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Calogero-Moser Spaces and Representation Theory

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*To **Ivan Todorov** with respect and admiration*

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Abstract. We characterize the phase spaces of both rational and trigonometric Calogero-Moser systems in terms of certain infinite-dimensional Lie algebras. The two versions of these algebras are defined here together with some of their natural highest weight representations. The construction makes use of the theory of bispectral operators. All needed notions and results are described with details but most of the proofs are omitted. Our final result is that the Calogero-Moser spaces (in both cases) coincide with the orbit of the vacuum of reasonably defined group GL_∞ in this representation.

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1 Introduction

Integrable systems provide a broad area of interaction of mathematics and physics. The method of inverse scattering for integration of KdV-type equations, the solution of the Riemann-Schottky problem [1] *via* the finite-gap solutions of Kadomtsev-Petviashvili equation, the intersection theory on moduli space of Riemann surfaces [2] by using a special solution of KdV, all use directly or implicitly ideas from modern physics.

For more than 15 years considerable part of my research is also occupied by integrable systems and their interplay with some important notions in quantum physics as Fock spaces, Virasoro and $W_{1+\infty}$ algebras and representation theory. I began this line of investigation jointly with my students B. Bakalov and M.

Yakimov. Later (maybe following the fashion), I focused on Calogero-Moser systems but within the above framework.

Calogero-Moser systems (in short CM) appeared first in quantum and classical integrable systems [3, 4]. Again in this context trigonometric and elliptic versions were introduced as well as other particle systems on one-dimensional manifolds. Later it was found out that they are closely related to different mathematical objects from algebraic geometry, representation theory, non-commutative geometry, etc. like rational solutions of KP-hierarchy [5, 6], Wilson's adelic Grassmannian [7,8]; isomorphism classes of right ideals of the Weyl algebra [9], representations of quivers, Cherednik algebras [10], to mention few of them.

In this paper I make a survey of my work on Calogero-Moser systems, some of it done in collaboration, see [11, 12]. I find this particularly appropriate for Ivan Todorov's fest, as one of the main tools in the present review comes from his paper (joint with B. Bakalov and L. Georgiev) on quantum field theory approach to $W_{1+\infty}$ [13].

The motion of Calogero-Moser particle [3,4] systems is governed by the Hamiltonians

$$H_n = \frac{1}{2}\sum_{i}^{n} p_i^2 - \sum_{i<j} \frac{1}{U(x_i - x_j)}, \tag{1}$$

where the potential U is one of the following functions:

1) $U(q) = q^2$ – rational CM systems;
2) $U(q) = \sinh^2(q)$ – trigonometric CM systems;
3) $U(q) = \mathfrak{p}(q)$, where $\mathfrak{p}$ is the Weierstrass $\mathfrak{p}$-function – elliptic CM systems.

In the original papers on the subject the variables are real and collisions ($x_i = x_j$) are avoided; however here we work with complex variables and do allow collisions, which will be explained below in the case of rational CM (see [8, 14] for more details). We use not only the flow of the CM Hamiltonian, but the commuting with it flows.

In [14] Kazhdan, Kostant and Sternberg used Hamiltonian reduction to describe in a simple way the Calogero-Moser flows. Let me recall their construction.

Consider the sets $\overline{C}_n, = 0, 1, 2\ldots$ of all pairs of complex matrices $\{X, Z\}$ subject to the condition that the matrix

$$[X, Z] + I \tag{2}$$

has rank one. Here I is the identity matrix. Let C_n be the quotient space $C_n = \overline{C}_n/GL(n, \mathbb{C})$, where the group $GL(n, \mathbb{C})$ acts by simultaneous conjugation. Denote by C'_N the subspace of pairs, such that X is diagonalizable. In that case the pair can be conjugated to a pair of the form

$$X = \mathrm{diag}(x_1, \ldots, x_n), \quad Z_{ii} = p_i,\ Z_{ij} = (x_i - x_j)^{-1},\ \text{for } i \neq j. \tag{3}$$

A matrix Z of that form is called *Calogero-Moser matrix*. The main result of [14] is that the Calogero-Moser flows are quotients of the simple flows $(X, Z) \to (X - t_k Z^k, Z)$, $k = 1, \ldots$ Here the variables t_k denote the commuting time variables. The last flows obviously make sense for any pair of matrices, not only to (3). For this reason the Calogero-Moser flows can be continued on the singular locus and the spaces C_n are completions of C'_N. One more remark. The CM system can be written in a Lax form

$$\dot{L} = [L, M],$$

where $L = Z$, $M = \mathrm{D}(\mathrm{tr}\, Z^2/2)$; Z is the above matrix Z. The last equation coincides with the equation defined by the Hamiltonian (1), which in fact is $H = \mathrm{tr}\, Z^2/2$.

Similar constructions exist for the other cases. I will recall briefly them for the trigonometric case.

In our recent paper [11] we defined a sub-Grassmannian $Gr^{\,\mathrm{trig}}$ of Sato's Grassmannian, which parametrizes the solutions of the bispectral problem studied in [15]. We called it the trigonometric Grassmannian and we showed its explicit connection with (trigonometric) Calogero-Moser pairs of matrices. More precisely, for $N = 1, 2, \ldots$, we denote

$$C_N^{\mathrm{trig}} = \Big\{ (X, Z) \in GL(N, \mathbb{C}) \times gl(N, \mathbb{C}) : \\ \mathrm{rank}\Big([X, Z] + X \Big) = 1 \Big\} / GL(N, \mathbb{C}). \tag{4}$$

Our objective is to describe the unions of the spaces C_n or C_n^{trig}. More precisely we construct infinite-dimensional Lie algebras $W^{\,\mathrm{ad}}$ and $DW^{\,\mathrm{ad}}$ and their highest weight modules. Then we show that $\bigcup C_n$ (respectively $\bigcup C_n^{\mathrm{trig}}$) coincides with the orbit of the vacuum under the action of the group $GL_{\,\infty}$ in this representations. This will be done by passing through an intermediate object – the so-called *bispectral operators*.

Bispectral operators have been introduced by F. A. Grünbaum in his work on medical imaging [16] (see also [17]). An ordinary differential operator $L(x, \partial_{\,x})$ is called bispectral if there exists an infinite-dimensional family of eigenfunctions $\psi(x, z)$, which are also eigenfunctions of another differential operator $\Lambda(z, \partial_z)$ in the spectral parameter z, *i.e.*, for which the following identities hold

$$L(x, \partial_x)\psi(x, z) = f(z)\psi(x, z), \tag{5}$$

$$\Lambda(z, \partial_z)\psi(x, z) = \theta(x)\psi(x, z), \tag{6}$$

with some non-constant functions $f(z)$ and $\theta(x)$. G. Wilson [7] has classified all bispectral operators of rank one (see the next Section for more details). Using slightly different terminology than in [7], they are all operators with rational coefficients that are Darboux transformations of operators with constant coefficients. Sato's theory associates with each operator (or rather with the maximal algebra of operators commuting with it) a plane in Sato's Grassmannian. The set of all planes corresponding to the rank one bispectral algebras of operators has been called by G. Wilson *an adelic Grassmannian* and denoted by $Gr^{\,\mathrm{ad}}$. Originally G. Wilson has characterized the rank one bispectral algebras A as those whose spectral curve Spec A is rational and its singularities are only cusps. Then in [8] he found an isomorphism between the disjoin union

$$\bigcup_{n\geq 0} C_n$$

and Gr^{ad}.

Thus we can consider that the points of the rational CM spaces are the rank one bispectral algebras.

About the same time we have characterized broad families of bispectral algebras [18] in terms of representation theory. More precisely we have built certain bosonic highest weight modules of $W_{1+\infty}$. Then the bispectral algebras whose spectral curves have exactly one cusp point with any degeneracy is in 1:1 correspondence with the tau-functions in the module.

Of course the restriction on the bispectral algebras seems artificial. In [18] we conjectured that a similar result should hold for the entire set of rank one bispectral operators.

Our results here [11, 19] give an affirmative answer to this conjecture.

Here are some of the ideas in our work. The first step is to put the problem into the framework of bispectral theory. The points of rational CM spaces are naturally transformed into pairs of bispectral operator, as explained above. The points of trigonometric CM spaces correspond to more tricky pairs. While the L operator remains differential (but with trigonometric coefficients) the Λ operator becomes difference. The complete collection of such pairs needs careful description along the above lines of the standard version. One has to point out a suitable generalization of the $W_{1+\infty}$-algebras. For the rational CM case the most natural candidate does the job – the algebra we look for is a central extension of the algebra of differential operators with rational coefficients. We call this new algebra *an adelic W-algebra*. For the trigonometric CM spaces the corresponding version is a central extension of the algebra of discrete operators. We call it discrete adelic algebra and denote it by $DW^{\,\mathrm{ad}}$. The next step is to construct a bosonic representation $\mathcal{M}^{\mathrm{ad}}$ of W^{ad} (respectively $\mathcal{DM}^{\mathrm{ad}}$ for the discrete adelic algebra) which is similar to a highest weight representation.

The upshot in our equivalence theorem is that the bispectral Darboux transformations are lowering operators in the above algebra. The opposite is also true: lowering operators when producing tau-functions are bispectral. In precise language our main result in the case of rational bispectral operators is the following

Theorem 1 *If an element $\tau \in \mathcal{M}^{ad}$ is a tau-function then the corresponding plane belongs to Gr^{ad}. Conversely, if $W \in Gr^{ad}$ then $\tau_W \in \mathcal{M}^{ad}$.*

Returning to the realization of Gr^{ad} as Calogero-Moser spaces we obtain immediately

Theorem 2 *The points of the rational Calogero-Moser spaces are in 1:1 correspondence with the tau-functions in $\mathcal{M}^{ad}$.*

Exactly in the same spirit we can treat the trigonometric case. The only substantial difference is the construction of the corresponding Lie algebra – DW^{ad}. This is a central extension of a natural subalgebra of the algebra of difference operators with integer coefficients. The corresponding module is denoted by $\mathcal{DM}^{\text{ad}}$. The analogues for trigonometric coefficients sounds similarly:

Theorem 3 *The points of the trigonometric Calogero-Moser spaces are in 1:1 correspondence with the tau-functions in $\mathcal{DM}^{ad}$.*

This yields

Theorem 4 *The points of the trigonometric Calogero-Moser spaces are in 1:1 correspondence with the tau-functions in $\mathcal{DM}^{ad}$.*

The paper is organized as follows. In the next Section 2 we introduce the version of bispectral Darboux transformations from [20]. It is sufficient for the rational CM systems. For the trigonometric ones we need the abstract version from [21]. We give a detailed account of the construction and immediately apply it to obtain bispectral Darboux transformations of operators with constant coefficients into operators with trigonometric coefficients. The needed notions from Sato's theory of KP hierarchy are given at the beginning of the Section.

In the following Section 3 we describe the rational and trigonometric subgrassmannians of Sato's Grassmannian.

In Section 4 we introduce the differential adelic Lie algebra W^{ad} and its discrete version – DW^{ad}. The modules $\mathcal{M}^{\text{ad}}$ and $\mathcal{DM}^{\text{ad}}$ are easily constructed from the vacuum.

Up to this point we present both versions in parallel. The last section deals only with the rational case. The point is that with the differences made clear the treatment of the trigonometric version goes essentially without new ingredients.

2 Darboux Transformations

Here I have collected some facts about Sato's theory, Darboux transformations and the bispectral problem, $W_{1+\infty}$-algebra. The last three subsections deal with the trigonometric problem and in particular with its Darboux transformations.

2.1 Sato's Theory of KP-Hierarchy

In this subsection we recall some facts and notation from Sato's theory of KP-hierarchy [22, 23] needed in the paper. We use the approach of V. Kac and D. Peterson [24] based on infinite wedge products (see *e.g.* [25]) and the survey paper by P. van Moerbeke [26].

Consider the infinite-dimensional vector space of formal series

$$\mathbb{V} = \Big\{ \sum_{k\in\mathbb{Z}} a_k v_k \Big| \, a_k = 0 \text{ for } k \gg 0 \Big\}.$$

Sato's Grassmannian Gr (more precisely – its big cell) [22] consists of all subspaces ("planes") $W \subset \mathbb{V}$ which have an admissible basis

$$w_k = v_k + \sum_{i<k} w_{ik} v_i, \quad k = 0, 1, 2, \ldots$$

The *fermionic Fock space* $F^{(0)}$ consists of formal infinite sums of semi-infinite wedge monomials

$$v_{i_0} \wedge v_{i_1} \wedge \ldots$$

such that $i_0 < i_1 < \ldots$ and $i_k = k$ for $k \gg 0$. The wedge monomial

$$\psi_0 = v_0 \wedge v_1 \wedge \ldots$$

plays a special role and is called the *vacuum*. The plane that corresponds to it will be denoted by W_0. There exists a well known linear isomorphism, called a *boson-fermion correspondence*:

$$\sigma\colon F^{(0)} \to B, \tag{7}$$

(see [25]), where $B = \mathbb{C}\,[[t_1, t_2, \ldots]]$ is the bosonic Fock space.

To any plane $W \in Gr$ one naturally associates a state $|W\rangle \in F^{(0)}$ as follows:

$$|W\rangle = w_0 \wedge w_1 \wedge w_2 \wedge \ldots,$$

where $w_0, w_1, \ldots$ form an admissible basis. One of the main objects of Sato's theory is the *tau-function* of W defined as the image of $|W\rangle$ under the boson-fermion correspondence (7)

$$\tau_W(t) = \sigma(|W\rangle) = \sigma(w_0 \wedge w_1 \wedge w_2 \wedge \ldots). \tag{8}$$

It is a formal power series in the variables $t_1, t_2, \ldots$, *i.e.* an element of $B := \mathbb{C}[[t_1, t_2, \ldots]]$. In particular the tau-function corresponding to the vacuum ψ_0 is $\tau_0 \equiv 1$. Using the tau-function one can define the other important function connected to W – the *Baker* or *wave function*

$$\Psi_W(t,z) = e^{\sum_{k=1}^{\infty} t_k z^k} \frac{\tau_W\left(t - [z^{-1}]\right)}{\tau_W(t)}, \tag{9}$$

where $[z^{-1}]$ is the vector $\left(z^{-1}, z^{-2}/2, \ldots\right)$. Introducing the vertex operator

$$X(t,z) = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{1}{kz^k} \frac{\partial}{\partial t_k}\right) \tag{10}$$

the above formula (9) can be written as

$$\Psi_W(t,z) = \frac{X(t,z)\tau(t)}{\tau(t)}. \tag{11}$$

We often use the formal series $\Psi_W(x,z) = \Psi_W(t,z)|_{t_1=x, t_2=t_3=\cdots=0}$, which we call again wave function. The wave function, corresponding to the vacuum is

$$\Psi_0(x,z) = e^{xz}.$$

The wave function $\Psi_W(x,z)$ contains the entire information about W and hence about τ_W, as the vectors $w_{-k} = \partial_x^k \Psi_W(x,z)\Big|_{x=0}$ form an admissible basis of W (if we take $v_k = z^k$ as a basis of $\mathbb{V}$).

2.2 Darboux Transformations and Rational Bispectral Operators

We shall recall a version of Darboux transformation from [20]

Definition 1 We say that a plane W (or the corresponding wave function $\Psi_W(x,z)$, the tau-function τ_W) is a *Darboux transformation* of the vacuum (respectively – of the wave function $\Psi_0(x,z)$, the tau-function τ_0) iff there exist polynomials $f(z)$, $g(z)$ and differential operators $P(x, \partial_x)$, $Q(x, \partial_x)$ such that

$$\Psi_W(x,z) = \frac{1}{g(z)} P(x, \partial_x) \Psi_0(x,z), \tag{12}$$

$$\Psi_0(x,z) = \frac{1}{f(z)} Q(x, \partial_x) \Psi_W(x,z). \tag{13}$$

The Darboux transformation is called *polynomial* iff the operators $P(x, \partial_x)$ and $Q(x, \partial_x)$ have rational coefficients.

Obviously

$$Q(x,\partial_x)P(x,\partial_x)\Psi_0 = g(z)f(z)\Psi_0, \tag{14}$$

Denoting the polynomial $g(z)f(z)$ by $h(z)$ and recalling that $\Psi_0 = e^{xz}$ we see that

$$Q(x,\partial_x)P(x,\partial_x) = h(\partial_x).$$

On the other hand the wave function Ψ_W is an eigen-function of the differential operator

$$L(x,\partial_x) = P(x,\partial_x)Q(x,\partial_x).$$

Notice that the operator L is a traditional Darboux transformation of the operator $h(\partial_x)$, which justifies the terminology of the definition. We will also say that the operator L is *a polynomial Darboux transformation* of the operator ∂_x.

We shall need a second definition of the polynomial Darboux transformation. In the above notation let the polynomial $h(\partial_x)$ factorizes as

$$h(\partial_x) = \prod_{j=1}^{m}(\partial_x - \lambda_j)^{d_j},$$

where λ_j are the different roots with multiplicities d_j. Then the kernel of $h(\partial_x)$ is given by

$$\ker h(\partial_x) = \oplus_{j=1}^{m} W_j,$$

where

$$W_j = \left\{e^{\lambda_j x}, xe^{\lambda_j x}, \ldots, x^{d_j-1}e^{\lambda_j x}\right\}.$$

Definition 2 The Darboux transformation is *polynomial* iff the kernel of P has the form

$$\ker P = \oplus_{j=1}^{m} K_j,$$

where K_j is a linear subspace of W_j.

The equivalence of the two definitions can be found in [27]. Each nonzero element $f \in K_j$ will be called (after Wilson) *condition supported at* λ_j. Finally we recall the bispectral involution b, which in this case maps the operators with polynomial coefficients in the x-variable into operators with polynomial coefficients in the z-variable by the formulas

$$b(\partial_x) = z, \qquad b(x) = \partial_z,$$

i.e., in this case b is the formal Fourier transformation. It will be used when the differential operators are applied to Ψ_0 as follows:

$$\partial_x\Psi_0 = z\Psi_0, \qquad x\Psi_0 = \partial_z\Psi_0$$

We end this subsection with the following important result of G. Wilson [7]:

Theorem 5 *Any polynomial Darboux transformation of ∂_x is a rank one bispectral operator and vice versa.*

This theorem is formulated by G. Wilson in a different terminology. See [28] for an exposition using Darboux transformations.

2.3 Abstract Form of Darboux Transformations

The above construction is enough to deal with the rational case. For the trigonometric CM systems we need more sophisticated version of the bispectral Darboux transformations. In this Section we introduce it, following [21]. Consider the field $\mathbb{C}(e^{-x})$ of rational functions over $\mathbb{C}$ in e^{-x} and the algebra $\mathbb{C}(e^{-x})[\partial_x]$ of differential operators with coefficients in $\mathbb{C}(e^{-x})$. Similarly, we consider the field $\mathbb{C}(z)$ of rational functions in z and the algebra $\mathbb{C}(z)[T^{-1}]$ of difference operators with rational coefficients from $\mathbb{C}(z)$ of the form

$$D = \sum_{j=0}^{m} a_j(z) T^{-j}, \tag{15}$$

where T^{-j} is the shift operator acting on functions of z by

$$T^{-j} f(z) = f(z-j). \tag{16}$$

Let $\psi_0(x,z)$ denote the function

$$\psi_0(x,z) = e^{xz}. \tag{17}$$

We are interested in the equation

$$P\psi_0 = Q\psi_0 \tag{18}$$

for $P \in \mathbb{C}(e^{-x})[\partial_x]$ and $Q \in \mathbb{C}(z)[T^{-1}]$. Introduce the following notation:

$$\mathcal{B}_1 = \{P \in \mathbb{C}(e^{-x})[\partial_x] : \exists Q \in \mathbb{C}(z)[T^{-1}] \text{ for which (18) holds}\} \tag{19}$$
$$\mathcal{B}_2 = \{Q \in \mathbb{C}(z)[T^{-1}] : \exists P \in \mathbb{C}(e^{-x})[\partial_x] \text{ for which (18) holds}\}. \tag{20}$$

It is clear that $\mathcal{B}_1$ and $\mathcal{B}_2$ are associative algebras without zero divisors. Obviously (18) defines an anti-isomorphism

$$b : \mathcal{B}_1 \to \mathcal{B}_2, \qquad b(P) = Q. \tag{21}$$

Introduce also the subalgebras

$$\mathcal{K}_1 = \mathcal{B}_1 \cap \mathbb{C}(e^{-x}), \quad \mathcal{K}_2 = \mathcal{B}_2 \cap \mathbb{C}(z), \tag{22}$$
$$\mathcal{A}_1 = b^{-1}(\mathcal{K}_2), \quad \mathcal{A}_2 = b(\mathcal{K}_1). \tag{23}$$

It is easy to check that

$$\mathcal{B}_1 = \mathbb{C}[e^{-x}][\partial_x], \quad \mathcal{B}_2 = \mathbb{C}[z][T^{-1}], \tag{24}$$

$$\mathcal{K}_1 = \mathbb{C}[e^{-x}], \quad \mathcal{K}_2 = \mathbb{C}[z], \tag{25}$$

$$\mathcal{A}_1 = \mathbb{C}[\partial_x], \quad \mathcal{A}_2 = \mathbb{C}[T^{-1}]. \tag{26}$$

In particular we have

$$\partial_x \psi_0 = z\psi_0 \quad \text{and} \quad T^{-1}\psi_0 = e^{-x}\psi_0.$$

The last equations show that the operators in the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are bispectral.

Following [21] we give a definition of bispectral Darboux transformations. We notice that the algebras $(\mathcal{B}_1, \mathcal{B}_2)$ being without zero divisors and Noethetian have corresponding skew-fields of fractions $\mathcal{F}_1$, $\mathcal{F}_2$. The anti-automorphism b naturally extends to $\mathcal{F}_1$.

Definition 3 *Let $(\mathcal{B}_1, \mathcal{B}_2)$ be dual algebras. Let $L_0 \in \mathcal{A}_1$ be a bispectral operator which can be factorized as*

$$L_0 = Q\theta^{-1}P \ \ \textit{with} \quad P,\, Q \in \mathcal{B}_1,\ \theta \in \mathcal{K}_1. \tag{27}$$

We call the operator

$$L = PQ\theta^{-1} \tag{28}$$

a bispectral Darboux transformation *of L_0.*

We reproduce the proof of the following theorem [21] as it is quite simple and instructive.

Theorem 6 *The operator L constructed in* (28) *is bispectral. More precisely, if $f = b(L_0) \in \mathcal{K}_2$ then*

$$L\psi = f\psi \tag{29}$$

and

$$\Lambda\psi = \theta\psi, \tag{30}$$

where

$$\Lambda = b(P)b(Q)f^{-1} \ \textit{and}\ \psi = P\psi_0. \tag{31}$$

Proof. As $\mathcal{B}_1$ has no zero divisors (27) implies

$$\theta = PL_0^{-1}Q, \quad L_0^{-1} \in \mathcal{F}_1.$$

Applying the anti-isomorphism b we obtain

$$\Lambda_0 = b(\theta) = b(Q)f^{-1}b(P).$$

This shows that Λ is a bispectral Darboux transformation of Λ_0. From (28) and (31) it follows that ψ satisfies (29) and (30).

Theorem 6 constructs bispectral algebras $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$ with common eigenfunction ψ, containing the operators L and Λ, respectively.

Remark 1 Let $(\mathcal{A}_1, \mathcal{A}_2)$ be bispectral algebras. There is a trivial way of producing new bispectral algebras from $(\mathcal{A}_1, \mathcal{A}_2)$ by fixing nonzero elements $g_1 \in \mathcal{K}_1$ and $g_2 \in \mathcal{K}_2$ and defining

$$\bar{\mathcal{A}}_j = g_j \mathcal{A}_j g_j^{-1} = \{g_j L g_j^{-1} : L \in \mathcal{A}_j\} \text{ for } j = 1, 2$$

with new common eigenfunction $\psi = g_1 g_2 \psi_0$. This corresponds to a trivial factorization in (27) by choosing $P = \theta \in \mathcal{K}_1$. We shall fix this freedom later by normalizing appropriately the common eigenfunction ψ.

2.4 Bispectral Operators with Trigonometric Coefficients

In order to apply Theorem 6, we need to describe the factorizations (27) for the dual algebras defined by (18). Consider an operator L_0 with constant coefficients, i.e. $L_0 \in \mathbb{C}[\partial_x]$. The key is to understand and parametrize factorizations of L_0 of the form

$$L_0 = Q_0 P_0, \quad \text{where} \quad P_0 \in \mathbb{C}(e^{-x})[\partial_x]. \tag{32}$$

Indeed, if (32) holds, then $Q_0 = L_0 P_0^{-1} \in \mathbb{C}(e^{-x})[\partial_x]$. Therefore, we can find $\theta_1, \theta_2 \in \mathbb{C}[e^{-x}] = \mathcal{K}_1$, such that

$$Q_0 = Q\frac{1}{\theta_1} \text{ and } P_0 = \frac{1}{\theta_2}P, \text{ where } Q, P \in \mathbb{C}[e^{-x}][\partial_x] = \mathcal{B}_1.$$

Thus equation (32) leads to a factorization of the form (27) if we take $\theta = \theta_1 \theta_2$. The freedom in choosing θ amounts to trivial conjugations of the bispectral algebras as explained in Remark 1. Conversely, if (27) holds we can simply take $Q_0 = Q$ and $P_0 = \theta^{-1} P$ to obtain (32).

Let us we focus on (32). Without any restriction we can assume that P_0 is a monic operator. Clearly, if (32) holds then $\ker(P_0) \subset \ker(L_0)$. Conversely, if $\ker(P_0) \subset \ker(L)$ then one can find Q_0 such that (32) holds. It is well known that a monic differential operator is uniquely determined by its kernel. Indeed if $\ker(P_0) = \mathrm{Span}\{\Phi_1, \Phi_2, \ldots, \Phi_k\}$ then it is easy to see that

$$P_0(\Phi) = \frac{\mathrm{Wr}_x(\Phi_1, \Phi_2, \ldots, \Phi_k, \Phi)}{\mathrm{Wr}_x(\Phi_1, \Phi_2, \ldots, \Phi_k)}, \tag{33}$$

where Wr_x denotes the Wronskian determinant with respect to x. Hence the factorization (32) amounts to picking a subspace of $\ker(L_0)$. The operator L_0

can be uniquely written as

$$L_0 = \prod_{r=1}^{n} \prod_{j=0}^{n_r} (\partial_x - \lambda_r + j)^{m_{r,j}}, \tag{34}$$

where the complex numbers $\{\lambda_r\}$ are such that $\lambda_r - \lambda_j \notin \mathbb{Z}$ for $r \neq j$, and $m_{r,j} \in \mathbb{N}_0$ is the multiplicity of the root $\lambda_r - j$ with $m_{r,0} > 0$. We can write the kernel of L_0 as

$$\ker(L_0) = \bigoplus_{r=1}^{n} W_r, \tag{35}$$

where

$$W_r = \mathrm{Span}\{x^s e^{(\lambda_r - j)x} : s = 0, 1, \ldots, m_{r,j} - 1, \quad j = 0, 1, \ldots, n_r\}. \tag{36}$$

The next proposition is analogous to Lemmas 2.8-2.9 in [27]. It allows us to parametrize the bispectral algebras of differential operators described in the previous section by a sub-Grassmannian Gr^{trig} of Sato's Grassmannian.

Proposition 1 *Let $L_0 \in \mathbb{C}[\partial_x]$ and P_0 be a monic differential operator, such that* (32) *holds. Then the following conditions are equivalent:*

(i) $P_0 \in \mathbb{C}(e^{-x})[\partial_x]$, *i.e. the coefficients of P_0 are rational functions of e^{-x};*

(ii) $\ker(P_0) = \oplus_{r=1}^{n} V_r$ *with $V_r \subset W_r$ having a basis which is a union of sets of the form*

$$\frac{\partial_y^l}{l!} \left(\sum_{j=0}^{n_r} \sum_{s=0}^{m_{r,j}} c_{r,j,s} y^s e^{(\lambda_r - j)x} \right) \Bigg|_{y=x}, \quad l = 0, 1, \ldots, l_0, \tag{37}$$

where $l_0 = \max\{s : c_{r,j,s} \neq 0$ for some $j\}$.

Following G. Wilson we will call the set of all planes $W \subset Gr$ that are polynomial Darboux transformations of W_0 *the adelic Grassmannian* and denote it by Gr^{ad}. In another paper [8] G. Wilson proved that there is a bijection

$$\beta : \bigcup_{n \geq 0} C_n \to Gr^{\mathrm{ad}}$$

between the union of all Calogero-Moser spaces and the adelic Grassmannian.

2.5 The Trigonometric Grassmannian

To each function of the form $\Phi(x) = \sum_{k,j} c_{k,j} x^k e^{\lambda_j x}$ corresponds a linear functional (condition) acting in the variable z defined by $c = \sum_{k,j} c_{k,j} \delta^{(k)}_{\lambda_j}$. Using this correspondence for $\ker(P_0)$ in Proposition 1 we define a finite dimensional space C of conditions. Thus we obtain $W \in \mathrm{Gr}^{\mathrm{rat}}$ and therefore for every $p \in A_W$ there exists an operator $L_p \in \bar{\mathcal{A}}_1 = \mathcal{A}_W$ such that (29) holds with $f = p$. Moreover, by Theorem 6 and Remark 1 there exists also a commutative algebra $\bar{\mathcal{A}}_2$ of difference operators in z, such that $\psi_W(x, z)$ is an eigenfunction for every operator in $\bar{\mathcal{A}}_2$ with eigenvalue depending only on x. Thus the operators in the algebras $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$ provide solutions of the differential-difference bispectral problem.

There is a unique choice for the polynomial in (44) so that the stationary wave function satisfies the condition

$$\lim_{e^x \to \infty} \psi_W(x, z) e^{-xz} = 1. \tag{38}$$

The Grassmannian $\mathrm{Gr}^{\mathrm{rat}}$ consists of all such spaces W. The formal definition is as follows.

Consider a finite set of complex numbers $\mu_1, \mu_2, \ldots, \mu_n$, not necessarily distinct. Let C be a subspace of conditions with basis $\{c_1, c_2, \ldots, c_n\}$, where

$$c_r = \sum_{j=0}^{n_r} \sum_{s=0}^{m_{r,j}} c_{r,j,s} \delta^{(s)}_{\mu_r - j} \text{ for } r = 1, 2, \ldots, n, \tag{39}$$

such that for every r the condition

$$c'_r = \sum_{j=0}^{n_r} \sum_{s=1}^{m_{r,j}} c_{r,j,s} s \delta^{(s-1)}_{\mu_r - j} \in C.$$

Equivalently, we can write $c_r = \sum_{j=0}^{n_r} d_{r,j}$, where $d_{r,j} = \sum_{s=0}^{m_{r,j}} c_{r,j,s} \delta^{(s)}_{\mu_r - j}$ are one-point conditions at $\mu_r - j$. Without any restriction we can assume that the conditions $\{d_{1,0}, d_{2,0}, \ldots, d_{n,0}\}$ are linearly independent. We define $W \in \mathrm{Gr}^{\mathrm{rat}}$ by

$$W = \prod_{r=1}^{n} \frac{1}{(z - \mu_r)} \left\{ g \in \mathbb{C}[z] : \langle c, g \rangle = 0 \text{ for all } c \in C \right\}. \tag{40}$$

The trigonometric Grassmannian $\mathrm{Gr}^{\mathrm{rat}}$ consists of all subspaces W obtained by the above construction. The commutative algebras $\mathcal{A}_W$ for $W \in \mathrm{Gr}^{\mathrm{rat}}$ are precisely the algebras $\bar{\mathcal{A}}_1$ described in Remark 1 in terms of the Darboux transformation.

3 Adelic W-Algebras

In this Section we introduce two Lie algebras and their particular representations. The latter form the framework of our characterization of CM spaces.

3.1 $W_{1+\infty}$-Algebra

First we recall the definition of $W_{1+\infty}$, and some of its bosonic representations introduced in [29]. For more details see [25]

The algebra w_∞ of the additional symmetries of the KP–hierarchy is isomorphic to the Lie algebra of regular polynomial differential operators on the circle

$$w_\infty \equiv \mathcal{D} = \mathrm{Span}\{z^\alpha \partial_z^\beta |\ \alpha, \beta \in \mathbb{Z},\ \beta \geq 0\}.$$

It was introduced in [30, 31] and was extensively studied by many authors (see, e.g. [32, 33], *etc*.). Its unique central extension is denoted by $W_{1+\infty}$.

Denote by c the central element of $W_{1+\infty}$ and by $W(A)$ the image of $A \in \mathcal{D}$ under the natural embedding $\mathcal{D} \hookrightarrow W_{1+\infty}$ (as vector spaces). The algebra $W_{1+\infty}$ has a basis

$$c,\ J_k^l = W(-z^{l+k}\partial_z^l), \qquad l, k \in \mathbb{Z},\ l \geq 0.$$

In [29] we constructed a family of highest weight modules of $W_{1+\infty}$. Here we need the most elementary one of them, for which the next theorem is an easy exercise.

Theorem 7 *The function τ_0 satisfies the constraints*

$$J_k^l \tau_0 = 0, \qquad k \geq 0,\ l \geq 0, \tag{41}$$

$$W\left(z^{-k} P_k(D_z) D_z^l\right)\tau_0 = 0, \qquad k > 0,\ l \geq 0, \tag{42}$$

where $P_k(D_z) = \prod_{j=0}^{k-1}(D_z - j), \quad D_z = z\partial_z.$

The first constraint means that τ_0 is the highest weight vector with highest weight $\lambda(J_0^l) = 0$ of a representation of $W_{1+\infty}$ in the module

$$\mathcal{M}_0 = \mathrm{Span}\left\{ J_{k_1}^{l_1} \cdots J_{k_p}^{l_p} \tau_0 \middle| k_1 \leq \ldots \leq k_p < 0 \right\}. \tag{43}$$

One easily checks that the central charge $c = 1$. The second constraint yields that the module $\mathcal{M}_0$ is quasifinite, *i.e.*, it is finite-dimensional in each level.

3.2 An Adelic W-Algebra

The Lie algebra constructed below (see [19]) is a main ingredient needed to characterize the rational CM spaces. I call it adelic W-algebra. Most of the definitions and constructions are similar to those of $W_{1+\infty}$. For that reason we list the facts but skip much of the arguments.

Instead of the Lie algebra w_∞ of regular operators on the circle we start with the Lie algebra $\mathcal{RD}$ of differential operators with rational coefficients on the complex line. We are going to use the following basis of $\mathcal{RD}$:

$$1) \quad z^{n+l}\partial_z^l, \quad n \in \mathbb{Z}, \quad l \geq 0; \tag{44}$$

$$2) \quad (z-a)^{-n+l}\partial_z^l, \quad -n+l<0, \quad l \geq 0, \quad a \in \mathbb{C}-\{0\}. \tag{45}$$

Usually we shall consider the elements from $\mathcal{RD}$ as differential operators with coefficients that are Laurent series in z^{-1} by expanding $(z-a)^{-n+l}$ around infinity. We would like to construct natural representations of $\mathcal{RD}$. We shall work with the space $\mathbb{V}$ where $v_k = z^k$. Obviously $\mathcal{RD}$ acts naturally on $\mathbb{V}$. Then we can associate with each operator $A \in \mathcal{RD}$ an infinite matrix having only finite number of diagonals below the principal one but having eventually infinite number above it. In2 other words the matrix $(a_{i,j})$, associated with A, has the property that $a_{i,j}=0$ for $i-j \gg 0$. The Lie algebra of such matrices will be denoted by a'_∞. It can be considered as a completion of the algebra a_∞ of matrices having only finite number of diagonals (see [25]). Now we explain how to construct representations in the fermionic Fock space $F^{(0)}$. We recall that in the case considered here $F^{(0)}$ consists of formal series of semi-infinite wedge monomials

$$z^{i_0} \wedge z^{i_1} \wedge z^{i_2} \wedge \ldots, \tag{46}$$

with $i_0 < i_1 < \ldots$ and $i_k = k$ for $k \gg 0$. We can define the action of $A \in a'_\infty$ by the standard definition (see [25]. First for matrices with only finite number of entries define

$$\begin{aligned} r(A)(z^{i_0} \wedge z^{i_1} \wedge \ldots) &= Az^{i_0} \wedge z^{i_1} \wedge \ldots \\ &+ z^{i_0} \wedge Az^{i_1} \wedge \ldots \\ &\ldots \end{aligned}$$

It is easy to check that if $A \in a'_\infty$ has no entries on the main diagonal $r(A)$ still makes sense, the image being infinite formal series. For matrices with infinite number of entries on the main diagonal the above definition is no longer meaningful. For that reason we need to modify it as follows. We put

$$\hat{r}(E_{i,j}) = r(E_{i,j}) \quad \mathrm{f}or \quad i \neq j \quad \mathrm{o}r \quad i=j>0; \tag{47}$$

$$\hat{r}(E_{i,i}) = r(E_{i,i}) - Id \quad \mathrm{f}or \quad i \leq 0. \tag{48}$$

See [25] for more details.

This defines a representation of the central extension $a'_\infty \oplus \mathbb{C}c$. The corresponding central extension of the subalgebra $\mathcal{RD}$ of a'_∞ will be called *adelic W-algebra*. We will use the notation W^{ad}. The terminology and the notation are chosen to be similar to those of the adelic Grassmannian Gr^{ad}. The main result of the present paper naturally connects the two objects.

We shall describe in some more details W^{ad}. By $W(A)$ we shall denote the image of the element $A \in \mathcal{RD}$ under the natural embedding $\mathcal{RD} \subset W^{ad}$ (as vector spaces). Then for $a \in \mathbb{C}$, $l \geq 0, \quad n \in \mathbb{Z}$ put

$$J_n^l(a) = W(-(z-a)^{n+l}\partial_z^l) \tag{49}$$

For $a = 0$ we also shall use the notation $J_n^l = J_n^l(0)$. When a is fixed the above operators (49) together with the central charge c form a copy of $W_{1+\infty}$, which we shall denote by $W_{1+\infty}(a)$. Recall that $W_{1+\infty}(a)$ has a grading: the elements $J_n^l(a)$ have weight n. The elements with nonnegative grading are common for all a. In fact the common part is much larger: for all $n + l \geq 0$ the elements $J_n^l(a)$ are common. Thus we have the following basis for W^{ad}:

$$1) \quad J_n^l(0), \quad l \geq 0 \quad n+l \geq 0; \tag{50}$$

$$2) \quad J_n^l(a), \quad n+l < 0, \quad a \neq 0; \tag{51}$$

$$3) \quad c. \tag{52}$$

In complete analogy to the case of $W_{1+\infty}$ we can construct a representation of W^{ad} in the Fock spaces using the vacuum. We formulate the needed properties in the bosonic picture.

Theorem 8 *The tau-function τ_0 satisfies the following constrains*

$$1) \quad J_n^l \tau_0 = 0, \quad l \geq 0; \quad n \geq 0 \tag{53}$$

$$2) \quad W((z-a)^{-k}P_k((z-a)\partial_z))((z-a)\partial_z)^l \tau_0 = 0, \tag{54}$$

where $P_k(u) = u(u-1)\dots(u-k+1)$.

We set

$$W_-^{ad} = span\{J_n^l, \quad a \in \mathbb{C}, \quad n < 0\}. \tag{55}$$

Then define the W_-^{ad}-module $\mathcal{M}^{ad}$ by

$$\mathcal{M}^{ad} = span\{J_{n_1}^{l_1}(a_1)\dots J_{n_m}^{l_m}(a_m)\tau_0\}, \tag{56}$$

where $n_j + l_j < 0$ for $a_j \neq 0$ and $n_j < 0$ for $a_j = 0$.

Corollary 1 *The vector space $\mathcal{M}^{ad}$ is a space of representation of the Lie algebra W^{ad}.*

Proof. is obvious.

3.3 Discrete Analog

In order to work out the representation-theoretic description of the trigonometric CM spaces we need to build something similar to the algebras $W_{1+\infty}$ and W^{ad} in the case of difference operators. First we define the Lie algebra $\mathfrak{D}$ which is spanned over $\mathbb{C}$ by difference operators with rational coefficients of the form

$$P = \sum_{j=1}^{m} p_j(z) T^{-j}, \text{ where } p_j(z) \in \mathbb{C}(z) \text{ for } j = 1, 2, \ldots, m. \qquad (57)$$

Again we look on the rational coefficients as Laurent series around $z = \infty$. Thus we can naturally define a central extension of $\mathfrak{D}$ which is the algebra $DW^{\,ad}$.

For $k, j \in \mathbb{Z}$ and $b \in \mathbb{C}$ we denote

$$D_k^{-j}(b) = W((z-b)^k T^{-j}). \qquad (58)$$

Then the Lie algebra $DW^{\,ad}$ has a basis

$$c, \quad D_k^{-j}(0), \quad k \in \mathbb{N}_0, \quad j \in \mathbb{N}; \qquad (59)$$

$$D_{-k}^{-j}(b), \quad 0 \neq b \in \mathbb{C}, \quad k, j \in \mathbb{N}. \qquad (60)$$

Using the definition of $\hat{r}$ one can easily establish the following analog of Theorem 7.

Proposition 2 *The function $\tau_0 = 1$ satisfies the constraints*

$$D_k^{-j}\, \tau_0 = 0 \quad \textit{for} \quad k \in \mathbb{N}_0, \quad j \in \mathbb{N}. \qquad (61)$$

As an immediate corollary of the above proposition, we see that if we set

$$DW_-^{ad} = \text{Span}\{D_{-k}^{-j}(b) : b \in \mathbb{C}, \quad k, j \in \mathbb{N}\}, \qquad (62)$$

we obtain a representation of the Lie algebra $DW^{\,ad}$ in the space

$$\mathcal{MD}^{\text{ad}} = \text{Span}\{d_1 \circ d_2 \circ \cdots \circ d_r(\tau_0) : d_j \in DW_-^{ad}\}. \qquad (63)$$

4 Sketch of the Proof of Theorem 1

In this Section we give the main steps of the proof of Theorem 1. The proofs for the trigonometric case are similar.

4.1 Tau-functions in the module $\mathcal{M}^{ad}$

We suppose that a tau-function $\tau \in \mathcal{M}^{\mathrm{ad}}$. We want to show that it is a Darboux transformation. The elements of $\mathcal{M}^{\mathrm{ad}}$ can be represented in the form $\tau_W = u\tau_0$ where u is an element of the universal enveloping algebra of W_{-}^{ad} of the form

$$u = \sum b_{k,l,a} J^{l_1}_{-k_1}(a_1) \dots J^{l_p}_{-k_p}(a_p) J^{l_{p+1}}_{-k_{p+1}} \dots J^{l_{p+r}}_{-k_{p+r}} \tag{64}$$

with $l_j < k_j$.

The representation itself is by definition, but the last restriction on the indices needs some work.

Next we present the wave function in the form

$$\Psi_W(t,z) = \frac{X(t,z)u\tau_0}{u\tau_0}|_{t_1=x,t_2=\ldots=0}. \tag{65}$$

We need to commute the vertex operator $X(t,z)$ with u. We only give the final result

$$\Psi_W = \frac{P(x,z,\partial_z)\Psi_0}{g(z)p(x)}$$

or using the bispectral involution we finally obtain:

$$\Psi_W = \frac{P_1(x,\partial_x)\Psi_0}{g(z)}, \tag{66}$$

where P_1 is an operator with rational coefficients.

The last step is to express Ψ_0 in terms of Ψ_W with similar formula. For achieving such a representation we use the well known adjoint involution a, which roughly speaking is continuation of the conjugation of differential operators to all objects in W^{ad}. Then we apply the above formula to the plane aW to obtain

$$\Psi_{aW} = \frac{P_2(x,\partial_x)\Psi_0}{g_2(z)}.$$

Again applying the involution a, we finally obtain

$$\Psi_0(x,z) = \frac{P_2^*(x,\partial_x)\Psi_W}{g_2(z)}, \tag{67}$$

where P_2^* is the adjoint operator to P. Here the manipulations with the adjoint involution are quite non-trivial and in particular use the free field realization of W_∞ from [13].

The last formula (67) together with (66) gives the proof of the first part of the theorem.

4.2 The Planes of the Adelic Grassmannian

In this Section we shall prove the inverse of Theorem 1, that is, if a plane $W \in Gr^{ad}$ then the corresponding tau-function $\tau_W \in \mathcal{M}^{ad}$. Here we suppose that Ψ_W is obtained via Darboux transformation

$$\Psi_W(x,z) = \frac{P(x,\partial_x)\Psi_0(x,z)}{g(z)} \tag{68}$$

$$\Psi_0(x,z) = \frac{Q(x,\partial_x)\Psi_W(x,z)}{f(z)}, \tag{69}$$

where $Q \circ P = h(\partial_x)$ with some polynomial h. Our aim is to transform the operator $P(x,\partial_x)$ into an element of the universal enveloping algebra of $W^{\,\mathrm{ad}}$.

Let $\lambda_1, \ldots, \lambda_m$ be the different points, where the conditions are supported. Then we can suppose that the polynomial h is

$$h(z) = \prod_{j=0}^{m} (z - \lambda_j)^{d_j}.$$

Denote the degree of h by d. Let the number of the conditions supported at the point λ_j be r_j. Then

$$g(z) = \prod_{j=0}^{m} (z - \lambda_j)^{r_j}.$$

Put also $\deg g(z) = r = r_1 + \ldots r_m$. Let $\{\Phi_i\}_{|i=1,\ldots,d}$ be the standard basis of $\ker h(z)$, *i.e.*

$$\{\Phi_i\} = \bigcup_{j=1}^{m} \{e^{\lambda_j x}, \ldots, x^{d_j-1}e^{\lambda_j x}\}$$

Denote by $f_1, \ldots, f_r$ the functions spanning the kernel of the operator P, *i.e.* defining the Darboux transformation (68) – (69). Then

$$f_l(x) = \sum_{i=1}^{d} a_{l,i}\Phi_i(x), \quad l = 1, \ldots, r \tag{70}$$

Introduce also the following set of functions from the kernel of $h(\partial_x)$:

$$\begin{gathered} \tilde{f}_1(x) = e^{\lambda_1 x}, \ldots, \tilde{f}_{r_1}(x) = x^{r_1-1}e^{\lambda_1 x} \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \tilde{f}_{r-r_m+1} = e^{\lambda_m x}, \ldots, \tilde{f}_r = x^{r_m-1}e^{\lambda_r x} \end{gathered}$$

and consider the functions f_l as a deformation of $\tilde{f}_l$

$$f_l(x,\varepsilon) = \varepsilon f_l(x) + (1-\varepsilon)\tilde{f}_l(x), \quad l = 1, \ldots, r.$$

This defines a family of planes $W(\varepsilon)$ in Gr^{ad} with wave functions:

$$\Psi_{W(\epsilon)}(x,z) = \frac{\mathrm{Wr}_x(e^{xz}, f_1(x,\varepsilon), \dots f_r(x,\varepsilon))}{g(z)\mathrm{Wr}_x(f_1(x,\varepsilon), \dots f_r(x,\varepsilon))}. \tag{71}$$

It is obvious that $\Psi_{W(0)} = e^{xz}$. Also

$$\Psi_{W(\epsilon)} = \frac{\Big(g(z) + \varepsilon P(x,z,\varepsilon)\Big)e^{xz}}{g(z)\Big(1 + \varepsilon R(x,z,\varepsilon)\Big)},$$

where P and R are polynomials in all variables.

Let us expand the above expression around $\varepsilon = 0$ and then use the bispectral involution to get rid of x-dependence. We get

$$\Psi_{W(\epsilon)} = (1 + \sum_{j=1}^{\infty} \varepsilon^j P_j(z, \partial_z))e^{xz}. \tag{72}$$

The important fact here is that all the operators $P_j(z,\partial_z) \in W_-^{ad}$. The standard basis of $W(0)$ is given by $w_k = \partial_x^k \Psi_0 = z^k$, $k = 0, 1, \dots$. We need to find expression for the basis of the plane $W(\varepsilon)$. We have

$$\partial_x^k \Psi_{W(\varepsilon)} = (1 + \sum_{j=1}^{\infty} \varepsilon^j P_j(z,\partial_z))w_k.$$

Using the boson-fermion correspondence σ we get the tau-function $\tau_{W(\varepsilon)}$

$$\begin{aligned}\tau_{W(\varepsilon)} &= \sigma\Big((1 + \sum_{j=1}^{\infty} \varepsilon^j P_j(z,\partial_z))w_0 \wedge (1 + \sum_{j=1}^{\infty} \varepsilon^j P_j(z,\partial_z))w_1 \wedge \dots\Big) \\ &= \tau_0 + \varepsilon r(P_1)\tau_0 + \varepsilon^2\Big(r(P_2) + \frac{1}{2}r(P_1)^2 - \frac{1}{2}r(P_1^2)\Big)\tau_0 + \dots\end{aligned}$$

Notice that the coefficients at the powers of ε are polynomials in $r(P_k^j)$, applied to τ_0. Hence all of them belong to the W^{ad}-module $\mathcal{M}^{ad}$. We need to show that the entire series belongs to it. We are going to use the Wronskian representation of the tau-function

$$\tau_{W(\varepsilon)} = \mathrm{Wr}(f_1(x,\varepsilon), \dots, f_r(x,\varepsilon)).$$

The point is that the tau-function is a polynomial and not an infinite series in ε proving that it belongs to the W^{ad}-module $\mathcal{M}^{ad}$.

5 Final Remarks

It would be interesting to find analogs of the present results for other bispectral operators – both continuous and discrete. For example there are particle systems connected to the bispectral operators from [28, 34]. Even more intriguing would be to consider particle systems coming from discrete bispectral operators [15, 35, 36]. In this respect it seems to me that the results of P. Iliev [37] will be very helpful.

The treatment of the elliptic CM spaces seems to need other tools. The point is that the elliptic CM matrices have not been characterized in terms of bispectral operators and quite possibly they cannot be. Hence the plan of the present paper is unlikely to be carried out.

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