} = \frac{1}{2} ((\partial_t u)^2 - (\partial_x u)^2) - \frac{m^2}{\beta^2} (1 - \cos \beta u). \quad (22)$$

3.1 The Transmission Matrix

Following the remarks in the previous Section, concerning the classical scattering of a soliton by a defect, where the topological charge of the defect will typically change by two units at a time, one expects two types of transmission matrix, one of them, ${}^{\text{even}}T$, referring to even-labelled defects and the other, ${}^{\text{odd}}T$, referring to odd-labelled defects. On the assumption that the lowest energy state corresponds to no discontinuity at all, the former is expected to be unitary while the latter is expected to be related to the absorption of a soliton, and not expected to be unitary since the excited defect is expected to decay quantum mechanically. Rather, ${}^{\text{odd}}T$ is expected to be related (*via* a bootstrap principle) to a complex bound state pole in ${}^{\text{even}}T$. In fact this is precisely what happens. It is worth remarking that the relevant transmission matrices were in fact described by Konik and LeClair some time ago [3], although no distinction between odd and even labels was made, nor did those authors note the complex bound state.

It is convenient to use roman labels to denote soliton states (taking the value ± 1), and Greek labels to label the charge on a defect. Then, assuming topological charge is conserved in every process, it is expected that both transmission matrices will satisfy ‘triangle’ compatibility relations with the bulk S -matrix, for example

$$S_{ab}^{cd}(\theta_1 - \theta_2) T_{d\alpha}^{f\beta}(\theta_1) T_{c\beta}^{e\gamma}(\theta_2) = T_{b\alpha}^{d\beta}(\theta_2) T_{a\beta}^{c\gamma}(\theta_1) S_{cd}^{ef}(\theta_1 - \theta_2). \quad (23)$$

Here, it is supposed the solitons are travelling along the positive x -axis ($\theta_1 > \theta_2 > 0$). The bulk S -matrix depends on the bulk coupling β via the quantity $\gamma = 8\pi/\beta^2 - 1$, and the conventions used are those adopted in [8]. The equations (23) are well known in many contexts involving the notion of integrability (see [17]), but were discussed first with reference to defects by Delfino, Mussardo and Simonetti [2]; if the possibility of reflection was to be allowed an alternative framework (such as the one developed by Mintchev, Ragoucy and Sorba [4]), might be more appropriate. However, in the present case the defect is expected to be purely transmitting.

The solution (for general β , and for even or odd labelled defects — note the labelling is never mixed by (23)) — is given by

$$T_{a\alpha}^{b\beta}(\theta) = f(q, x) \begin{pmatrix} \nu^{-1/2} Q^\alpha \delta_\alpha^\beta & q^{-1/2} e^{\gamma(\theta-\eta)} \delta_\alpha^{\beta-2} \\ q^{-1/2} e^{\gamma(\theta-\eta)} \delta_\alpha^{\beta+2} & \nu^{1/2} Q^{-\alpha} \delta_\alpha^\beta \end{pmatrix}. \quad (24)$$

The solution was derived in [8] and found to be essentially unique and equivalent to the earlier result of Konik and LeClair. A block form has been adopted with the labels a, b labelling the four block elements on the right hand side, and where ν is a free parameter, as is η (to be identified with the defect parameter introduced in the previous Section), and

$$q = e^{i\pi\gamma}, \quad x = e^{\gamma\theta}, \quad Q^2 = -q = e^{4\pi^2 i/\beta^2}. \quad (25)$$

In addition, ${}^{\text{even}}T$ is a unitary matrix (for real θ), and both types of transmission matrix must be compatible with soliton–anti-soliton annihilation as a virtual process. Here, the thinking is equivalent to that of Konik and LeClair, but expressed rather differently. This transmission matrix represents a process with the incoming particle meeting the defect from the left and the process with a particle arriving from the right will be different (though related by crossing). These two requirements place the following restrictions on the overall factor for the even transmission matrix, ${}^e f(q, x)$, (and henceforth $e \equiv$ ‘even’):

$$\begin{cases} {}^e \bar{f}(q, x) = {}^e f(q, qx), \\ {}^e f(q, x) {}^e f(q, qx) (1 + e^{2\gamma(\theta-\eta)}) = 1. \end{cases} \quad (26)$$

These do not determine ${}^e f(q, x)$ uniquely but the ‘minimal’ solution determined by Konik–LeClair has

$${}^e f(q, x) = \frac{e^{i\pi(1+\gamma)/4} r(x)}{1 + i e^{\gamma(\theta-\eta)} \bar{r}(x)}, \quad (27)$$

with ($z = i\gamma(\theta - \eta)/2\pi$),

$$r(x) = \prod_{k=0}^{\infty} \frac{\Gamma(k\gamma + \frac{1}{4} - z) \Gamma((k+1)\gamma + \frac{3}{4} - z)}{\Gamma((k + \frac{1}{2})\gamma + \frac{1}{4} - z) \Gamma((k + \frac{1}{2})\gamma + \frac{3}{4} - z)}. \quad (28)$$

It is worth noting that the apparent pole in (27) at $1 + ie^{\gamma(\theta-\eta)} = 0$ is actually cancelled by a pole at the same location in $\bar{r}(x)$. However, there is another pole at

$$\theta = \eta - \frac{i\pi}{2\gamma} \rightarrow \eta \text{ as } \beta \rightarrow 0, \quad (29)$$

uncancelled by a zero, and this does actually represent the expected unstable bound state alluded to in the first Section.

Several brief remarks are in order. It is clear, on examining (24), that the processes in which a classical soliton would inevitably convert to an anti-soliton are clearly dominant even in the quantum theory, yet suppressed if a classical soliton is merely delayed. This much is guaranteed by the factor $e^{\gamma(\theta-\eta)}$ appearing in the off-diagonal terms. A curious feature is the different way solitons and anti-solitons are treated by the diagonal terms in (24). They are treated identically by the bulk S -matrix yet one should not be surprised by differences shown up in the transmission matrix since the classical defect conditions (5) do not respect all the usual discrete symmetries. Indeed, the dependence of the diagonal entries on the bulk coupling can be demonstrated to follow from the classical picture by using a functional integral type of argument, as explained more fully in [8].

The sine-Gordon spectrum contains bound states (breathers), and it is interesting to calculate their transmission factors. This much has been done [8]. One interesting fact is that the ‘transmission factor’ for the lightest breather has precisely the same form as the transmission factor in the linearised version of the model (the exercise set earlier). This strongly suggests it would also be interesting to attempt to match these breather transmission factors to perturbative calculations. However, this has not yet been done.

There are also open questions concerning how to treat defects in motion. From a classical perspective it seems quite natural that defects might move and scatter [8], however it is less clear how to describe this in the quantum field theory, although an attempt has already been made to do so, or indeed to understand what these objects really are. For example could they also be described by local fields? Might they correspond to objects that are limiting cases in the standard theory? For example, it is well-known there is no two-soliton solution where the two solitons share the same rapidity, yet in the standard theory their rapidities can be arbitrarily close.

4 The Affine Toda Field Theories

This Section will focus on a subset of affine Toda field theories [18], namely those associated with the root data of the Lie algebras a_r . Apart from having the most symmetrical root/weight systems, these are the models for which classically integrable defects have been described in detail [6], whose complex solitons are easy to describe [19], and whose full set of S -matrices are relatively

easy to calculate using the bootstrap [20].

In the bulk, $-\infty < x < \infty$, an affine Toda field theory corresponding to the root data of the Lie algebra a_r is described conveniently by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{m^2}{\beta^2} \sum_{j=0}^r (e^{\beta \alpha_j \cdot \phi} - 1), \quad (30)$$

where m and β are constants, and r is the rank of the algebra. The vectors α_j with $j = 1, \dots, r$ are simple roots (with the convention $|\alpha_j|^2 = 2$), and α_0 is the lowest root, defined by

$$\alpha_0 = - \sum_{j=1}^r \alpha_j.$$

The field $\phi = (\phi_1, \phi_2, \dots, \phi_r)$ takes values in the r -dimensional Euclidean space spanned by the simple roots $\{\alpha_j\}$. The extra root α_0 distinguishes between the massive affine and the massless non-affine Toda field theories. The massive affine theories are integrable, possessing infinitely many conserved charges, a Lax pair representation, and many other interesting properties, both classically and in the quantum domain. The simplest choice ($r = 1$) coincides with the sinh-Gordon model. For further details concerning the affine Toda field theories, see [18] where further references can be found.

After quantisation, provided the coupling constant β is real, and the fields are restricted to be real, the a_r affine Toda field theory describes r interacting scalars, also known as fundamental Toda particles, whose classical mass parameters are given by

$$m_a = 2m \sin\left(\frac{\pi a}{h}\right), \quad a = 1, 2, \dots, r, \quad (31)$$

where $h = r + 1$ is the Coxeter number of the algebra. On the other hand, if the fields are permitted to be complex each affine Toda field theory possesses classical ‘soliton’ solutions [19]. Conventionally, complex affine Toda field theories are described by the Lagrangian density (30) in which the coupling constant β is replaced with $i\beta$. Once complex fields are allowed it is clear that the potential appearing in the Lagrangian density (30) vanishes whenever the field ϕ is constant and equal to

$$\phi = \frac{2\pi w}{\beta} \quad \text{with} \quad \alpha_j \cdot w \in \mathbf{Z}, \quad \text{i.e.} \quad w \in \Lambda_W(a_r), \quad (32)$$

where $\Lambda_W(a_r)$ is the weight lattice of the Lie algebra a_r . These constant field configurations have zero energy and correspond to stationary points of the affine Toda potential. Soliton solutions smoothly interpolate between these vacuum configurations as x runs from $-\infty$ to ∞ . It is natural to define the ‘topological charges’ characterizing such solutions as follows:

$$Q = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \phi = \frac{\beta}{2\pi} [\phi(\infty, t) - \phi(-\infty, t)], \quad (33)$$

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and these lie in the weight lattice $\Lambda_W(a_r)$. Assuming $\phi(-\infty, t) = 0$, static solitons may be found for which $\phi(\infty, t)$ lies in a subset of the weight lattice. In particular, there are static solutions corresponding to weights within each of the representations with the highest weight w_a , $a = 1, \dots, r$, satisfying

$$\alpha_i \cdot w_a = \delta_{ia}, \quad i, a = 1, \dots, r. \tag{34}$$

Explicitly boosted solutions of this type that correspond to the representation labelled by a have the form

$$\phi^{(a)} = \frac{m^2 i}{\beta} \sum_{j=0}^r \alpha_j \ln(1 + E_a \omega^{aj}), \quad E_a = e^{a_a x - b_a t + \xi_a}, \quad \omega = e^{2\pi i/h}, \tag{35}$$

where $(a_a, b_a) = m_a (\cosh \theta, \sinh \theta)$, ξ_a is a complex parameter, and θ is the soliton rapidity. Despite the solutions (35) being complex, Hollowood [19] has shown their total energy and momentum is actually real and requires masses for static single solitons proportional to the mass parameters of the real scalar theory. These are given by

$$M_a = \frac{2h m_a}{\beta^2}, \quad a = 1, 2, \dots, r. \tag{36}$$

Moreover, for each $a = 1, \dots, r$ there are several solitons whose topological charges lie in the set of weights of the fundamental a^{th} representation of a_r [21]. However, apart from the two extreme cases, $a = 1$ and $a = r$, not every weight belonging to one of the other representations corresponds to a static soliton (still something of a mystery). The number of possible charges for the representation with label a is exactly equal to the greatest common divisor of a and h , the relevant weights being orbits of the Coxeter element, and explicit expressions for them may be found in [21]. The parameter ξ_a is almost arbitrary but clearly has to be chosen so that there are no singularities in the solution as x, t vary; shifting ξ_a by $2\pi i a/h$ changes the topological charge. For the two extreme representations (with $a = 1$ or $a = r$), it is clear repeated use of this translation steps the charges successively through the full set of weights.

There are several types of integrable defect for a_r affine Toda field theory and the distinctions between them are explained in [6]. A single defect located at $x = 0$ may be described by the following modified Lagrangian density:

$$\mathcal{L}_d = \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi + \delta(x)\mathcal{D}(\phi, \psi), \tag{37}$$

with

$$\mathcal{D}(\phi, \psi) = \frac{1}{2}(\phi \cdot E \partial_t \phi + \phi \cdot D \partial_t \psi - \partial_t \phi \cdot D \psi + \psi \cdot E \partial_t \psi) - \mathcal{B}(\phi, \psi), \tag{38}$$

where E is an antisymmetric matrix, $D = 1 - E$,

$$\mathcal{L}_\phi = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi + \frac{m^2}{\beta^2} \sum_{j=0}^r (e^{i\beta \alpha_j \cdot \phi} - 1), \tag{39}$$

and

$$\mathcal{B} = -\frac{m}{\beta^2} \sum_{j=0}^r \left(\sigma e^{i\beta\alpha_j \cdot (D^T \phi + D\psi)/2} + \frac{1}{\sigma} e^{i\beta\alpha_j \cdot D(\phi - \psi)/2} \right). \quad (40)$$

Here, ϕ and ψ are the fields on the left and on the right of the defect, respectively, and σ is the defect parameter. The matrix D satisfies the following constraints:

$$\alpha_k \cdot D\alpha_j = \begin{cases} 2 & k = j, \\ -2 & k = \pi(j), \\ 0 & \text{otherwise,} \end{cases} \quad D + D^T = 2, \quad (41)$$

where $\pi(j)$ indicates a permutation of the simple roots. Choosing the ‘clockwise’ cyclic permutation,

$$\alpha_{\pi(j)} = \alpha_{j-1}, \quad j = 1, \dots, r, \quad \alpha_{\pi(0)} = \alpha_r,$$

the set of constraints (41) is satisfied by the choice,

$$D = 2 \sum_{a=1}^r w_a (w_a - w_{a+1})^T, \quad (42)$$

where the vectors w_a , $a = 1, \dots, r$ are the fundamental highest weights of the Lie algebra a_r , with the added convention $w_0 \equiv w_{r+1} = 0$. Note, the ‘anticlockwise’ cyclic permutation used in [6] is effected by substituting the matrix (42) by its transpose.

Given the modified Lagrangian density (37) the corresponding equations of motion and defect conditions are, respectively,

$$\begin{aligned} \partial^2 \phi &= \frac{m^2 i}{\beta} \sum_{j=0}^r \alpha_j e^{i\beta\alpha_j \cdot \phi} & x < 0, \\ \partial^2 \psi &= \frac{m^2 i}{\beta} \sum_{j=0}^r \alpha_j e^{i\beta\alpha_j \cdot \psi} & x > 0, \end{aligned} \quad (43)$$

$$\begin{aligned} \partial_x \phi - E \partial_t \phi - D \partial_t \psi + \partial_\phi \mathcal{B} &= 0 & x = 0, \\ \partial_x \psi - D^T \partial_t \phi + E \partial_t \psi - \partial_\psi \mathcal{B} &= 0 & x = 0. \end{aligned} \quad (44)$$

Many basic properties of (44) have been noted elsewhere [6] [10]. Shifting the fields ϕ , ψ by roots yields another solution with the same energy and momentum. This is because both the bulk and defect potentials are invariant under the translations

$$\phi \rightarrow \phi + 2\pi r/\beta, \quad \psi \rightarrow \psi + 2\pi s/\beta, \quad (45)$$

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where r, s are any two elements of the root lattice. In particular, constant fields

$$(\phi, \psi) = 2\pi(r, s)/\beta \quad (46)$$

all have the same energy and momentum despite having a discontinuity at the location of the defect. Writing $\sigma = e^{-\eta}$, the energy-momentum of each of these configurations is

$$(\mathcal{E}_0, \mathcal{P}_0) = -\frac{2hm}{\beta^2}(\cosh \eta, -\sinh \eta). \quad (47)$$

Other constant configurations are possible and, because of the invariance under translations by roots, it is enough to consider configurations $(\phi, \psi) = 2\pi(w_p, w_q)/\beta$, where w_p, w_q are fundamental highest weights. As is the case for the sine-Gordon model there is a conserved momentum associated with the defect.

The system described by the Lagrangian density (37) is neither invariant under parity nor under time reversal. By convention, a soliton with positive rapidity will travel from the left to the right and, at some time, it will meet the defect located at $x = 0$. The soliton ψ emerging on the right will be similar to ϕ , but delayed. It is described by

$$\psi^{(a)} = \frac{m^2 i}{\beta} \sum_{j=0}^r \alpha_j \ln(1 + z_a E_a \omega^{aj}). \quad (48)$$

The expression for the delay z_a was derived in [6] for the ‘anticlockwise’ permutation. To obtain the delay for the present situation it is enough to send the a^{th} soliton to the $(h - a)^{\text{th}}$ soliton in the formula appearing in [6]. Therefore the delay is given by

$$z_a = \left(\frac{e^{-(\theta-\eta)} + i e^{-i\gamma_a}}{e^{-(\theta-\eta)} + i e^{i\gamma_a}} \right), \quad \gamma_a = \frac{\pi a}{h}. \quad (49)$$

The delay is generally complex with exceptions being self-conjugate solitons, corresponding to $a = h/2$ (with r odd), for which the delay is real. In such cases, the delay is equal to the delay found for the sine-Gordon model [5] described earlier.

Note also that the delays experienced by a soliton, labelled a , and its associated antisoliton, labelled $\bar{a} = h - a$, are complex conjugates since $z_{\bar{a}} = \bar{z}_a$. For this reason, solitons and antisolitons are expected to behave differently as they pass a defect.

It was also pointed out in [10] that the argument of the phase of the delay (49) is given by

$$\tan(\arg z_a) = - \left(\frac{\sin 2\gamma_a}{e^{-2(\theta-\eta)} + \cos 2\gamma_a} \right), \quad (50)$$

implying that the phase shift produced by the defect can vary between zero (as $\theta \rightarrow -\infty$) and $-2\gamma_a$ (as $\theta \rightarrow \infty$), decreasing if necessary through $-\pi/2$ if $\cos 2\gamma_a < 0$. On the other hand, the boundaries between the different topological charge sectors in terms of the imaginary part of ξ_a (eq(35)) are separated by exactly $2\gamma_a$. This means that a soliton might convert to one of the adjacent solitons as it passes the defect provided $\arg z_a$ is sufficiently large. In effect, the defect imposes a rather severe selection rule on the possible topological charges of the emerging soliton. In the quantised theory, it is expected that either the transition matrix has zeroes to reflect this selection rule, or severely suppressed matrix elements to represent tunnelling between classically disconnected configurations. In the sine-Gordon model such an effect was never evident because the basic representation includes just two states and transitions between them are always permitted.

The delay (49) diverges when

$$\theta = \eta + \frac{i\pi}{2} \left(1 - \frac{2a}{h}\right), \quad (51)$$

and, with the exception of self-conjugate solitons having $a = h/2$ (including the sine-Gordon model where $(a, h) = (1, 2)$), this implies a soliton with real rapidity cannot be absorbed by a defect. For the sine-Gordon model it has been noted already that a classical defect can absorb a soliton and, within the quantum theory, this phenomenon implies the existence of unstable bound states. Once the affine Toda field theories are quantised, however, poles in locations given by (51) may correspond to additional states that possess no classical counterpart. The positions of the poles are expected to depend on the coupling and it might be the case that there is a range of couplings for which a bound state exists without the range including the classical limit. A phenomenon rather like this does actually occur in the a_2 model and is reported in detail in [10].

More generally, the delay (49) satisfies a classical bootstrap [10] in the sense that when two particles a, b in the real quantum field theory have a bound state \bar{c} the corresponding pole in their S-matrix will occur at rapidities

$$\theta_a = \theta_c - i\bar{U}_{ac}^b, \quad \theta_b = \theta_c + i\bar{U}_{bc}^a, \quad (52)$$

and the corresponding delays (49) in the complex classical theory satisfy

$$z_a(\theta - i\bar{U}_{ac}^b) z_b(\theta + i\bar{U}_{bc}^a) = z_{\bar{c}}(\theta). \quad (53)$$

These observations motivated a study of the triangular equations (23) in the context of the a_2 affine Toda field theory. This is already substantially more intricate than the similar analysis for the sine-Gordon model since there are a number of independent solutions that need to be matched to the Lagrangian starting point *via* suitable semi-classical arguments and some that need to be discarded. The

solutions together with their interpretation, an analysis of the bootstrap and its compatibility with the triangular relations, an investigation of bound states and breather transmission factors may all be found in Ref. [10]. Unfortunately there is no room for these details here.

5 Concluding Remark

It is quite remarkable that the simple-looking question concerning integrable shocks asked at the beginning has led to an interesting avenue of enquiry. The specific question does not appear to have been explored previously, yet it links with earlier results, such as (24), and it is not yet exhausted. The next steps will be to start a classification of the triangular compatibility relations, first in the context of other a_r affine Toda field theories, then afterwards more generally. The transmission matrices are infinite dimensional, with components labelled by roots, as far as the defects are concerned, and weights for the solitons, and it will be interesting to see the variety of possibilities and how they contrive to match the classical data (if indeed they do). There are several puzzles to be resolved, one of them being that the a_r models appear to be special in the defect context [6]. In all models it remains to be seen if the defects themselves can be regarded consistently as scattering states. This appears to be entirely plausible in the sine-Gordon case, but not yet explored for any other systems.

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