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Schrödinger Equation with Modified Hulthén plus Scarf Potential

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Abstract. The general solutions of the Schrödinger equation with the modified Hulthén plus Scarf potential are obtained in terms of Jacobi polynomials by using Nikiforov-Uvarov method. We have also looked for exact solutions of the Schrödinger equation for the $\mathcal{PT}$ and non $\mathcal{PT}$-symmetric potentials of the kind mentioned above. The results are compared with ones obtained before.

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1 Introduction

The exact solution of the Schrödinger equation with any potential is an important subject in nonrelativistic quantum physics. Many authors have studied the Schrödinger equation for various potentials such as the Eckart potential[1], the Scarf potential[2], the Pöschl-Teller potential[3], the parabolic type potential[4], the Morse potential[5] *etc*. Different techniques have been used in solving the above mentioned potential cases. Recently, an alternative method which is known as the Nikiforov-Uvarov method has been introduced in solving the Schrödinger equation.

In this paper, our aim is to solve the Schrödinger equation for the generalized Hulthén plus Scarf potential *via* the Nikiforov-Uvarov method, for which the corresponding eigenvalue problem can be solved exactly. In this study we have discussed the $\mathcal{PT}$[6-15] and non $\mathcal{PT}$-symmetric potentials. We have deduced the energy eigenvalues of Hulthén potential[16] and Scarf potential[17] separately. We have also shown that for particular cases the result is consistent with ones obtained before.

The organization of this paper consists of eight sections: in Section 2, we review the Nikiforov-Uvarov method briefly. In Section 3, we consider the Hulthén plus

Scarf potential and obtain its eigenvalue and eigenfunctions for the corresponding Hamiltonian by the Nikiforov-Uvarov method. In Section 4, we study the solution of $\mathcal{PT}$-symmetric Hulthén potential plus Scarf potential. In Section 5, we discuss the solution of non $\mathcal{PT}$-symmetric Hulthén plus Scarf potential. In Sections 6 and 7, we obtain the eigenvalues and eigenfunctions of the Hulthén potential and the Scarf potential respectively. Section 8 contains some concluding remarks.

2 Nikiforov-Uvarov Method

The differential equations whose solutions are the special functions of hypergeometric type can be solved by using the Nikiforov-Uvarov method which has been developed by Nikiforov and Uvarov[18]. In this method, the one-dimensional Schrödinger equation is reduced to an equation by an appropriate coordinate transformation $x = x(s)$,

$$\psi''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials, at most of second degree, and $\widetilde{\tau}(s)$ is a polynomial, at most of first degree. In order to obtain a particular solution to Eq.(1), we set the following wave function as a multiple of two independent parts:

$$\psi(s) = \phi(s)y(s). \tag{2}$$

Using Eq. (1) and Eq. (2) we have

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0, \tag{3}$$

which demands that the following conditions be satisfied:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{4}$$

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s), \qquad \tau'(s) < 0. \tag{5}$$

The condition $\tau'(s) < 0$ helps to generate energy eigenvalues and corresponding eigenfunctions. The condition $\tau'(s) > 0$ has been widely discussed in [19]. The λ in Eq.(3) satisfies the following second-order differential equation:

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \qquad n = 0, 1, 2, \dots \tag{6}$$

The polynomial $\tau(s)$ with the parameter s and prime factors show the differentials at first degree be negative. It is to be noted that λ or λ_n are obtained from a particular solution of the form $y(s) = y_n(s)$ which is a polynomial of degree

n. The second part $y_n(s)$ of the wavefunction (2) is the hypergeometric-type function whose polynomial solutions are connected by Rodrigues relation

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n}[\sigma^n(s)\rho(s)], \tag{7}$$

where C_n is the normalization constant and the weight function $\rho(s)$ satisfies the relation as

$$\frac{d}{ds}[\sigma(s)\rho(s)] = \tau(s)\rho(s). \tag{8}$$

On the other hand, in order to find the eigenfunctions, $\phi_n(s)$ and $y_n(s)$ in Eqs.(4) and (7) and eigenvalues λ_n in (6), we need to calculate the functions

$$\pi(s) = \left(\frac{\sigma' - \widetilde{\tau}}{2}\right) \pm \sqrt{\left(\frac{\sigma' - \widetilde{\tau}}{2}\right)^2 - \widetilde{\sigma} + k\sigma} \tag{9}$$

$$k = \lambda - \pi'(s). \tag{10}$$

In principle, since $\pi(s)$ has to be a polynomial of degree at most one, the expression under the square root sign in Eq. (9) can be put into order to be the square of a polynomial of first degree [18], which is possible only if its discriminant is zero. Thus, the equation for k obtained from the solution of Eq. (9) can be further substituted in Eq. (10). In addition, the energy eigenvalues are obtained from Eqs. (6) and (10).

3 Modified Hulthén Potential Plus the Scarf Potential

The modified Hulthén potential plus the Scarf potential is given by

$$V(x) = -(V_1 + iV_2)\frac{e^{-2ax}}{1 - qe^{-2ax}} + V_3 \mathrm{cosech}_q^2 ax. \tag{11}$$

The Schrödinger equation becomes

$$\frac{d^2\psi}{dx^2} + \left[E + (V_1 + iV_2)\frac{e^{-2ax}}{1 - qe^{-2ax}} - V_3 \mathrm{cosech}_q^2 ax\right]\psi = 0, \tag{12}$$

where $\hbar = 2m = 1$. Setting the following notations:

$$\varepsilon = -\frac{E}{4a^2}, \qquad \alpha = \frac{V_1}{4a^2}(> 0), \qquad \beta = \frac{V_2}{4\alpha^2}(> 0)$$

$$\gamma = \frac{V_3}{4a^2}(> 0) \qquad \text{and} \qquad s = e^{-2ax} \tag{13}$$

with $\varepsilon > 0 (E < 0)$ for bound states, Eq. (12) becomes

$$\frac{d^2\psi}{ds^2} + \frac{1 - qs}{s - qs^2}\frac{d\psi}{ds} + \frac{1}{(s - qs^2)^2}\Big[- (q^2\varepsilon + (\alpha + i\beta)q)s^2$$

$$(\alpha + i\beta - 4\gamma + 2q\varepsilon)s - \varepsilon\Big]\psi = 0. \tag{14}$$

After the comparison of Eq. (14) with Eq. (1), we have

$$\widetilde{\tau}(s) = 1 - qs, \qquad \sigma(s) = s - qs^2,$$
$$\widetilde{\sigma}(s) = -(q^2\varepsilon + (\alpha + i\beta)q)s^2 + (\alpha + i\beta - 4\gamma + 2q\varepsilon)s - \varepsilon. \tag{15}$$

Substituting these polynomials into Eq. (9), we have

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2}\left[(2\sqrt{\varepsilon} - \mu P)qs - 2\sqrt{\varepsilon}\right] \tag{16}$$

if

$$k = (\alpha + i\beta) - 4\gamma + \mu q\sqrt{\varepsilon}P,$$

where

$$\mu = +1, -1 \qquad \text{and} \qquad P = \sqrt{1 + \frac{4V_3}{qa^2}}.$$

For bound state solutions, it is necessary to choose

$$\pi(s) = -\frac{qs}{2} - \frac{1}{2}\left[(2\sqrt{\varepsilon} - \mu P)qs - 2\sqrt{\varepsilon}\right] \tag{17}$$

if

$$k = (\alpha + i\beta) - 4\gamma + \mu q\sqrt{\varepsilon}P.$$

The following track in this selection is to achieve the condition $\tau'(s) < 0$ Therefore $\tau(s)$ becomes

$$\tau(s) = 1 + 2\sqrt{\varepsilon} - \left[2 + 2\sqrt{\varepsilon} - \mu P\right] qs \tag{18}$$

and then its negative derivatives become

$$\tau'(s) = -\left[2 + 2\sqrt{\varepsilon} - \mu P\right] q. \tag{19}$$

A particular solution can be obtained by using Eqs. (6), (10), (17) and (18)

$$\lambda = \lambda_n = n\left[2 + 2\sqrt{\varepsilon} - \mu P\right] q + n(n-1)q \tag{20}$$

and

$$\lambda = (\alpha + i\beta) - 4\gamma + \mu q\sqrt{\varepsilon}P - \frac{q}{2} - \frac{q}{2}(2\sqrt{\varepsilon} - \mu P). \tag{21}$$

Comparing Eqs. (20) and (21), we have

$$n\left[2 + 2\sqrt{\varepsilon} - \mu P\right] q + n(n-1)q = (\alpha + i\beta) - 4\gamma + \mu q\sqrt{\varepsilon}P - \frac{q}{2} - \frac{q}{2}(2\sqrt{\varepsilon} - \mu P)$$

$$\Rightarrow (2n + 1 - \mu P)\sqrt{\varepsilon} + n(n+1) + \frac{1}{2} - \left(n + \frac{1}{2}\right)\mu P = \frac{(\alpha + i\beta) - 4\gamma}{q}$$

$$\Rightarrow 4(2n+1-\mu P)\sqrt{\varepsilon}+(2n+1-\mu P)^2=\frac{4(\alpha+i\beta)-16\gamma}{q}-1+P^2.$$

Substituting the values of α, β, γ, ε and P we obtain the energy eigenvalues:

$$E_n=-\frac{a^2}{4}\left[\left(2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}}\right)-\frac{(V_1+iV_2)}{qa^2\left(2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}}\right)}\right]^2, \quad (22)$$

where $(n=0,1,2,.......)$, $q\geq 1$. From Eqs. (5), (8), and (15) we obtain the weight function

$$\rho(s)=s^{2\sqrt{\varepsilon}}(1-qs)^{-\mu P} \quad (23)$$

and from Eqs. (4), (15), and (17), we have

$$\phi(s)=s^{\sqrt{\varepsilon}}\,(1-qs)^{\frac{1}{2}(1-\mu P)}. \quad (24)$$

Now using the properties of Jacobi Polynomial [20,21]

$$P_n^{(c,d)}(x)=\frac{(-1)^n(1-x)^{-c}(1+x)^{-d}}{2^n n!}\frac{d^n}{dx^n}\left[(1-x)^{n+c}(1+x)^{n+d}\right]$$
$$P_n^{(2\sqrt{\varepsilon},-\mu P)}(1-2qs)=\frac{(-2q)^n s^{-2\sqrt{\varepsilon}}(1-qs)^{\mu P}}{n!}\frac{d^n}{ds^n}\left[s^{n+2\sqrt{\varepsilon}}(1+x)^{n-\mu P}\right]. \quad (25)$$

The wave functions are obtained from Eqs. (2), (7), (23-25)

$$\psi_n(s)=N_n s^{\sqrt{\varepsilon}}\,(1-qs)^{\frac{1}{2}\left(1-\mu\sqrt{1+\frac{4V_3}{qa^2}}\right)}\,P_n^{\left(2\sqrt{\varepsilon},-\mu\sqrt{1+\frac{4V_3}{qa^2}}\right)}(1-2qs), \quad (26)$$

where N_n is normalization constant to be determined from the normalization condition

$$\int_0^1|\psi_n(s)|^2ds=1. \quad (27)$$

Two different forms of Jacobi Polynomials [20,21] are

$$P_n^{(c,d)}(x)=2^{-n}\sum_{p=0}^{n}(-1)^{n-p}\binom{n+c}{p}\binom{n+d}{n-p}(1-x)^{n-p}(1+x)^p \quad (28)$$

$$P_n^{(c,d)}(x)=\frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+c+d+r+1)}{\Gamma(r+c+1)}\left(\frac{x-1}{2}\right)^r, \quad (29)$$

where

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}=\frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$$

$$P_n^{(2c,2d)}(1-2qs) = (-1)^n \Gamma(n+2c+1)\Gamma(n+2d+1) \times \sum_{p=0}^{n} \frac{(-1)^p q^{n-p}}{p!(n-p)!} \frac{s^{n-p}(1-qs)^p}{\Gamma(p+2d+1)\Gamma(n+2c-p+1)} \quad (30)$$

$$P_n^{(2c,2d)}(1-2qs) = (-1)^n \frac{\Gamma(n+2c+1)}{\Gamma(n+2c+2d+1)} \times \sum_{r=0}^{n} \frac{(-1)^r q^r}{r!(n-r)!} \frac{\Gamma(n+2c+2d+r+1)}{\Gamma(2c+r+1)} s^r \quad (31)$$

$$1 = N_q^2 (-1)^n \frac{\Gamma(n+2c+1)^2 \Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times \sum_{p,r=0}^{n} \frac{(-1)^{p+r} q^{n-p+r} \Gamma(n+2c+2d+r+1) I_{nq}(p,r)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)\Gamma(r+2d+1)\Gamma(n+2c-p+1)}, \quad (32)$$

where

$$I_{nq}(p,r) = \int_0^1 s^{n+2\sqrt{\varepsilon}+r-p}(1-qs)^{p-\mu P+1} ds,\ c=\sqrt{\varepsilon},\ d=-\frac{1}{2}\mu P. \quad (33)$$

Using the following integral of hypergeometric function:

$$\int_0^1 s^{a-1}(1-s)^{c-a-1}(1-qs)^{-b} ds = {}_2F_1(a,b;c;q) \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}, \quad (34)$$

provided

$$Re(c) > Re(a) > 0,\ |arg(1-q)| < \pi,$$

which gives

$$\int_0^1 s^{a-1}(1-qs)^{-b} ds = \frac{1}{a}\, {}_2F_1(a,b;a+1;q). \quad (35)$$

Using Eq. (33) and Eq. (35), we have

$$I_{nq}(p,r) = \frac{1}{(n+2\sqrt{\varepsilon}+r-p+1)} \times {}_2F_1(n+2\sqrt{\varepsilon}+r-p+1, \mu P-p-1; n+2\sqrt{\varepsilon}+r-p+2, q) \quad (36)$$

Now when $V_2 = 0$, and $V_1, V_2, a \in \mathbb{R}$ the real eigenvalues are

$$E_n = -\frac{a^2}{4}\left[\left(2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}}\right) - \frac{V_1}{qa^2\left(2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}}\right)}\right]^2, \quad n \geq 0, \quad q \geq 1. \quad (37)$$

4 $\mathcal{PT}$-symmetric non-Hermitian case

In this case, we set the potential parameters in Eq.(11) as $V_2 = 0$, $q\in\mathbb{R}$ and $a\rightarrow ia$, then Eq.(11) becomes

$$V(x) = -V_1 \frac{\cos 2ax - q - i\sin 2ax}{1+q^2-2q\cos 2ax} + 4V_3 \frac{-2q+(1+q^2)\cos 2ax - i(1-q^2)\sin 2ax}{(1+q^2-2q\cos 2ax)^2}. \quad (38)$$

Then $V(x)$ satisfies the relation $(\mathcal{PT})V(x)(\mathcal{PT})^{-1} = V(x)$, where $\mathcal{P}x\mathcal{P}^{-1} = -x$; $\mathcal{P}p\mathcal{P}^{-1} = -p = \mathcal{T}p\mathcal{T}^{-1}$; $\mathcal{T}iI\mathcal{T}^{-1} = -iI$,$x^\dagger = x$, $i^\dagger = -i$, $p^* = -p$, $p^\dagger = p$. The energy eigenvalues and eigenfunctions of the potential (38) are

$$E_n = \frac{a^2}{4}\left[\left(2n+1-\mu\sqrt{1-\frac{4V_3}{qa^2}}\right) + \frac{V_1}{qa^2\left(2n+1-\mu\sqrt{1-\frac{4V_3}{qa^2}}\right)}\right]^2, \quad (39)$$

where

$$n < \frac{1}{2}\sqrt{\frac{4V_1}{qa^2}} + \frac{\mu}{2}\sqrt{1-\frac{4V_3}{qa^2}} - \frac{1}{2}, \quad (40)$$

$$\psi_n(s) = A_n s^{\sqrt{\varepsilon}}(1-qs)^{\frac{1}{2}(1-\mu\sqrt{1-\frac{4V_3}{qa^2}})} P_n^{(2\sqrt{\varepsilon},-\mu\sqrt{1-\frac{4V_3}{qa^2}})}(1-2qs), \quad (41)$$

where the normalization constant A_n is connected by the relation

$$1 = A_n^2(-1)^n \frac{\Gamma(n+2c+1)^2\Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times \sum_{p,r=0}^{n} \frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d+r+1)I_{nq}(p,r)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \quad (42)$$

with

$$I_{nq}(p,r) = \frac{1}{(n+2\sqrt{\varepsilon}+r-p+1)} \times {}_2F_1(n+2\sqrt{\varepsilon}+r-p+1, \mu P-p-1; n+2\sqrt{\varepsilon}+r-p+2, q),$$

$$c = \sqrt{\varepsilon}, \qquad d = -\frac{1}{2}\mu P, \qquad P = \sqrt{1-\frac{4V_3}{qa^2}}. \quad (43)$$

5 Non $\mathcal{PT}$-Symmetric Non-Hermitian Case

Now let us consider the case, $V_1 = 0$, $a \to ia$, such potentials are written as complex functions

$$V(x) = -V_2 \frac{\sin 2ax + i(\cos 2ax - q)}{1 + q^2 - 2q\cos 2ax} + 4V_3 \frac{-2q + (1+q^2)\cos 2ax - i(1-q^2)\sin 2ax}{(1 + q^2 - 2q\cos 2ax)^2}. \quad (44)$$

This type of potential is non $\mathcal{PT}$-symmetric. From Eq. (22), one can easily obtain the energy eigenvalues as

$$E_n = -\frac{a^2}{4}\left[\frac{V_2^2}{q^2 a^4 \left(2n+1-\mu\sqrt{1-\frac{4V_3}{qa^2}}\right)^2} - \left(2n+1-\mu\sqrt{1-\frac{4V_3}{qa^2}}\right)^2 - i\frac{2V_2}{qa^2}\right]. \quad (45)$$

The wave functions are

$$\psi_n(s) = B_n s^{\sqrt{\varepsilon}}(1-qs)^{\frac{1}{2}(1-\mu\sqrt{1+\frac{4V_3}{qa^2}})} P_n^{(2\sqrt{\varepsilon}, -\mu\sqrt{1+\frac{4V_3}{qa^2}})}(1-2qs), \quad (46)$$

and the normalization constant B_n is connected by the relation

$$1 = B_n^2(-1)^n \frac{\Gamma(n+2c+1)^2\Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times \sum_{p,r=0}^{n} \frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d+r+1)I_{nq}(p,r)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \quad (47)$$

with

$$I_{nq}(p,r) = \frac{1}{(n+2\sqrt{\varepsilon}+r-p+1)} \times {}_2F_1(n+2\sqrt{\varepsilon}+r-p+1, \mu P - p - 1; n+2\sqrt{\varepsilon}+r-p+2, q),$$

$$c = \sqrt{\varepsilon}, \qquad d = -\frac{1}{2}\mu P, \qquad P = \sqrt{1+\frac{4V_3}{qa^2}} \quad (48)$$

6 Hulthén Potential

Setting $V_3 = 0$, the potential (11) becomes Hulthén potential

$$V(x) = -(V_1 + iV_2)\frac{e^{-2ax}}{1 - qe^{-2ax}}. \quad (49)$$

The energy eigenvalues and wave functions of this potential are obtained from Eqs. (22) and (26), by setting $\mu = -1$

$$E_n = -a^2 \left[(n+1) - \frac{(V_1 + iV_2)}{4qa^2\,(n+1)} \right]^2, \qquad n \geq 0,\ q \geq 1, \tag{50}$$

$$\psi_n(s) = D_n s^{\sqrt{\varepsilon}}\,(1-qs)\,P_n^{\left(2\sqrt{\varepsilon},1\right)}(1-2qs), \tag{51}$$

where the normalization constant D_n is given by

$$1 = D_n^2 (-1)^n \frac{\Gamma(n+2c+1)^2 \Gamma(n)}{\Gamma(2c)} \times \sum_{p,r=0}^{n} \frac{(-1)^{p+r} q^{n-p+r} \Gamma(n+2c+r) I_{nq}(p,r)}{p!r!(n-p)!(n-r)!\Gamma(p)\Gamma(r)\Gamma(n+2c-p+1)} \tag{52}$$

with

$$I_{nq}(p,r) = \frac{1}{(n+2\sqrt{\varepsilon}+r-p+1)} \times {}_2F_1(n+2\sqrt{\varepsilon}+r-p+1, \mu-p-2; n+2\sqrt{\varepsilon}+r-p+2, q),\ c=\sqrt{\varepsilon}. \tag{53}$$

We are now going to consider different forms of generalized Hulthén potential, viz at least one of the parameters is purely imaginary. When $a \to ia$ and $V_2 = V_3 = 0$, then $V(x)$ becomes

$$V(x) = V_1 \frac{q - \cos 2ax}{1 + q^2 - 2q\cos 2ax} + iV_1 \frac{\sin 2ax}{1 + q^2 - 2q\cos 2ax}. \tag{54}$$

Then $(\mathcal{PT})V(x)(\mathcal{PT})^{-1} = V(x)$. The real energy eigenvalues are given by

$$E_n = a^2 \left(n + 1 + \frac{V_1}{4qa^2(n+1)} \right)^2, \quad q \geq 1, \tag{55}$$

where

$$n = 0, 1, 2, \ldots < \frac{1}{2}\sqrt{\frac{V_1}{a^2 q}} - 1 \tag{56}$$

we set $V_1 = V_3 = 0$, in Eq.(11), and $a \to ia$, then Eq. (11) takes the form

$$V(x) = -V_2 \frac{\sin 2ax}{1 + q^2 - 2q\cos 2ax} + iV_2 \frac{q - \cos 2ax}{1 + q^2 - 2q\cos 2ax}. \tag{57}$$

Such a potential is called as non-$\mathcal{PT}$-symmetric potential. The complex energy eigenvalues are given by

$$E_n = -a^2 \left[\frac{V_2^2}{16q^2a^4(n+1)^2} - (n+1)^2 - i\frac{V_2}{2qa^2} \right]. \tag{58}$$

But it has real plus imaginary energy spectra. When we consider the real part of energy eigenvalues an acceptable result is obtained when

$$n < \frac{1}{2}\sqrt{\frac{V_1}{a^2 q}} - 1$$

condition. However, the energy spectrum is not seen at the imaginary part of energy eigenvalues, since it is independent of n.

7 Scarf Potential

Assuming $V_1 = V_2 = 0$, the potential (11) turns into Scarf potential

$$V(x) = V_3 \mathrm{cosech}_q^2 ax. \tag{59}$$

The energy eigenvalues and wave functions of this potential by setting $\mu = 1$

$$E_n = -\frac{a^2}{4}\left[2n + 1 - \sqrt{1 + \frac{4V_3}{qa^2}}\right]^2, \tag{60}$$

$$\psi_n(s) = L_n s^{\sqrt{\varepsilon}}\,(1 - qs)^{\frac{1}{2}\left(1 - \sqrt{1 + \frac{4V_3}{qa^2}}\right)}\, P_n^{\left(2\sqrt{\varepsilon}, -\sqrt{1 + \frac{4V_3}{qa^2}}\right)}(1 - 2qs). \tag{61}$$

The normalization constant L_n is connected by the relation

$$\begin{aligned} 1 = {} & L_n^2 (-1)^n \frac{\Gamma(n + 2c + 1)^2 \Gamma(n + 2d + 1)}{\Gamma(2c + 2d + 1)} \\ & \times \sum_{p,r=0}^{n} \frac{(-1)^{p+r} q^{n-p+r} \Gamma(n + 2c + 2d + r + 1) I_{nq}(p, r)}{p!r!(n - p)!(n - r)!\Gamma(p+2d+1)\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \end{aligned} \tag{62}$$

with

$$\begin{aligned} I_{nq}(p, r) = {} & \frac{1}{(n + 2\sqrt{\varepsilon} + r - p + 1)} \\ & \times {}_2F_1(n + 2\sqrt{\varepsilon} + r - p + 1, \mu P - p - 1; n + 2\sqrt{\varepsilon} + r - p + 2, q) \end{aligned}$$

$$c = \sqrt{\varepsilon}, \qquad d = -\frac{1}{2}\mu P, \qquad P = \sqrt{1 + \frac{4V_3}{qa^2}}. \tag{63}$$

When $a \to ia$ and $V_1 = V_2 = 0$, then $V(x)$ satisfies the relation

$$(\mathcal{PT}) V(x) (\mathcal{PT})^{-1} = V(x).$$

The positive energy eigenvalues are then given by

$$E_n = \frac{a^2}{4}\left[2n+1-\sqrt{1-\frac{4V_3}{qa^2}}\right]^2, \qquad n < 0,1,2,\ldots,\frac{1}{2}\left(\sqrt{1-\frac{4V_3}{qa^2}}-1\right). \tag{64}$$

Next we set $V_3 \rightarrow iV_3$ and $q \rightarrow iq$ in (11), then the potential is non-$\mathcal{PT}$-symmetric potential. The energy eigenvalues are given by

$$E_n = -\frac{a^2}{4}\left[2n+1-\sqrt{1-\frac{4V_3}{qa^2}}\right]^2. \tag{65}$$

8 Conclusion

In this paper, the Schrödinger equation with the Hulthén potential plus the Scarf potential have been solved by using the Nikiforov-Uvarov method. Some interesting results including $\mathcal{PT}$-symmetric and non-$\mathcal{PT}$-symmetric versions of the Hulthén potential and the Scarf potential have also been discussed. The energy eigenvalues of the Hulthén potential and the Scarf potential have been presented separately. It is interesting to see that Eqs.(50) and (51) are consistent with[22], and for $V_2 = 0$, $\delta = 2a$ and $V_1 = Z^2 e\delta$, Eq.(50) is consistent with[23] also for $\hbar = 2m = 1$ and $A_1 = 0$, Eq.(60) is consistent with[17]. Under some restrictions of the potential parameters, we have shown that the non-$\mathcal{PT}$-symmetric Hulthén and Scarf potentials have real energy spectra. In Figures 1-5, the, Hulthén plus Scarf potential, Hulthén potentials and Scarf, are presented for various potential parameters.

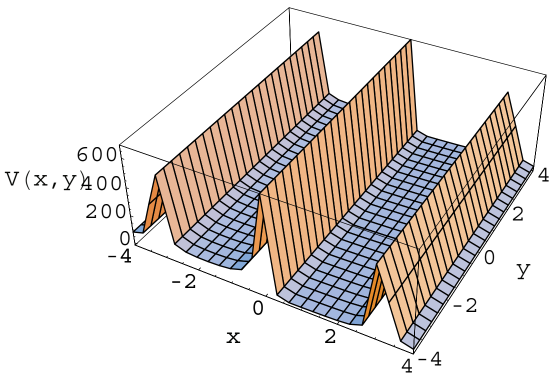

Figure 1. Hulthén plus Scarf potential for $V_1 = 64, V_2 = 0,\ a = q = 1$.

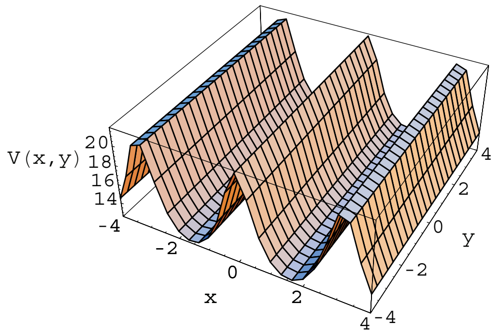

Figure 2. Real part of $\mathcal{PT}$-symmetric Hulthén plus Scarf potential for $V_1 = 64, V_3 = 0.64,\ a = 1,\ q = 4$.

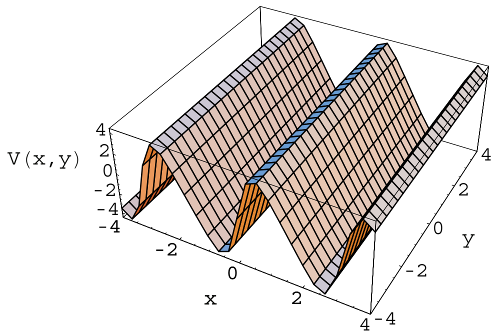

Figure 3. Imaginary part of $\mathcal{PT}$-symmetric Hulthén plus Scarf potential for $V_1 = 64, V_3 = 0.64,\ a = 1,\ q = 4$.

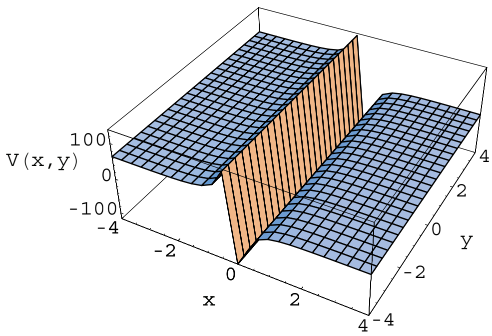

Figure 4. Hulthén potential for $V_1 = 64, V_2 = 0,\ a = q = 1$

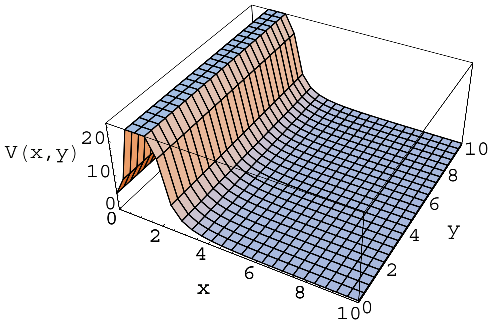

Figure 5. Scarf potential for $V_3 = 10,\ a = 1,\ q = 4$

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