Bulg. J. Phys. **36** (2009) 17–34

Schrödinger Equation with Woods-Saxon Plus Pöschl–Teller Potential

S. Meyur[1], **S. Debnath**[2]

[1]Tansuk Rai Ganapat Rai Khemka High School, 23, Rabindra Sarani, Liluah, Howrah-711204, India

[2]Department of Mathematics, Jadavpur University, Kolkata-700032, India

Received *26 March 2009*

Abstract. The Schrödinger equation is solved for Woods–Saxon plus Pöschl-Teller potential. Nikiforov–Uvarov method is used to obtain energy eigenvalues and the corresponding eigenfunctions. The $\mathcal{PT}$ and non $\mathcal{PT}$-symmetric solutions for this potential are presented.

PACS number: 03.65.Ge; 03.65.Db

1 Introduction

The Schrödinger equation is one of the fundamental wave equations in quantum physics. In non-relativistic quantum mechanics, one usually chooses a Hermitian potential to derive real energy eigenvalues of the corresponding Schrödinger equation [1,2]. Recently, the exact solution of the Schrödinger equation for the potentials which have complex spectrum are generally of interest due to the discovery of $\mathcal{PT}$-symmetry [3]. Potentials having this symmetry are complex and non-Hermitian. The interesting property of the $\mathcal{PT}$-symmetric quantum mechanics is that the eigenvalues of these complex valued Hamiltonians are real and positive. It is also well-known that $\mathcal{PT}$-symmetry does not lead to completely real eigenvalues, because there are several potentials where part or all of the energy eigenvalues are complex. In particular, the energy eigenvalues of the Schrödinger equation are real when $\mathcal{PT}$-symmetry is unbroken, where as they come in complex conjugate pair when $\mathcal{PT}$-symmetry is spontaneously broken. Non-Hermitian but $\mathcal{PT}$-symmetric models have applications in different fields, such as condensed matter [4], population biology [5], optics [6], nuclear physics [7] and quantum field theory [8].

The purpose of the present paper is to further pursue the development of $\mathcal{PT}$-symmetry and to solve the one-dimensional time-independent Schrödinger equation for some complex potentials. In view of the $\mathcal{PT}$-symmetric formulation, we will apply the Nikiforov–Uvarov method [9] to solve the time-independent Schrödinger equation for Woods-Saxon plus Pöschl–Teller potential. We will also obtain the eigenvalues and eigenfunctions of the Woods-Saxon potential [10] and the Pöschl–Teller potential [11].

The arrangement of the present paper is as follows. After a brief introductory discussion of the Nikiforov-Uvarov method in Section 2, we obtain the energy eigenvalues and eigenfunctions for real and complex cases of Woods–Saxon plus Pöschl–Teller potential in Section 3. In Sections 4 and 5, we discuss the solution of $\mathcal{PT}$-symmetric and non $\mathcal{PT}$-symmetric Woods-Saxon plus Pöschl-Teller potential. In Sections 6 and 7, we discuss, the eigenvalues and eigenfunctions of Woods-Saxon potential and Pöschl-Teller potential respectively. Finally, conclusions and remarkable facts are discussed in the last section.

2 Nikiforov–Uvarov Method

The conventional Nikiforov–Uvarov method [9], which received much interest, has been introduced for solving Schrödinger equation [12,13], Klein-Gordon and Dirac [14-18] equations.

The differential equations whose solutions are the special functions of hypergeometric type can be solved by using the Nikiforov-Uvarov method which has been developed by Nikiforov and Uvarov [9]. In this method, the one dimensional Schrödinger equation is reduced to an equation by an appropriate coordinate transformation $x = x(s)$

$$\psi''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials, at most of second degree, and $\widetilde{\tau}(s)$ is a polynomial, at most of first degree. In order to obtain a particular solution to Eq. (1), we set the following wave function as a multiple of two independent parts:

$$\psi(s) = \phi(s)y(s). \tag{2}$$

Using Eq. (1) and Eq. (2) we have

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0, \tag{3}$$

which demands that the following conditions be satisfied:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}, \tag{4}$$

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s), \qquad \tau'(s) < 0. \tag{5}$$

The condition $\tau'(s) < 0$ helps to generate energy eigenvalues and corresponding eigenfunctions. The condition $\tau'(s) > 0$ has widely been discussed in [19]. The

λ in (3) satisfies the following second-order differential equation:

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \qquad n = 0, 1, 2, ... \tag{6}$$

The polynomial $\tau(s)$ with the parameter s and prime factors show the differentials at first degree be negative. It is to be noted that λ or λ_n are obtained from a particular solution of the form $y(s) = y_n(s)$ which is a polynomial of degree n. The second part $y_n(s)$ of the wave function Eq. (2) is the hypergeometric-type function whose polynomial solutions are connected by the Rodrigues relation [20-21]

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n}[\sigma^n(s)\rho(s)], \tag{7}$$

where C_n is normalization constant and the weight function $\rho(s)$ satisfies the relation as

$$\frac{d}{ds}[\sigma(s)\rho(s)] = \tau(s)\rho(s). \tag{8}$$

On the other hand, in order to find the eigenfunctions, $\phi_n(s)$ and $y_n(s)$ in Eqs. (4) and (7) and eigenvalues λ_n in Eq. (6), we need to calculate the functions

$$\pi(s) = \left(\frac{\sigma' - \widetilde{\tau}}{2}\right) \pm \sqrt{\left(\frac{\sigma' - \widetilde{\tau}}{2}\right)^2 - \widetilde{\sigma} + k\sigma}\,, \tag{9}$$

$$k = \lambda - \pi'(s)\,. \tag{10}$$

In principle, since $\pi(s)$ has to be a polynomial of degree at most one, the expression under the square root sign in Eq. (9) can be put into order to be the square of a polynomial of first degree [9], which is possible only if its discriminant is zero. Thus, the equation for k obtained from the solution of Eq. (9) can be further substituted in Eq. (10). In addition, the energy eigenvalues are obtained from Eqs. (6) and (10).

3 Woods–Saxon Plus Pöschl–Teller Potential

The Woods-Saxon potential plus Pöschl-Teller potential is given by

$$V(x) = -V_1\frac{e^{-2ax}}{1+qe^{-2ax}} + V_2\frac{e^{-4ax}}{(1+qe^{-2ax})^2} - V_3\mathrm{sech}_q^2 ax\,. \tag{11}$$

The Schrödinger equation becomes

$$\frac{d^2\psi}{dx^2} + \left[E + V_1\frac{e^{-2ax}}{1+qe^{-2ax}} - V_2\frac{e^{-4ax}}{(1+qe^{-2ax})^2} + V_3\mathrm{sech}_q^2 ax\right]\psi, \tag{12}$$

where $\hbar = 2m = 1$. Setting the following notations

$$\varepsilon = -\frac{E}{4a^2}, \quad \beta_i = \frac{V_i}{4a^2}(>0), \quad (i=1,2,3), \quad \text{and } s = -e^{-2ax} \tag{13}$$

with $\varepsilon > 0 (E < 0)$ for bound states, equation (12) becomes

$$\frac{d^2\psi}{ds^2} + \frac{1-qs}{s-qs^2}\frac{d\psi}{ds} + \frac{1}{(s-qs^2)^2}\Big[-\{q^2\varepsilon - q\beta_1 + \beta_2\}s^2 + \{2q\varepsilon - (\beta_1 + 4\beta_3)\}s - \varepsilon\Big]\psi = 0\,. \quad (14)$$

After the comparison of Eq. (14) with Eq. (1), we have

$$\begin{aligned}&\widetilde{\tau}(s) = 1 - qs, \quad \sigma(s) = s - qs^2,\\ &\widetilde{\sigma}(s) = -\{q^2\varepsilon - q\beta_1 + \beta_2\}s^2 + \{2q\varepsilon - (\beta_1 + 4\beta_3)\}s - \varepsilon.\end{aligned} \quad (15)$$

Substituting these polynomials into Eq. (9), we have

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2}\left[(2\sqrt{\varepsilon} - \mu P)qs - 2\sqrt{\varepsilon}\right] \quad (16)$$

if $k = -(\beta_1 + 4\beta_3) + \mu q\sqrt{\varepsilon}P$, where

$$\mu = +1, -1 \quad \text{and} \quad P = \sqrt{1 + \frac{16\beta_3}{q} + \frac{4\beta_2}{q^2}} = \sqrt{1 + \frac{4V_3}{qa^2} + \frac{V_2}{q^2a^2}}\,.$$

For bound state solutions, it is necessary to choose

$$\pi(s) = -\frac{qs}{2} - \frac{1}{2}\left[(2\sqrt{\varepsilon} - \mu P)qs - 2\sqrt{\varepsilon}\right] \quad (17)$$

if $k = -(\beta_1 + 4\beta_3) + \mu q\sqrt{\varepsilon}P$.

The following track in this selection is to achieve the condition $\tau'(s) < 0$. Therefore, $\tau(s)$ becomes

$$\tau(s) = 1 + 2\sqrt{\varepsilon} - \left[2 + 2\sqrt{\varepsilon} - \mu P\right]qs \quad (18)$$

and then its negative derivatives become

$$\tau'(s) = -\left[2 + 2\sqrt{\varepsilon} - \mu P\right]q\,. \quad (19)$$

Therefore, from Eqs. (6) and (10), we have

$$\lambda = \lambda_n = n\left[2 + 2\sqrt{\varepsilon} - \mu P\right]q + n(n-1)q \quad (20)$$

and

$$\lambda = -(\beta_1 + 4\beta_3) + \mu q\sqrt{\varepsilon}P - \frac{q}{2} - \frac{q}{2}(2\sqrt{\varepsilon} - \mu P). \quad (21)$$

Comparing Eqs. (20) and (21), we have

$$\begin{aligned}&n[2+2\sqrt{\varepsilon} - \mu P]q + n(n-1)q = -(\beta_1+4\beta_3) + \mu q\sqrt{\varepsilon}P - \frac{q}{2} - \frac{q}{2}(2\sqrt{\varepsilon} - \mu P)\\ &\Rightarrow (2n+1-\mu P)\sqrt{\varepsilon} + n(n+1) + \frac{1}{2} - \left(n + \frac{1}{2}\right)\mu P = -\frac{(\beta_1+4\beta_3)}{q}\\ &\Rightarrow 4(2n+1-\mu P)\sqrt{\varepsilon} + (2n+1-\mu P)^2 = -\frac{4(\beta_1+4\beta_3)}{q} - 1 + P^2. \quad (22)\end{aligned}$$

Substituting the values of β_1, β_2, β_3, ε and P, we obtain the energy eigenvalues:

$$E_n = -\frac{a^2}{4}\left[\left(2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}\right) - \frac{\frac{V_2}{q^2a^2}-\frac{V_1}{qa^2}}{2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}}\right]^2, \quad (23)$$

where $(n = 0, 1, 2, ...)$, $q \geq 1$. Again Eq. (22) can be expressed as

$$(2n+1-\mu P)\sqrt{\varepsilon} + n(n+1) + \frac{1}{2} - \left(n+\frac{1}{2}\right)\mu P = -\frac{(\beta_1+4\beta_3)}{q}$$

$$\Rightarrow (2n+1-\mu P)\sqrt{\varepsilon} + n(n+1) + \frac{1}{2} + \frac{1}{2}(2n+1-\mu P)(2n+1) - \frac{1}{2}(2n+1)^2 = -\frac{(\beta_1+4\beta_3)}{q}$$

$$\Rightarrow (2n+1-\mu P)\sqrt{\varepsilon} + (2n+1-\mu P)(2n+1) = n(n+1) - \frac{(\beta_1+4\beta_3)}{q}.$$

Substituting the values of β_1, β_2, β_3, ε and P, we write the energy eigenvalues in another form

$$E_n = -4a^2\left[\frac{2n+1}{2} - \frac{n(n+1)-\frac{V_1+4V_3}{4qa^2}}{2n+1-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}}\right]^2, \quad (24)$$

where $(n = 0, 1, 2, ...)$, $q \geq 1$. From Eqs. (5), (8), and (15), we obtain the weight function

$$\rho(s) = s^{2\sqrt{\varepsilon}}(1-qs)^{-\mu P} \quad (25)$$

and from Eqs. (4), (15), and (17), we have

$$\phi(s) = s^{\sqrt{\varepsilon}}(1-qs)^{\frac{1}{2}(1-\mu P)}. \quad (26)$$

Now using the properties of Jacobi Polynomial [20,21]

$$P_n^{(c,d)}(x) = \frac{(-1)^n(1-x)^{-c}(1+x)^{-d}}{2^n n!}\frac{d^n}{dx^n}\left[(1-x)^{n+c}(1+x)^{n+d}\right]$$
$$P_n^{(2\sqrt{\varepsilon},-\mu P)}(1-2qs) = \frac{(-2q)^n s^{-2\sqrt{\varepsilon}}(1-qs)^{\mu P}}{n!}\frac{d^n}{ds^n}\left[s^{n+2\sqrt{\varepsilon}}(1-qs)^{n-\mu P}\right]. \quad (27)$$

The wave functions are obtained from Eqs. (2), (7), (25-27)

$$\psi_n(s) = N_n s^{\sqrt{\varepsilon}}(1-qs)^{\frac{1}{2}\left(1-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}\right)} \times P_n^{\left(2\sqrt{\varepsilon},-\mu\sqrt{1+\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}\right)}(1-2qs), \quad (28)$$

where N_n is normalization constant to be determined from the normalization condition

$$\int_0^1 |\psi_n(s)|^2 ds = 1. \quad (29)$$

Two different forms of Jacobi Polynomials [20,21] are

$$P_n^{(c,d)}(x) = 2^{-n}\sum_{p=0}^{n}(-1)^{n-p}\binom{n+c}{p}\binom{n+d}{n-p}(1-x)^{n-p}(1+x)^p, \quad (30)$$

$$P_n^{(c,d)}(x) = \frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+c+d+r+1)}{\Gamma(r+c+1)}\left(\frac{x-1}{2}\right)^r, \quad (31)$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)},$$

$$P_n^{(2c,2d)}(1-2qs) = (-1)^n\Gamma(n+2c+1)\Gamma(n+2d+1) \times \sum_{p=0}^{n}\frac{(-1)^p q^{n-p}}{p!(n-p)!}\frac{s^{n-p}(1-qs)^p}{\Gamma(p+2d+1)\Gamma(n+2c-p+1)}, \quad (32)$$

$$P_n^{(2c,2d)}(1-2qs) = (-1)^n\frac{\Gamma(n+2c+1)}{\Gamma(n+2c+2d+1)} \times \sum_{r=0}^{n}\frac{(-1)^r q^r}{r!(n-r)!}\frac{\Gamma(n+2c+2d+r+1)}{\Gamma(2c+r+1)}s^r, \quad (33)$$

$$1 = N_q^2(-1)^n\frac{\Gamma(n+2c+1)^2\Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times \sum_{p,r=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d+r+1)I_{nq}(p,r)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)\Gamma(r+2d+1)\Gamma(n+2c-p+1)}, \quad (34)$$

where

$$I_{nq}(p,r) = \int_0^1 s^{n+2\sqrt{\varepsilon}+r-p}(1-qs)^{p-\mu P+1}ds, \quad c=\sqrt{\varepsilon}, \quad d=-\frac{1}{2}\mu P. \quad (35)$$

Using the following integral of hypergeometric function:

$$\int_0^1 s^{a-1}(1-s)^{c-a-1}(1-qs)^{-b}ds = {}_2F_1(a,b;c;q)\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} \qquad (36)$$

provided

$$\text{Re}(c) > \text{Re}(a) > 0, \quad |\text{arg}(1-q)| < \pi,$$

which gives

$$\int_0^1 s^{a-1}(1-qs)^{-b}ds = \frac{1}{a}{}_2F_1(a,b;a+1;q)\,. \qquad (37)$$

Using Eq. (35) and Eq. (37) we have

$$\begin{aligned} I_{nq}(p,r) = {} & \frac{1}{(n+2\sqrt{\varepsilon}+r-p+1)} \\ & \times {}_2F_1(n+2\sqrt{\varepsilon}+r-p+1, \mu P-p-1; n+2\sqrt{\varepsilon}+r-p+2, q). \end{aligned} \qquad (38)$$

4 $\mathcal{PT}$-Symmetric Non-Hermitian Case

In this case, we set the potential parameters in equation (11) as $V_1, V_2, V_3, V_4, q \in \mathbb{R}$ and $a \in \mathrm{I}\mathbb{R}$ $(a \to ia)$, then Eq. (11) becomes

$$\begin{aligned} V(x) = & -V_1\frac{\cos 2ax + q - i\sin 2ax}{1+q^2+2q\cos 2ax} \\ & + V_2\frac{(q^2+2q\cos 2ax+\cos 4ax)-i(2q\sin 2ax+\sin 4ax)}{(1+q^2+2q\cos 2ax)^2} \\ & - 4V_3\frac{2q+(q^2+1)\cos 2ax+i(q^2-1)\sin 2ax}{(1+q^2+2q\cos 2ax)^2}\,. \end{aligned} \qquad (39)$$

Then $V(x)$ satisfies the relation

$$(\mathcal{PT})V(x)(\mathcal{PT})^{-1} = V(x),$$

where

$$\mathcal{P}x\mathcal{P}^{-1} = -x; \quad \mathcal{P}p\mathcal{P}^{-1} = -p = \mathcal{T}p\mathcal{T}^{-1}; \quad \mathcal{T}iI\mathcal{T}^{-1} = -iI$$
$$x^\dagger = x, \quad i^\dagger = -i, \quad p^* = -p, \quad p^\dagger = p.$$

The energy eigenvalues of the potential (39) are

$$E_n = 4a^2\left[\frac{2n+1}{2} - \frac{n(n+1)+\dfrac{V_1+4V_3}{4qa^2}}{2n+1-\mu\sqrt{1-\dfrac{4V_3}{qa^2}-\dfrac{V_2}{q^2a^2}}}\right]^2, \quad q \geq 1, \qquad (40)$$

condition for n is

$$n < \frac{1}{2}\sqrt{\frac{V_1}{qa^2} - \frac{V_2}{q^2a^2}} + \frac{\mu}{2}\sqrt{1 - \frac{4V_3}{qa^2} - \frac{V_2}{q^2a^2}} - \frac{1}{2}\,. \qquad (41)$$

Hence eigenvalues are always real for $V_2 \leq q^2a^2 - 4qV_3$ and complex for $V_2 > q^2a^2 - 4qV_3$. The corresponding eigenfunctions are

$$\psi_n(s) = A_n s^{\sqrt{\varepsilon}}(1-qs)^{\frac{1}{2}\left(1-\mu\sqrt{1-\frac{4V_3}{qa^2}-\frac{V_2}{q^2a^2}}\right)} \times P_n^{\left(2\sqrt{\varepsilon},-\mu\sqrt{1-\frac{4V_3}{qa^2}-\frac{V_2}{q^2a^2}}\right)}(1-2qs)\,, \qquad (42)$$

where the normalization constant A_n is given by the relation

$$1 = A_n^2(-1)^n\frac{\Gamma(n+2c+1)^2\Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times \sum_{p,r=0}^{n}\frac{(-1)^{p+r}q^{n-p+r}\Gamma(n+2c+2d+r+1)I_{nq}(p,r)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \qquad (43)$$

with

$$I_{nq}(p,r) = \frac{1}{(n+2\sqrt{\varepsilon}+r-p+1)} \times {}_2F_1(n+2\sqrt{\varepsilon}+r-p+1, \mu P-p-1; n+2\sqrt{\varepsilon}+r-p+2, q), \qquad (44)$$

$$c=\sqrt{\varepsilon}, \quad d=-\frac{1}{2}\mu P, \quad P=\sqrt{1-\frac{4V_3}{qa^2}-\frac{V_2}{q^2a^2}}\,.$$

5 Non $\mathcal{PT}$-Symmetric Non-Hermitian Case

Now let us consider the case, $V_2, V_4, \in\mathbb{R}$, and $V_1, V_3, q, a \in \mathrm{I}\mathbb{R}$, $(V_1 \to iV_1, V_3 \to iV_3, q \to iq, a \to ia)$. Then potential (11) takes the form

$$V(x) = -V_1\frac{\sin 2ax+q+i\cos 2ax}{1+q^2+2q\sin 2ax} + V_2\left[\frac{\cos 2ax - i(\sin 2ax+q)}{1+q^2+2q\sin 2ax}\right]^2 - 4V_3\frac{(q^2-1)\cos 2ax + i\{(q^2+1)\sin 2ax+2q\}}{(1+q^2+2q\sin 2ax)^2}\,. \qquad (45)$$

This kind of potential is non $\mathcal{PT}$-symmetric. From Eq. (24), the energy eigenvalues are

$$E_n = 4a^2 \left[\frac{2n+1}{2} - \frac{n(n+1) + \dfrac{V_1 + 4V_3}{4qa^2}}{2n+1-\mu\sqrt{1-\dfrac{4V_3}{qa^2}+\dfrac{V_2}{q^2a^2}}} \right]^2 , \tag{46}$$

where

$$n < \frac{1}{2}\sqrt{\frac{V_1}{qa^2}+\frac{V_2}{q^2a^2}} + \frac{\mu}{2}\sqrt{1-\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}} - \frac{1}{2}\,. \tag{47}$$

The corresponding eigenfunctions are

$$\psi_n(s) = B_n s^{\sqrt{\varepsilon}} (1-qs)^{\frac{1}{2}\left(1-\mu\sqrt{1-\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}\right)} \times P_n^{\left(2\sqrt{\varepsilon},\,-\mu\sqrt{1-\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}\right)}(1-2qs)\,, \tag{48}$$

where the normalization constant B_n is given by the relation

$$1 = B_n^2(-1)^n \frac{\Gamma(n+2c+1)^2\Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times \sum_{p,r=0}^{n} \frac{(-1)^{p+r} q^{n-p+r}\Gamma(n+2c+2d+r+1) I_{nq}(p,r)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \tag{49}$$

with

$$I_{nq}(p,r) = \frac{1}{(n+2\sqrt{\varepsilon}+r-p+1)} \times {}_2F_1(n+2\sqrt{\varepsilon}+r-p+1, \mu P-p-1; n+2\sqrt{\varepsilon}+r-p+2, q), \tag{50}$$

$$c = \sqrt{\varepsilon}, \quad d = -\frac{1}{2}\mu P, \quad P = \sqrt{1-\frac{4V_3}{qa^2}+\frac{V_2}{q^2a^2}}\,.$$

6 Woods–Saxon Potential

For $V_3 = 0$, the potential (11) becomes Woods-Saxon potential [10]

$$V(x) = -V_1 \frac{e^{-2ax}}{1+qe^{-2ax}} + V_2 \frac{e^{-4ax}}{(1+qe^{-2ax})^2}\,. \tag{51}$$

The energy eigenvalues and eigenfunctions of this potential are obtained from Eqs. (24) and (28) by setting $\mu = -1$

$$E_n = -4a^2 \left[\frac{2n+1}{2} - \frac{n(n+1) - \dfrac{V_1}{4qa^2}}{2n+1+\sqrt{1+\dfrac{V_2}{q^2a^2}}} \right]^2 \qquad n \geq 0, \quad q \geq 1, \qquad (52)$$

$$\psi_n(s) = D_n s^{\sqrt{\varepsilon}} (1-qs)^{\frac{1}{2}\left(1+\sqrt{1+\frac{V_2}{q^2a^2}}\right)} P_n^{\left(2\sqrt{\varepsilon}, \sqrt{1+\frac{V_2}{q^2a^2}}\right)}(1-2qs)\,, \qquad (53)$$

where D_n, the normalization constant, is given by

$$1 = D_n^2 (-1)^n \frac{\Gamma(n+2c+1)^2 \Gamma(n+2d+1)}{\Gamma(2c+2d+1)} \times \sum_{p,r=0}^{n} \frac{(-1)^{p+r} q^{n-p+r} \Gamma(n+2c+2d+r+1) I_{nq}(p,r)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \qquad (54)$$

with

$$I_{nq}(p,r) = \frac{1}{(n+2\sqrt{\varepsilon}+r-p+1)} \times {}_2F_1(n+2\sqrt{\varepsilon}+r-p+1, \mu P-p-1; n+2\sqrt{\varepsilon}+r-p+2, q), \qquad (55)$$

$$c = \sqrt{\varepsilon}, \quad d = -\frac{1}{2}\mu P, \quad P = \sqrt{1+\frac{V_2}{q^2a^2}}\,.$$

We are now going to consider different forms of generalized Woods–Saxon potential, viz at least one of the parameters is purely imaginary. When $a \to ia$ and $V_1, V_2 \in \mathbb{R}$, then $V(x)$ becomes

$$V(x) = -V_1 \frac{\cos 2ax + q}{1+q^2+2q\cos 2ax} + V_2 \frac{q^2+2q\cos 2ax + \cos 4ax}{(1+q^2+2q\cos 2ax)^2} + i\left[V_1 \frac{\sin 2ax}{1+q^2+2q\cos 2ax} - V_2 \frac{2q\sin 2ax + \sin 4ax}{(1+q^2+2q\cos 2ax)^2} \right]. \qquad (56)$$

Then $(\mathcal{PT})V(x)(\mathcal{PT})^{-1} = V(x)$. The real positive energy eigenvalues are given by

$$E_n = 4a^2 \left[\frac{2n+1}{2} - \frac{n(n+1) + \dfrac{V_1}{4qa^2}}{2n+1+\sqrt{1-\dfrac{V_2}{q^2a^2}}} \right]^2 \qquad n \geq 0, \qquad q \geq 1 \qquad (57)$$

if and only if

$$n < \frac{1}{2}\sqrt{\frac{V_1}{qa^2} - \frac{V_2}{q^2a^2}} - \frac{1}{2}\sqrt{1 - \frac{V_2}{q^2a^2}} - \frac{1}{2}\,. \tag{58}$$

The eigenvalues are always positive real when $V_2 = 0$ and condition for n is $n < \frac{1}{2}\sqrt{\frac{V_1}{qa^2}} - 1$ but can be complex for $V_2 > q^2a^2$.

Next let us take $V_1 \to iV_1$, $V_2 \in \mathbb{R}$ and $a \to ia$, then (11) takes the form

$$\begin{aligned} V(x) = &-V_1 \frac{\sin 2ax}{1 + q^2 + 2q\cos 2ax} + V_2 \frac{q^2 + 2q\cos 2ax + \cos 4ax}{(1 + q^2 + 2q\cos 2ax)^2} \\ &-i\left[V_1 \frac{\cos 2ax + q}{1 + q^2 + 2q\cos 2ax} + V_2 \frac{2q\sin 2ax + \sin 4ax}{(1 + q^2 + 2q\cos 2ax)^2}\right]. \end{aligned} \tag{59}$$

Such a potential is non-$\mathcal{PT}$-symmetric. The complex energy eigenvalues are given by

$$E_n = 4a^2\left[\frac{2n+1}{2} - \frac{n(n+1) + \dfrac{iV_1}{4qa^2}}{2n + 1 + \sqrt{1 - \dfrac{V_2}{q^2a^2}}}\right]^2 \qquad n \geq 0, \qquad q \geq 1. \tag{60}$$

7 Pöschl–Teller Potential

Assuming $V_1 = V_2 = 0$, the potential (11) turns into Pöschl–Teller potential [11]

$$V(x) = -V_3 \mathrm{sech}_q^2 ax\,. \tag{61}$$

The energy eigenvalues and eigenfunctions of this potential are obtained from Eqs. (23) and (28) by setting $\mu = 1$

$$E_n = -\frac{a^2}{4}\left[2n + 1 - \sqrt{1 + \frac{4V_3}{qa^2}}\right]^2 \qquad n \geq 0, \qquad q \geq 1 \tag{62}$$

$$\psi_n(s) = L_n s^{\sqrt{\varepsilon}} (1 - qs)^{\frac{1}{2}\left(1 - \sqrt{1 + \frac{4V_3}{qa^2}}\right)} P_n^{\left(2\sqrt{\varepsilon}, -\sqrt{1 + \frac{4V_3}{qa^2}}\right)}(1 - 2qs), \tag{63}$$

where L_n, the normalization constant is given by

$$\begin{aligned} 1 = &L_n^2 (-1)^n \frac{\Gamma(n + 2c + 1)^2 \Gamma(n + 2d + 1)}{\Gamma(2c + 2d + 1)} \\ &\times \sum_{p,r=0}^{n} \frac{(-1)^{p+r} q^{n-p+r} \Gamma(n+2c+2d+r+1) I_{nq}(p, r)}{p!r!(n-p)!(n-r)!\Gamma(p+2d+1)\Gamma(r+2d+1)\Gamma(n+2c-p+1)} \end{aligned} \tag{64}$$

with

$$I_{nq}(p,r)=\frac{1}{(n+2\sqrt{\varepsilon}+r-p+1)} \times {}_2F_1(n+2\sqrt{\varepsilon}+r-p+1,\mu P-p-1;n+2\sqrt{\varepsilon}+r-p+2,q), \quad (65)$$

$$c=\sqrt{\varepsilon}, \quad d=-\frac{1}{2}\mu P, \quad P=\sqrt{1+\frac{4V_3}{qa^2}}\,.$$

We are now going to consider different forms of generalized Pöschl–Teller potential, viz at least one of the parameters is purely imaginary. When $a\rightarrow ia$ and $V_3,\ q\in\mathbb{R}$, then $V(x)$ becomes

$$V(x)=-4V_3\frac{2q+(q^2+1)\cos 2ax}{(1+q^2+2q\cos 2ax)^2}-4iV_3\frac{(q^2-1)\sin 2ax}{(1+q^2+2q\cos 2ax)^2}\,. \quad (66)$$

Obviously $V(-x)^*=V(x)$ ($\mathcal{PT}$). The positive energies are

$$E_n=\frac{a^2}{4}\left[2n+1-\sqrt{1-\frac{4V_3}{qa^2}}\right]^2 \qquad n\geq 0, \qquad q\geq 1\,. \quad (67)$$

Next let us take $V_3\rightarrow iV_3$, $q\rightarrow iq$ and $a\rightarrow ia$, then (11) takes the form

$$V(x)=-4V_3\frac{(q^2-1)\cos 2ax}{(1+q^2+2q\sin 2ax)^2}-4iV_3\frac{(q^2+1)\sin 2ax+2q}{(1+q^2+2q\sin 2ax)^2}\,. \quad (68)$$

In this case $V(x)$ is non $\mathcal{PT}$-symmetric. The eigenvalues are

$$E_n=\frac{a^2}{4}\left[2n+1-\sqrt{1-\frac{4V_3}{qa^2}}\right]^2 \qquad n\geq 0, \qquad q\geq 1. \quad (69)$$

8 Conclusion

In this paper, the Schrödinger equation with Woods-Saxon plus Pöschl–Teller potential have been solved by using the Nikiforov–Uvarov method. Some interesting results including $\mathcal{PT}$-symmetric and non-$\mathcal{PT}$-symmetric versions of the Woods-Saxon plus Pöschl–Teller potential have also been discussed. The energy eigenvalues of the Woods–Saxon plus Pöschl–Teller potential have been presented separately. It is interesting to see that Eqs. (52), (53), (57), (60) are consistent with [10], and Eqs. (62) and (63) are consistent with [11]. Under some restrictions of the potential parameters, we have shown that the non-$\mathcal{PT}$-symmetric Woods–Saxon plus Pöschl-Teller potentials have real energy spectra. In Figures 1–14, the Woods–Saxon plus Pöschl–Teller potential, Woods-Saxon potential and Pöschl–Teller potential, are presented for various potential parameters.

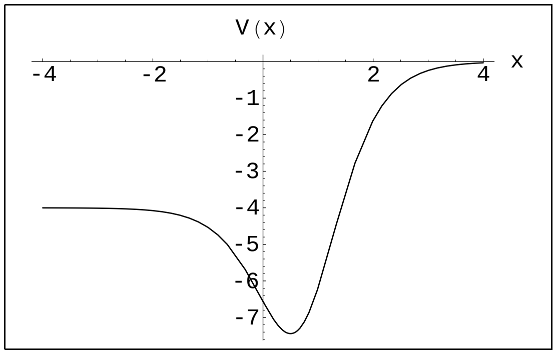

Figure 1. Woods–Saxon plus Pöschl–Teller potential for $V_1 = 40$, $V_2 = 96$, $V_3 = 60$, $a = 1$, $q = 4$.

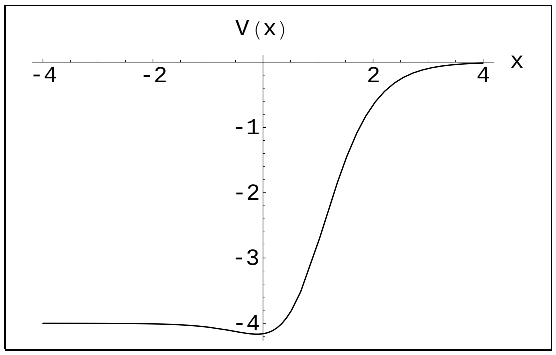

Figure 2. Woods–Saxon potential for $V_1 = 40$, $V_2 = 96$ $a = 1$, $q = 4$.

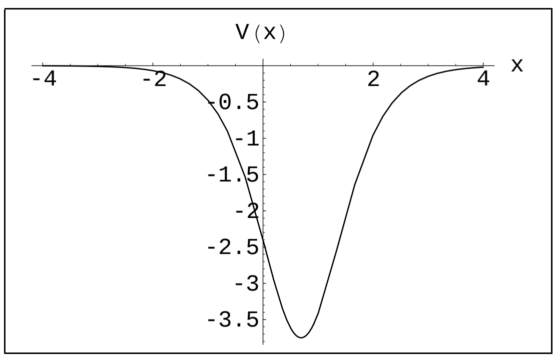

Figure 3. Pöschl–Teller potential for $V_3 = 60$, $a = 1$, $q = 4$.

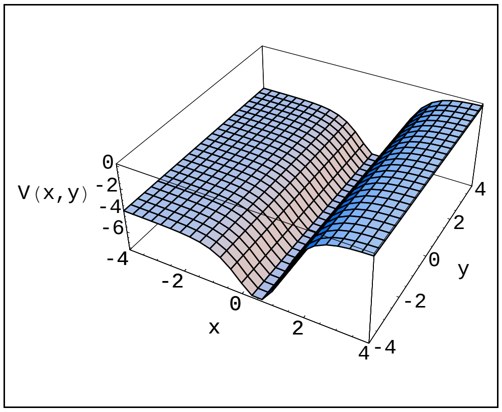

Figure 4. Woods–Saxon plus Pöschl–Teller potential for $V_1 = 40$, $V_2 = 96$, $V_3 = 64$, $a = 1$, $q = 4$.

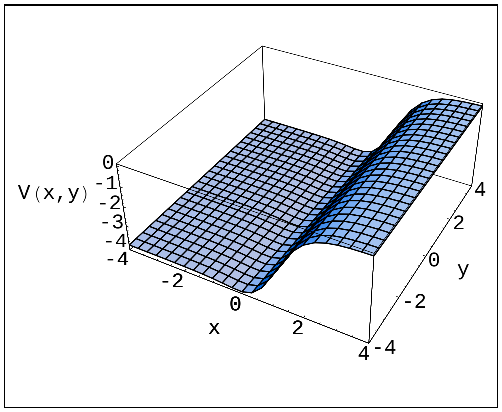

Figure 5. Woods–Saxon potential for $V_1 = 40$, $V_2 = 96$, $a = 1$, $q = 4$.

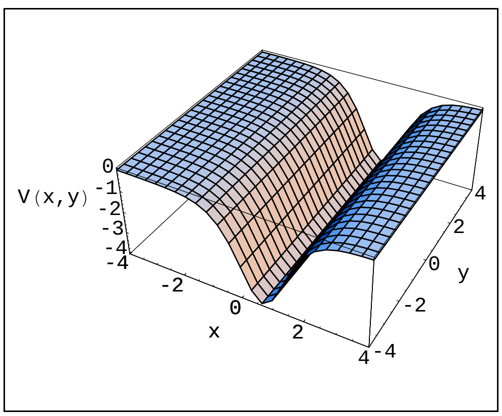

Figure 6. Pöschl–Teller potential for $V_3 = 64$, $a = 1$, $q = 4$.

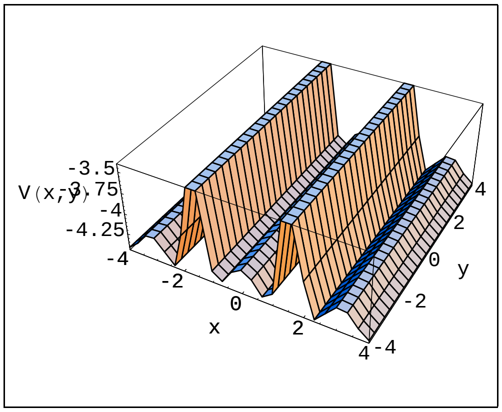

Figure 7. Real part of $\mathcal{PT}$-symmetric Woods–Saxon potential for $V_1 = 40$, $V_2 = 96$, $a = 1$, $q = 4$.

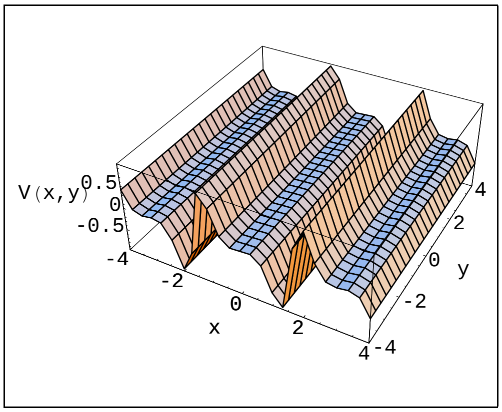

Figure 8. Imaginary part of $\mathcal{PT}$-symmetric Woods–Saxon potential for $V_1 = 40$, $V_2 = 96$, $a = 1$, $q = 4$.

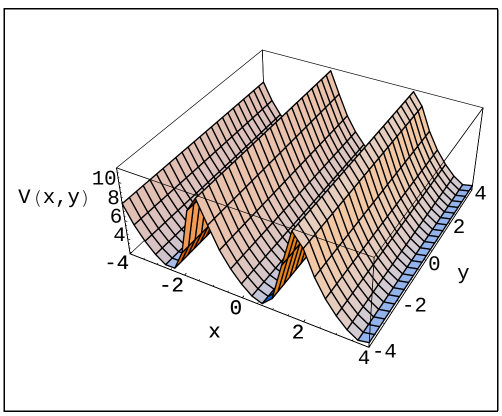

Figure 9. Real part of non $\mathcal{PT}$-symmetric Woods–Saxon potential for $V_1 = 40$, $V_2 = 96$, $a = 1$, $q = 4$.

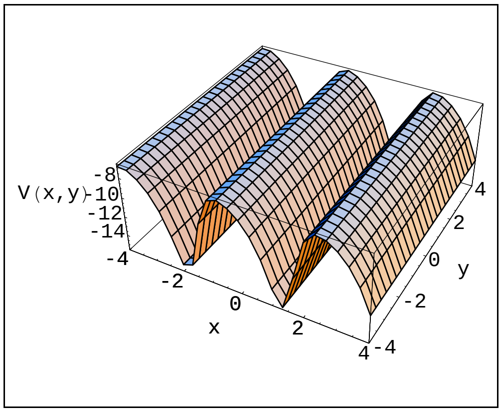

Figure 10. Imaginary part of non $\mathcal{PT}$-symmetric Woods–Saxon potential for $V_1 = 40$, $V_2 = 96$, $a = 1$, $q = 4$.

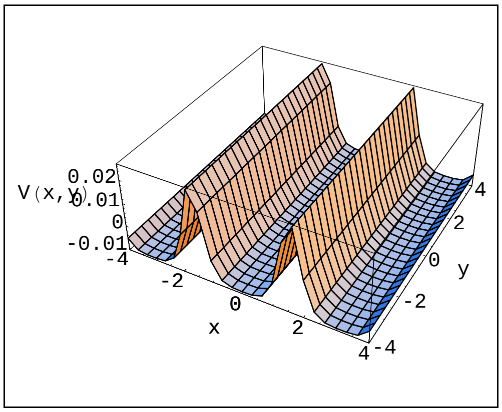

Figure 11. Real part of $\mathcal{PT}$-symmetric Pöschl-Teller potential for $V_3 = 0.25$, $a = 1$, $q = 4$.

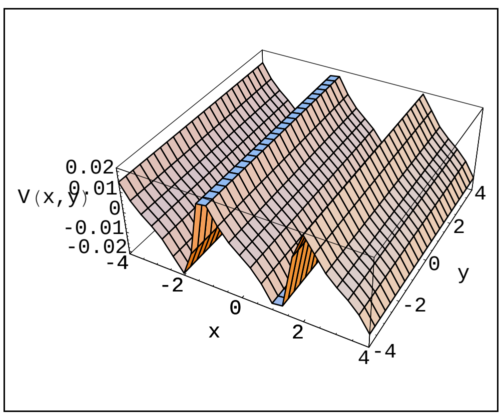

Figure 12. Imaginary part of $\mathcal{PT}$-symmetric Pöschl–Teller potential for $V_3 = 0.25$, $a = 1$, $q = 4$.

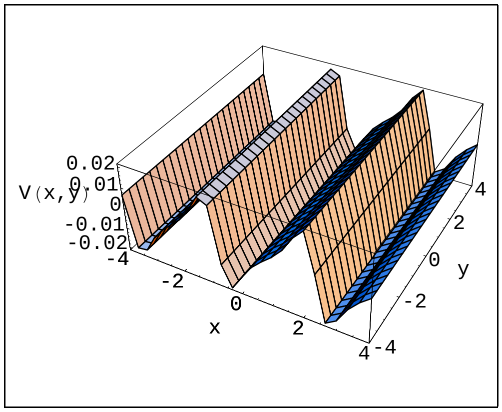

Figure 13. Real part of non $\mathcal{PT}$-symmetric Pöschl–Teller potential for $V_3 = 0.25$, $a = 1$, $q = 4$

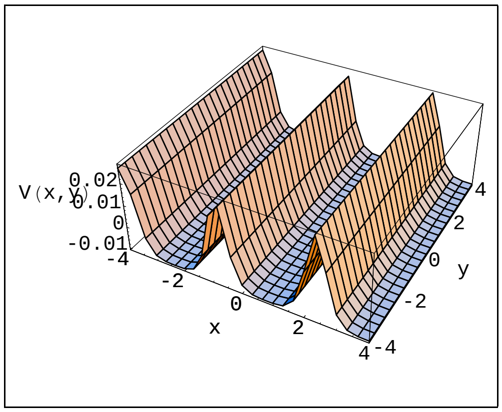

Figure 14. Imaginary part of non $\mathcal{PT}$-symmetric Pöschl–Teller potential for $V_3 = 0.25$, $a = 1$, $q = 4$.

References

[1] C.M. Bender (2005) *Contemp. Phys.* **46** 277.

[2] A. Khare, B.P. Mondal (2000) *Phys. Lett. A* **272** 53.

[3] C.M. Bender, S. Boettcher (1998) *Phys. Rev. Lett.* **80** 5243.

[4] N. Hatano, D.R. Nelson (1996) *Phys. Rev. Lett.* **77** 570.

[5] N. Hatano, N.M. Shnerb (1998) *Phys. Rev. E* **58** 1383.

[6] S.K. Moayedi, A. Rostami (2003) *Eur. Phys. J. B* **36** 359.

[7] R.N. Deb, A. Khare, B.D. Roy (2003) *Phys. Lett. A* **307** 215.

[8] C.M. Bender, K.A. Milton, V.M. Savage (2000) *Phys. Rev. D* **62** 085001.

[9] A.F. Nikiforov, V.B. Uvarov (1988) *Special Functions of Mathematical Physics*, Birkhäuser, Basel.

[10] S.M. Ikhdair, R. Sever (2007) *Int. J. Theor. Phys.* **46** 1643.

[11] Ö. Yeşiltaş *et al* (2003) *Phys. Scr.* **67** 472.

[12] A. Berkdemir *et al* (2005) *Phys. Rev. C* **72** 027001.

[13] Ö. Yeşiltaş (2007) *Phys. Scr.* **75** 41.
[14] H. Egrifes, R. Sever (2005) *Phys. Lett.* **A344** 117.
[15] S.M. Ikhdair and R. Sever (2007) *Ann. Phys.* **16** 218.
[16] M. Şimşek *et al.* (2004) *J. Phys. A. Math. Gen.* **37** 4379.
[17] A.S. Castro (2005) *Phys. Lett.* **A338** 81.
[18] F. Cooper, A. Khare, U. Sukhatme (1995) *Phys. Rep.* **251** 267.
[19] B. Gönül, K. Köksal (2007) *Phys. Scr.* **75** 686.
[20] M. Abramowitz, I. Stegun (1964) *Handbook of Mathematical Function with Formulas, Graphs and Mathematical Tables*, Dover, New York.
[21] W. Magnus, F. Oberhettinger, R.P. Soni (1966) *Formulas and Theorems for the Special Function of Mathematical Physics*, 3rd ed., Springer, Berlin.