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The $\mathcal{C}$-Operator for a 2×2 Matrix Hamiltonian

S. Meyur[1], **S. Debnath**[2]

[1]Tansuk Rai Ganapat Rai Khemka High School, 23, Rabindra Sarani, Liluah, Howrah-711204, India

[2]Department of Mathematics, Jadavpur University, Kolkata-700032, India

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Abstract. We obtain the charge conjugation operator and the positive definite operator for a 2×2 non-Hermitian matrix Hamiltonian.

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1 Introduction

In the past few years there has been a considerable work on non-Hermitian Hamiltonian. The main reason for this is that the energy eigenvalues of a number of complex potentials turned out to be real (at least partly), which contradicted the usual expectations regarding non-Hermitian systems. This unusual behavior of the energy eigenvalues was attributed to the so-called $\mathcal{PT}$-symmetry[1-10]. It was shown that the energy eigenvalues are real when $\mathcal{PT}$-symmetry is unbroken, whereas the eigenvalues occur in complex conjugate pairs when $\mathcal{PT}$-symmetry is spontaneously broken. At the unbroken $\mathcal{PT}$-symmetry stage, every eigenfunction of a Hamiltonian H is also an eigenfunction of the $\mathcal{PT}$-operator. This condition guarantees that the eigenvalues are real. However, a significant barrier to the physical interpretation of such theories was that the natural metric in Hilbert space $\mathcal{H}$ is indefinite. Recently, Bender and his co-workers[11] in their very noteworthy work have found that the class of non-Hermitian Hamiltonian having an unbroken $\mathcal{PT}$-symmetry also possesses a further symmetry, called complex linear operator $\mathcal{C}$, similar to the charge conjugation operator. The probabilistic interpretation of quantum mechanics can be restored with the construction of new inner product using the $\mathcal{CPT}$-symmetry. The operator $\mathcal{C}$ satisfies three important relations $[\mathcal{C}, \mathcal{PT}] = 0$, $\mathcal{C}^2 = \mathbf{1}$, $[\mathcal{C}, \mathcal{H}] = 0$, that is $\mathcal{C}$ is new time-independent $\mathcal{PT}$-symmetric reflection operator. This recipe gives the $\mathcal{C}$-operator as a product of the exponential of an antisymmetric Hermitian operator and the parity operator. The $\mathcal{PT}$ and $\mathcal{CPT}$ inner products [11,12] have been defined as

$$\langle\psi|\phi\rangle = [\mathcal{PT}\psi]^T \cdot \phi \tag{1}$$

and

$$\langle\psi|\phi\rangle = [\mathcal{CPT}\psi]^T \cdot \phi, \tag{2}$$

which is positive definite and for all $\psi, \phi \in \mathcal{H}$.

A complementary approach in constructing physically meaningful theories with non-Hermitian Hamiltonian admitting real spectrum is to introduce the notion of pseudo-Hermiticity[13]. A Hamiltonian is said to be η-pseudo Hermitian, if there exists a linear, Hermitian, invertible operator η for which it satisfies the relations $H^\dagger = \eta H \eta^{-1}$. The non-Hermitian Hamiltonians admitting real spectrum is shown to be pseudo-Hermitian and are invariant under an anti-linear symmetry which reduces to the standard $\mathcal{PT}$-symmetry for some cases. Mostafazadeh has claimed that the $\mathcal{PT}$ inner product is just the pseudo inner product and the $\mathcal{CPT}$ inner product is just the η_+(a positive definite operator) inner product[14]

$$\langle\psi,\phi\rangle_\eta = \langle\psi,\eta\phi\rangle \tag{3}$$

and

$$\langle\psi|\phi\rangle = \langle\psi,\eta_+\phi\rangle, \tag{4}$$

where $\langle\psi|\phi\rangle = \psi^\dagger \cdot \phi$. This construction contains both conventional Hermiticity$(\eta = 1)$ and $\mathcal{PT}$-symmetry$(\eta = \mathcal{P})$ as a special case. In our present article, we give a $\mathcal{PT}$-symmetric Hamiltonian and calculate the $\mathcal{C}$ operator. We also obtain the positive definite operator.

The plan of the paper is as follows. We obtain the eigenvalue and eigenfunctions of the matrix Hamiltonian in Section 2. In Section 3, we calculate the $\mathcal{C}$ operator for this Hamiltonian. Section 4 is kept for conclusions and discussions.

2 2×2 Matrix Hamiltonian

Hamiltonian with four free real parameters has discussed in[7,12]. In this paper, we consider the Hamiltonian

$$H = \begin{pmatrix} \lambda + \cosh_q(\alpha - i\beta) & i\sinh_q(\alpha - i\beta) \\ i\sinh_q(\alpha - i\beta) & \lambda - \cosh_q(\alpha - i\beta) \end{pmatrix}, \tag{5}$$

where λ, q, α and β are four free real parameters. Hamiltonian (5) is non-Hermitian and $\mathcal{PT}$-symmetric with respect to the parity

$$\mathcal{P} = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix} \tag{6}$$

and the time reversal operator acts as complex conjugation where the deformed hyperbolic functions are defined as

$$\sinh_q x = \frac{e^x - qe^{-x}}{2}, \cosh_q x = \frac{e^x + qe^{-x}}{2}, \tanh_q x = \frac{\sinh_q x}{\cosh_q x}.$$

The eigenvalues for H in (5) are

$$\psi_{\pm} = \lambda \pm \sqrt{q}, \quad (7)$$

where $q > 0$ is the region of unbroken $\mathcal{PT}$-symmetry. When $q > 0$, the eigenstates of the operators H and $\mathcal{PT}$ are

$$\langle \psi_+ | = \frac{1}{\sqrt{2(1-\sqrt{q}\,\mathrm{sech}_q\alpha)\sqrt{q}\,\mathrm{sech}_q\alpha}} \times \begin{pmatrix} \tanh_q \alpha \cos\frac{\beta}{2} - i(1-\sqrt{q}\,\mathrm{sech}_q\alpha)\sin\frac{\beta}{2} \\ \tanh_q \alpha \sin\frac{\beta}{2} + i(1-\sqrt{q}\,\mathrm{sech}_q\alpha)\cos\frac{\beta}{2} \end{pmatrix} \quad (8)$$

and

$$\langle \psi_- | = \frac{1}{\sqrt{2(1+\sqrt{q}\,\mathrm{sech}_q\alpha)\sqrt{q}\,\mathrm{sech}_q\alpha}} \times \begin{pmatrix} \tanh_q \alpha \cos\frac{\beta}{2} - i(1+\sqrt{q}\,\mathrm{sech}_q\alpha)\sin\frac{\beta}{2} \\ \tanh_q \alpha \sin\frac{\beta}{2} + i(1+\sqrt{q}\,\mathrm{sech}_q\alpha)\cos\frac{\beta}{2} \end{pmatrix}. \quad (9)$$

With respect to the inner product (1), the followings are true

$$\langle \psi_+ | \psi_+ \rangle = 1,\ \langle \psi_- | \psi_- \rangle = -1,\ \langle \psi_+ | \psi_- \rangle = \langle \psi_- | \psi_+ \rangle = 0. \quad (10)$$

The eigenvectors of $H^\dagger$ are $\psi_\pm^*$ since $H^\dagger = H^*$. Setting $\phi_\pm = \pm\psi_\pm^*$, we have two linearly independent eigenvectors of $H^\dagger$

$$\langle \phi_\pm | = \frac{1}{\sqrt{2(1-\sqrt{q}\,\mathrm{sech}_q\alpha)\sqrt{q}\,\mathrm{sech}_q\alpha}} \times \begin{pmatrix} \tanh_q \alpha \cos\frac{\beta}{2} + i(1\mp\sqrt{q}\,\mathrm{sech}_q\alpha)\sin\frac{\beta}{2} \\ \tanh_q \alpha \sin\frac{\beta}{2} - i(1\mp\sqrt{q}\,\mathrm{sech}_q\alpha)\cos\frac{\beta}{2} \end{pmatrix}. \quad (11)$$

3 $\mathcal{C}$-Operator

In the Hilbert space $\mathcal{H} = \mathbb{R}^2$, $\psi_\pm$ and $\phi_\pm$ form a complete bi-orthonormal system $\{\psi_\pm, \phi_\pm\}$ and $\psi_+ \cdot \phi_+^\dagger + \psi_- \cdot \phi_-^\dagger = I$, where I is the identity matrix. The charge conjugation and positive definite operator are given by [12,14]

$$\mathcal{C} = \psi_+ \cdot \phi_+^\dagger - \psi_- \cdot \phi_-^\dagger = \frac{1}{\sqrt{q}} \begin{pmatrix} \cosh_q(\alpha - i\beta) & i\sinh_q(\alpha - i\beta) \\ i\sinh_q(\alpha - i\beta) & -\cosh_q(\alpha - i\beta) \end{pmatrix} \quad (12)$$

and

$$\eta_+ = \phi_+ \cdot \phi_+^\dagger + \phi_- \cdot \phi_-^\dagger = \frac{1}{\sqrt{q}} \begin{pmatrix} \cosh_q \alpha & i\sinh_q \alpha \\ i\sinh_q \alpha & -\cosh_q \alpha \end{pmatrix}, \quad (13)$$

where we have used the relations

$$\sinh_q(x+iy) = \sinh_q x \cos y + i \cosh_q x \sin y$$

$$\cosh_q(x+iy) = \cosh_q x \cos y + i \sinh_q x \sin y.$$

With respect to the inner product (2) we have

$$\langle \psi_+ | \psi_+ \rangle = 1,\ \langle \psi_- | \psi_- \rangle = 1. \tag{14}$$

One can easily verify the following equations:

$$[\mathcal{C}, \mathcal{PT}] = 0,\ [\mathcal{C}, \mathcal{P}] \neq 0,\ [\mathcal{C}, \mathcal{T}] \neq 0,\ (\mathcal{CP}) = (\mathcal{CP})^*, \tag{15}$$

$$\mathcal{C}^2 = \mathbf{1}, \tag{16}$$

and

$$[\mathcal{C}, H] = 0. \tag{17}$$

4 Conclusion

In this paper, we have obtained the charge conjugation operator and positive definite operator. We have found that the Hamiltonian (5) coincides with the charge conjugation operator $\mathcal{C}$ for $q = 1$, $\lambda = 0$. We have also shown that the energy eigenvalues depend on the deformation parameter q.

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