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3-Point Tachyon Correlators in the Open ZZ Non-Critical String

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Dedicated to ***Ivan Todorov*** *on the occasion of his 75th birthday*

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Abstract. The 3-point tachyon correlators of the open non-critical string theory with ZZ Liouville branes are found. The construction includes the determination of the matter 3-point boundary coefficients for non-degenerate $c < 1$ Virasoro representations.

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1 Introduction

In [1] Al. Zamolodchikov and A. Zamolodchikov initiated the study of a $c > 25$ Virasoro (Vir) CFT interpreted as a quantisation of the classical Liouville equation on a non-compact manifold, the 2d Euclidean pseudosphere. In contrast with the FZZ Liouville theory on the disk [2], the assumed in [1] factorisation condition makes the boundary bootstrap equations and their solutions similar to those first considered by Cardy and Lewellen [3–5] in the $c < 1$ (rational) Virasoro CFT. Thus the ZZ type conformal boundary conditions are described by the set of degenerate Vir representations. For generic values of the central charge c_L and scaling dimensions $\triangle_L(\alpha)$

$$c_L = 13 + 6(b^2 + \frac{1}{b^2})\,,\ \triangle_L(\alpha) = \alpha(Q-\alpha)\,,\ Q = b + \frac{1}{b} \qquad (1)$$

with real parameter b, this infinite set is parametrised by the charges $\alpha = \alpha_{2m+1,2n+1}$

$$\alpha = -mb - \frac{n}{b},\ \text{or},\ \alpha = Q + mb + \frac{n}{b},\ ,2m, 2n \in \mathbb{Z}_{\geq 0}. \qquad (2)$$

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Furthermore, in the multiplicity ${n_{\alpha a}}^{b}$ in the cylinder partition function Z_{ba} with boundary conditions a and b of this type, the representation label α runs in the same set (2) as well. This fusion multiplicity determines the possible boundary fields ${}^{b}V_{\alpha}^{a}(x)$ in the ZZ Liouville theory and it corresponds to the quasi-rational $sl(2)\times sl(2)$ fusion multiplicity (finite irreps tensor product decomposition multiplicity). The bulk-boundary correlators for ZZ boundary fields have been computed in [6, 7].

The Liouville ZZ and FZZ branes were exploited in studies [8–11] of the open non-critical string. This is a quantum Liouville gravity theory with matter described by a dual Vir theory of $c < 1$ central charge

$$c_M = 13 - 6(b^2 + \frac{1}{b^2})\,,\ \triangle_M(e) = e(e_0 - e)\,,\ e_0 = \frac{1}{b} - b \qquad (3)$$

so that $c_M + c_L = 26$. Adding a pair of reparametrisation ghosts $(\mathbf{b}, \mathbf{c})$ of dimensions $(2, -1)$ and central charge $c_{gh} = -26$ cancels the total charge of the theory.

Our aim in this paper is to construct the 3-point correlators of boundary tachyon operators. Recall that these operators $T_{\beta}^{(\epsilon)} = {}^{(\bar\sigma_2,\sigma_2)}\,{T_{(e,\beta)}}^{(\bar\sigma_1,\sigma_1)}$ of chirality $\epsilon = \pm 1$ and boundary Liouville σ_i (respectively matter $\bar\sigma_i$) parameters, are constructed as products of matter and Liouville boundary operators times a ghost field, altogether of zero total scaling dimension

$$T_e^{(\epsilon)}(x) = b^{\epsilon}\Gamma(b^{\epsilon}(Q - 2\beta))\,\mathbf{c}(x)\,e^{2ie\chi(x)}\,e^{2\beta\phi(x)}\,, \qquad (4)$$

$$\triangle_M(e) + \triangle_L(\beta = \epsilon e + b^{\epsilon}) = 1 = -\triangle_{\text{ghost}\,\mathbf{c}}. \qquad (5)$$

The 3-point correlators factorise, the coordinate dependence cancels: the resulting correlator numbers are presented by the product of 3-point boundary coefficients of the matter and Liouville correlators

$$\hat{C}^{(\bar\sigma_3,\sigma_3),(\bar\sigma_2,\sigma_2),(\bar\sigma_1,\sigma_1)}_{(e_3,\beta_3),(e_2,\beta_2),(e_1,\beta_1)} = \prod_i b^{\epsilon_i}\Gamma(b^{\epsilon_i}(Q-2\beta))\ {}^{M}C^{\bar\sigma_3,\bar\sigma_2,\bar\sigma_1}_{e_3,e_2,e_1}\ {}^{L}C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} \qquad (6)$$

where the pairs (e_i, β_i) are subject of the mass-shell condition in (5). The leg factor normalisation in (4) is used as usually for convenience.

In [12] we have solved the analogous problem in the case when the dressing Liouville boundary correlator corresponds to FZZ type boundary conditions. The charges e_i of the matter boundary fields in that case correspond to degenerate $c < 1$ Vir representations, or, equivalently the Liouville charges $\alpha_i = e_i + b$, or $\alpha_i = -e_i + \frac{1}{b}$ of the tachyons are given by

$$\alpha_i = b + m_i b - \frac{n_i}{b}\,, \quad 2m_i, 2n_i \in \mathbb{Z}_{\geq 0} \qquad (7)$$

or, any of the reflected values, $Q - \alpha_i$. Both the Liouville and matter factors are solutions of pentagon equations. The integral formula of Teschner-Ponsot

(PT) [13], derived originally for the continuous series of $c > 25$ representations $\alpha_i = \frac{Q}{2} + iP_i$ with real P_i, can be continued to the real discrete set (7). The integral rewrites as a (double) infinite sum. Instead, a compact expression for the Liouville 3-point factor for the charges (7) was obtained in [12] by directly solving recursively the Liouville pentagon equation. It is meromorphic in the variables

$$c_i = c(\sigma_i) = 2\cos\pi b(b - 2\sigma_i)\,, \tilde{c}_i = \tilde{c}(\sigma) = 2\cos\frac{\pi}{b}(\frac{1}{b} - 2\sigma_i) \qquad (8)$$

paramerising the boundary cosmological constants. This expression generalises a special (thermal) case in [14] derived from the PT formula, examples of which have been earlier obtained in matrix model approaches [15], see also [16]. On the other hand the matter solution of the pentagon equations is a $c < 1$ Coulomb gas type boundary correlator with the three representation indices e_i restricted by a charge conservation condition: it is an extension for generic b of the expression in the rational case, given by the quantum 6j symbols up to a gauge, with the boundary labels $\bar{\sigma}_i$ identified with representation charges e_i. Alternatively one can recover this expression starting from the universal PT formula: for the three parameters α_i satisfying a charge conservation condition

$$\sum_i \alpha_i = Q - mb - \frac{n}{b}\,, \quad m, n \in \mathbb{Z}_{\geq 0} \qquad (9)$$

or any of its reflected counterparts, the PT formula develops poles. The resudue reproduces the Coulomb gas (CG) Liouville correlator, which in turn can be continued analytically to the region $c < 1$ so that (9) goes to the matter charge conservation condition.

The ZZ case under consideration in this paper is somewhat dual to the FZZ case considered in [12], since now the Liouville boundary fields are described by degenerate representations parametrised by the charges (2). Their 3-point boundary correlators are obtained from the $c > 25$ Coulomb gas correlators, setting in addition to (9) the Liouville charges to the values (2). This determines the Liouville factor ${}^{L}C = {}^{(CG)}C$ in (6). According to the tachyon mass-shell condition (5) the matter charges are described then by

$$e_i = -b - m_i b - \frac{n_i}{b}\,, \qquad 2m_i,\ 2n_i \in \mathbb{Z}_{\geq 0} \qquad (10)$$

or any of the dual values obtained by the reflection $e_i \to e_0 - e_i$.

The question arises how to describe the matter CFT labelled by these (non-degenerate) representations. There is no $c < 1$ analog of the integral PT formula for generic charges e_i and the latter formula given in terms of the double Gamma functions $\Gamma_b(x)$ and their ratios $S_b(x) = \Gamma_b(x)/\Gamma_b(Q - b)$ cannot be continued analytically. We shall use the expression for the Liouville correlator found in [12] to solve this complementary problem. Namely, unlike the PT formula, it admits an analytic continuation to the region $c < 1$ for the values (10) and leads to the 3-point correlator of boundary matter operators V_{e_i} of charges e_i given by the values (10). The boundary parameters $\{\bar\sigma_i\}$ are kept generic.

The values (10) correspond to the labels of the singular vectors of the related $c < 1$ Vir Verma modules of highest weight described by degenerate values, *i.e.*,

$$\triangle_M(-(m+1)b - \frac{n}{b}) = \triangle_M(mb - \frac{n}{b}) + (2m+1)(2n+1). \tag{11}$$

The corresponding vertex operators V_e can be seen as the images of differential operators acting on logarithmic field counterparts of the primaries of degenerate dimensions $\triangle_M(mb - \frac{n}{b}) = \triangle_M(e_0 - mb + \frac{n}{b})$. They are $c < 1$ analogs of the logarithmic Liouville fields, the bulk counterparts of which were exploited in [17, 18].

The content of the paper: We first describe the matter reflection amplitude for the values (10). Then we recall the Liouville Coulomb gas expression which provides the dressing factor in our case. In Section 4 we present the matter 3-point coefficients obtained by analytic continuation of the meromorphic expression in [12]. The Appendix contains some of the technical details.

2 Matter Reflection Amplitude

We first recall the Liouville reflection amplitude [2]

$$\begin{aligned} {}^{\sigma_2}V_{\beta}^{\sigma_1} &= S(\sigma_2,\beta,\sigma_1)\, {}^{\sigma_2}V_{Q-\beta}^{\sigma_1}\,, \\ S(\sigma_2,\beta,\sigma_1)\,(S(\sigma_2,Q-\beta,\sigma_1)) &= 1\,, \end{aligned}$$

which determines the non-trivial term in the Liouville 2-point function. It reads

$$\begin{aligned} &\Gamma(\frac{1}{b}(Q-2\beta))\Gamma(b(Q-2\beta))\, S(\sigma_2,\beta,\sigma_1) = \frac{2\pi b\, \lambda_{_L}{}^{\frac{Q-2\beta}{2b}}}{Q-2\beta}\bar G(\sigma_2,\beta,\sigma_1)\,, \\ &\bar G(\sigma_2,\beta,\sigma_1)\frac{S_b(b)}{S_b(2\beta-Q)} = \frac{G^{(-)}(\sigma_2,\beta,\sigma_1)}{G^{(-)}(\sigma_2,Q-\beta,\sigma_1)}, \end{aligned} \tag{12}$$

where

$$G^{(-)}(\sigma_2,\beta,\sigma_1) = S_b(-\beta+\sigma_2+\sigma_1)S_b(Q-\beta+\sigma_2-\sigma_1)\,. \tag{13}$$

For the values (7) the ratio in the r.h.s. of (12) can be written as [12]

$$\frac{\bar G(\sigma_2,\beta_2{=}b{+}m_2 b{-}n_2/b,\sigma_3)}{S_b(2\beta_2-Q)S_b(\frac{1}{b})} = \frac{\tilde B(\sigma_2,\sigma_3)^{(2n_2;p(2m_2))}}{B(\sigma_2,\sigma_3)^{(2m_2;p(2n_2))}}, \tag{14}$$

where

$$
\begin{aligned}
B(\sigma_2,\sigma_1)^{(k;p(n))} &= \frac{G^{(-)}\Big(\sigma_2, -\dfrac{kb}{2}-\dfrac{n}{2b},\sigma_1\Big)}{G^{(-)}\Big(\sigma_2, b+\dfrac{kb}{2}-\dfrac{n}{2b},\sigma_1\Big)} \\
&= (-1)^{(k+1)(n+1)} B(\sigma_1,\sigma_2)^{(k;p(n))}\,, \\
\tilde{B}(\sigma_2,\sigma_1)^{(n;p(k))} &= \frac{G^{(-)}\Big(\sigma_2, -\dfrac{n}{2b}-\dfrac{kb}{2},\sigma_1\Big)}{G^{(-)}\Big(\sigma_2, \dfrac{1}{b}+\dfrac{n}{2b}-\dfrac{kb}{2},\sigma_1\Big)} \\
&= (-1)^{(k+1)(n+1)} \tilde{B}(\sigma_1,\sigma_2)^{(n;p(k))}\,. \qquad (15)
\end{aligned}
$$

Here $p(n)$ refers to the parity of the integer n and $B(\sigma_2,\sigma_1)^{(k;p(n))}$ is a $k+1$-order polynomial of the parameters $\{c_1, c_2\}$, related to the boundary cosmological constants (8), while $\tilde{B}(\sigma_2,\sigma_1)^{(n;p(k))}$ is a $n+1$-order polynomial of the dual parameters $\{\tilde{c}_1,\tilde{c}_2\}$, see the Appendix for more explicit formulae.

To continue $\bar{G}$ analytically we first rewrite it as a ratio of finite products of sin's functions, using the relations satisfied by the function $S_b(x)$

$$
S_b(x+b^{\pm 1}) = 2\sin \pi b^{\pm 1} x\, S_b(x)\,,\ S_b(b)=b,
$$

then replace

$$
b^2 \to -b^2\,,\ b\beta \to be\,,\ b\sigma \to b\bar{\sigma}\,,\ \lambda_{_L} \to \lambda_M
$$

and again represent the result in a compact form in terms of the S_b functions. For instance, for nonnegative integer $2m, 2n$

$$
\begin{aligned}
&\frac{S_b\Big((2m+1)b-\dfrac{2n}{b}\Big)}{S_b(b)} \\
&\qquad \to (-1)^{2n2m+2m+2n}\frac{S_b((2m+1)b)}{S_b\Big(\dfrac{2n+1}{b}\Big)} \\
&\qquad\qquad = (-1)^{2m+2n}\frac{S_b\Big((2m+1)b)-\dfrac{2n}{b}\Big)}{S_b(b)}\,.
\end{aligned}
$$

The polynomials in (15) are continued straightforwardly so that the matter analogs are given by the same polynomials with the Liouville cosmological constants $c_i, \tilde{c}_i$ replaced by

$$
c_i \to c_i^M = 2\cos \pi b(b+2\bar{\sigma}_i) = c(-\bar{\sigma}_i) = c(\bar{\sigma}_i+b)\,,
$$

$$
\tilde{c}_i \to \tilde{c}_i^M = 2\cos\frac{\pi}{b}(\frac{1}{b}-2\bar{\sigma}_i) = \tilde{c}(\bar{\sigma}_i) = \tilde{c}(\bar{\sigma}_i+b)\,,
$$

i.e.,

$$B(\sigma_2,\sigma_1)^{(k;p(n))} \to {}^M B(\bar\sigma_2,\bar\sigma_1)^{(k;p(n))} := B(-\bar\sigma_2,-\bar\sigma_1)^{(k;p(n))}\,,$$
$$\tilde B(\sigma_2,\sigma_1)^{(n;p(k))} \to {}^M \tilde B(\bar\sigma_2,\bar\sigma_1)^{(n;p(k))} := \tilde B(\bar\sigma_2,\bar\sigma_1)^{(n;p(k))}\,. \qquad (16)$$

We finally obtain for the matter reflection amplitude S^M

$$\Gamma(\frac{1}{b}(2e-e_0))\Gamma(b(e_0-2e))\, S^M(\bar\sigma_2, e=-(m+1)b-\frac{n}{b},\bar\sigma_1) = \frac{2\pi b\, \lambda_M^{\frac{e_0-2e}{2b}}}{2e-e_0}\bar G^M(\bar\sigma_2,e,\bar\sigma_1) \qquad (17)$$

with

$$\bar G(\sigma_2,(m+1)b-\frac{n}{b},\sigma_1) \to \bar G^M(\bar\sigma_2,e,\bar\sigma_1) := (-1)^{2n+2m+1}\bar G(\bar\sigma_2+b,(m+1)b-\frac{n}{b},\bar\sigma_1+b) \qquad (18)$$

and

$$S^M(\bar\sigma_2,e_0-e,\bar\sigma_1) = (S^M(\bar\sigma_2,e,\bar\sigma_1))^{-1}\,.$$

The amplitude (17) satisfies the two $c<1$ functional equations. In the simplest example $e=-b$ it corresponds to the 2-point function of one of the operators defining the matter screening charges,

$$S^M(\bar\sigma_2,-b,\bar\sigma_1) = \lambda_M^{\frac{Q}{2b}}\frac{\Gamma\Big(1+\dfrac{1}{b^2}\Big)}{\Gamma(1+b^2)}\frac{\tilde c_2^M-\tilde c_1^M}{c_2^M-c_1^M} = \frac{2\pi\lambda_M^{\frac{Q}{2b}}\bar G^M(\bar\sigma_2,-b,\bar\sigma_1)}{\Gamma\Big(-\dfrac{1}{b^2}\Big)\Gamma(1+b^2)}\,.$$

If the chiralities are chosen as $(+-)$, the factors of Γ in (17) can be rewritten again as leg factors $\Gamma(\frac{1}{b}(Q-2\beta))\Gamma(b(Q-2\beta))$. Then the r.h.s. of (17) defines the tachyon reflection amplitude

$$\bar G^M(\sigma_2,e,\sigma_1):\ T^{(-)}_{e_0-e}\to T^{(+)}_e \qquad (19)$$

in the notation in (4), with e taking the values in (10).

3 Liouville CG 3-Point Boundary Correlator

From the PT integral formula [13] it is straightforward to compute for non-negative integers m,n

$${}^{CG}C^{\sigma_3\sigma_2\sigma_1}_{\beta_3\beta_2\beta_1} = 2\pi Res_{\beta_{123}-Q+mb+n/b=0}\ {}^{(PT)}C^{\sigma_3\sigma_2\sigma_1}_{\beta_3\beta_2\beta_1} =$$

$$
\begin{aligned}
&= \Pi^{(LCG)}(\beta_3\,,\beta_2,\beta_1)\,\frac{S_b(2\beta_1)}{S_b(Q-\beta_{123}+2\beta_1)}\,\lambda_{_L}^{\,\frac{m}{2}}\tilde{\lambda}_L^{\frac{n}{2}}(-1)^{m+n}\\
&\quad\times\sum_{k=0}^{m}\sum_{p=0}^{n}\frac{(-1)^{(m-k)(n-p)}}{S_b((k+1)b)S_b((m-k+1)b)S_b(\frac{p+1}{b})S_b(\frac{n-p+1}{b})}\\
&\quad\times\frac{\bar{G}(\sigma_3+\frac{(m-k)b}{2}+\frac{n-p}{2b},Q-\beta_3+\frac{(k-m)b}{2}+\frac{p-n}{2b}\,,\sigma_1)}{\bar{G}(\sigma_3,Q-\beta_3\,,\sigma_1)}\\
&\quad\times\frac{\bar{G}(\sigma_3-\frac{kb}{2}-\frac{p}{2b},Q-\beta_2-\frac{kb}{2}-\frac{p}{2b}\,,\sigma_2)}{\bar{G}(\sigma_3,Q-\beta_2\,,\sigma_2)}\\
&= \frac{\Pi^{(LCG)}(\beta_3\,,\beta_2,\beta_1)S_b(2\beta_1)\lambda_{_L}^{\,\frac{m}{2}-\frac{n}{2b^2}}}{(-1)^{m+n}S_b(mb+\frac{n}{b}+2\beta_1)}\\
&\quad\times P^{\sigma_3,\sigma_2,\sigma_1}_{Q-\beta_3,Q-\beta_2}(m;b)\,P^{\sigma_3,\sigma_2,\sigma_1}_{Q-\beta_3,Q-\beta_2}(n;\frac{1}{b})\,,\\
&\quad\Pi^{(LCG)}(\beta_3\,\beta_2,\beta_1)\\
&\quad= b^{e_0(Q-\beta_{123})}\,\prod_{i=1}^{3}\frac{\Gamma_b(Q-\beta_{123}+2\beta_i)}{\Gamma_b(2\beta_i)}\frac{\Gamma_b(2Q-\beta_{123})}{\Gamma_b(Q)},
\end{aligned}
\tag{20}
$$

where $\lambda_{_L}$ and $\tilde{\lambda}_L$ are the bulk cosmological constants and we have used the notation $\bar{G}$ defined in (12). In the second factorised form of (20) enter polynomials P of the parameters $\{c_i = c(\sigma_i)\}$ and $\{\tilde{c}_i = c(\tilde{c}_i)\}$ expressed by particular basic hypergeometric functions of type ${}_4\phi_3$, see formula (28) of the Appendix. The boundary parameters do not distinguish the degenerate values (2) $\sigma = -mb - n/b$ and the (singular vector) values $\sigma = (m+1)b - n/b$ in (7). Related symmetries of the 6j symbols were discussed in [19] in the rational $c < 1$ case. In this expression we furthermore set the values of $\beta_i = \epsilon_i e_i + b^{\epsilon_i}$ with e_i given in (10). We choose the chiralities so that $\sum_i \epsilon_i = 1$ - this implies that $m = \sum_i \epsilon_i m_i\,, n = \sum_i \epsilon_i n_i$ in (20).

As a particular example of (20) for $m = 1, n = 0, \sigma_3 = \sigma_2 \pm b/2$ one recovers from (20) the fundamental boundary OPE coefficients ${}^{CG}C^{\sigma_2\pm b/2,\sigma_2,\sigma_1}_{Q-\beta_1-b/2,-b/2,\beta_1}$ computed in [2]. Another example is given setting one of the boundary parameters, *e.g.*, σ_1 to the basic ZZ value $\sigma_1 = 0$; then the other two are dictated by the fusion rules of the chiral vertex operators, *i.e.*, $\sigma_2 = \beta_1\,, \sigma_3 = Q - \beta_3$. The fundamental pentagon equation simplifies to two (instead of three terms) and accordingly the expression (20) simplifies since the sum reduces to one term.

4 Matter 3-Point Boundary Correlator

We recall the solution of the Liouville pentagon equations found in [12] for the values of β_i as in (7)

$$
\begin{aligned}
&{}^{(FZZ)}C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} \\
&= -\frac{\lambda_L^{\frac{Q-\beta_{123}}{2b}}\,\Pi'_L(\beta_3,\beta_2,\beta_1)}{B(\sigma_1,\sigma_2)^{(2m_1;p(2n_1))}B(\sigma_2,\sigma_3)^{(2m_2;p(2n_2))}B(\sigma_3,\sigma_1)^{(2m_3;p(2n_3))}} \\
&\quad\times(-1)^{2m_2 2n_1}\Big((-1)^{2m_1+2n_2}\tilde{B}(\sigma_2,\sigma_3)^{(2n_2;p(2m_2))}\tilde{P}^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3} \\
&\times B(\sigma_3,\sigma_1)^{(2m_3;p(2n_3))}P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} - (-1)^{2m_2+2n_1}\tilde{B}(\sigma_3,\sigma_1)^{(2n_3;p(2m_3))} \\
&\qquad\times\tilde{P}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}B(\sigma_2,\sigma_3)^{(2m_2;p(2n_2))}P^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3}\Big). \qquad (21)
\end{aligned}
$$

The polynomials $P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}$ of $\{c_i\}$ are of the type in (20)

$$
P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} = \frac{S_b((2m_1+1)b)S_b((2m_2+1)b)S_b(\frac{1}{b})^3}{(-1)^{m_{123}(1+2n_3)+2m_1}S_b(Q-(m^3_{12}+1)b)} \times P^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1}(m^3_{12};b) \qquad (22)
$$

and

$$
\tilde{P}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} = \frac{(-1)^{n_{123}(1+2m_3)+2n_1}S_b(\frac{2n_1+1}{b})S_b(\frac{2n_2+1}{b})}{S_b(\frac{1}{b})^3 S_b(Q-\frac{n^3_{12}+1}{b})} \times P^{\sigma_2,\sigma_1,\sigma_3}_{Q-\beta_2,Q-\beta_1}(n^3_{12};\frac{1}{b}). \qquad (23)
$$

Here $m_{123}=\sum_{i=1}^3 m_i$, $m^k_{ij}=m_i+m_j-m_k$ are nonnegative integers as dictated by the fusion rule for the matter degenerate representations. The prefactor $\Pi'_L(\beta_3,\beta_2,\beta_1)$ is recalled in the Appendix.

Following the same steps as in the derivation of (16) for the analytic continuation of (22), (23), leads to the same polynomials with $c_i,\tilde{c}_i$ replaced by c^M_i, $\tilde{c}^M_i$ respectively, up to an overall sign

$$
\begin{aligned}
&P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} \to {}^{M}P^{\bar\sigma_3,\bar\sigma_2,\bar\sigma_1}_{e_3,e_2,e_1} \\
&\quad=\frac{S_b((2m_1+1)b)S_b((2m_2+1)b)S_b(\frac{1}{b})^3}{(-1)^{m_{123}2n_3+2m_1}S_b(Q-(m^3_{12}+1)b)}P^{-\bar\sigma_2,-\bar\sigma_1,-\bar\sigma_3}_{-e_2-\frac{n_2}{b},-e_1-\frac{n_1}{b}}(m^3_{12};b), \\
&\tilde{P}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} \to {}^{M}\tilde{P}^{\bar\sigma_3,\bar\sigma_2,\bar\sigma_1}_{e_3,e_2,e_1} \\
&\quad=\frac{S_b(\frac{2n_1+1}{b})S_b(\frac{2n_2+1}{b})S_b(b)^3}{(-1)^{n_{123}2m_3+2n_1}S_b(Q-\frac{n^3_{12}+1}{b})}P^{\bar\sigma_3,\bar\sigma_2,\bar\sigma_1}_{e_0-e_2-m_2b,e_0-e_1-m_1b}(n^3_{12};\frac{1}{b}). \qquad (24)
\end{aligned}
$$

The non-negative integers m_i , m^k_{ij} now determine the bounds as dictated by the fusion rule for the Liouville degenerate representations.

What remains to be continued is the prefactor $\Pi'_L(\beta_3,\beta_2,\beta_1)$ in (21), see the Appendix for details, we give here the final result

$$\Pi' \to \Pi'_M(e_3, e_2, e_1)$$
$$= \frac{i\,(-1)^{(m_{123}+1)(n_{123}+1)} b^{Q(e_{123}-e_0)} \prod_{i=1}^3 \Gamma_b(\frac{1}{b} - 2e_i)\Gamma_b(\frac{1}{b})}{\prod_{i=1}^3 S_b(\frac{2n_i+1}{b})\Gamma_b(\frac{1}{b} - e_{123} + 2e_i)\,\Gamma_b(\frac{1}{b} + e_0 - e_{123})}\,. \quad (25)$$

The final expression

$$ {}^{L}C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} \;\to\; {}^{M}C^{\bar\sigma_3,\bar\sigma_2,\bar\sigma_1}_{e_3,e_2,e_1} \quad (26)$$

is obtained inserting (16), (24), (25). Furthermore, the matter reflection amplitude (17) has to be used to obtain the 3-point correlators with some of the charges e_i taken in the dual region $e = mb + (n+1)/b$ instead of (10). The simplest example is the correlator of the matter screening charge, $e_i = -b$,

$${}^{M}C^{\bar\sigma_3,\bar\sigma_2,\bar\sigma_1}_{-b,-b,-b} = \frac{1}{ib\Gamma^2(1+b^2)\Gamma(-1-\frac{1}{b^2})}$$
$$\times \frac{2\pi S_b(b)\lambda_M^{\frac{-(Q+b)}{2b}}}{S_b(\frac{2}{b})} \frac{\tilde c_1^M(c_2^M - c_3^M) + \tilde c_2^M(c_3^M - c_1^M) + \tilde c_3^M(c_1^M - c_2^M)}{(c_2^M - c_1^M)(c_1^M - c_3^M)(c_3^M - c_2^M)}\,. \quad (27)$$

5 Tachyon Correlator

Finally we insert in (6) the ZZ Liouville factor ${}^{L}C = {}^{CG}C$ from (20) and the matter continuation ${}^{M}C$ (26) to get the tachyon 3-point correlation number. In this normalised product for any choice of the chiralities all Γ_b functions disappear combining into the ratios of S_b functions, see the Appendix for details. Alternatively one can continue directly the overall prefactor of the tachyon correlator computed in [12]. If a power of b is included in the leg factor normalisation for the FZZ expression, as in (4), then in the ZZ 3-point expression it has to be replaced by ib. E.g., in the tachyon counterpart of (27), for which the Liouville CG factor is trivial, the prefactor in the r.h.s of the first line is canceled by the leg factor normalisation.

6 Summary and Discussion

We have constructed the 3-point tachyon boundary correlators in the non-critical string theory with ZZ type Liouville boundary conditions. In this case the Liouville boundary fields are described by degenerate $c > 25$ Vir representations - the same set that parametrises the ZZ boundary conditions. This implies that the matter boundary fields dressed by such operators correspond to charges (10). These values are beyond the usual set (also in the case of rational b^2). As it is standard in cases of indecomposable representations, these fields labelled by singular vectors, originate from logarithmic fields of degenerate dimensions, with correlators not annihilated by the corresponding differential operators. We constructed the matter 3-point correlators for such charges by analytic continuation of the general Liouville correlators found in [12], with charges given in (7), and keeping generic the boundary labels σ_i. Accordingly these correlators represent new solutions of the $c<1$ pentagon equations, valid for the range (10), which differ from the standard $c<1$ expressions, derived for the degenerate representations, which are polynomial in the boundary parameters $c_i^M\,\tilde{c}_i^M$.

Finally the functional matter-Liouville equations in [12] can be used to represent the general tachyon 3-point correlation numbers as linear combination of correlators in which the Liouville dressing factor has trivial boundary conditions, *i.e.*, one of the boundaries is the basic ZZ boundary condition $\sigma=0$ while the other two are taking the values dictated by the fusion rules of the Liouville chiral vertex operators.

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Appendix

• The polynomials in (20) are defined as

$$P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2}(m;b) := \sum_{k=0}^{m} \frac{S_b(b)^2}{S_b((k+1)b)S_b((m-k+1)b)} \times \frac{\bar{G}(\sigma_3+\frac{(m-k)b}{2},\beta_3+\frac{(k-m)b}{2},\sigma_1)}{\bar{G}(\sigma_3,\beta_3,\sigma_1)} \frac{\bar{G}(\sigma_3-\frac{kb}{2},\beta_2-\frac{kb}{2},\sigma_2)}{\bar{G}(\sigma_3,\beta_2,\sigma_2)}\,. \quad (A.1)$$

Applying (A.1) for degenerate values both of the charges β_i and the boundary labels $\sigma_i=-s_ib-r_i/b\,, 2s_i, 2r_i\in\mathbb{Z}_{\geq 0}$ and using that the fusion rules dictate integer values for the various combinations $\pm r_1\pm r_3\pm n_3\,, \pm r_2\pm r_3\pm n_2$, the dependence on $r_i\,, n_i$ is canceled in the sum (A.1) up to an overall mixing sign $(-1)^{2r_3m}$.

To make connection with the standard notation for the basic hypergeometric functions (q - hypergeometries) ${}_4\phi_3(a_1,a_2,a_3,a_4;b_1,b_2,b_3;q;q)$ use the formulae

$$\frac{S_b((m+1)b)}{S_b(b)} = (2\sin\pi b^2)^m\,[m]_q!\,,\quad [m]_q! := \prod_{k=1}^{m}\frac{\sin\pi b^2k}{\sin\pi b^2}\,,$$
$$\frac{S_b(kb+\alpha)}{S_b(\alpha)} = (-2i\,q^{(k-1)/2}\,e^{\pi i\alpha b})^{-k}\quad (a;q)_k\,,\quad q=e^{2\pi i b^2}$$
$$(a;q)_k := \prod_{p=0}^{k-1}(1-aq^p)\,,\quad a=e^{2\pi i b\alpha}=q^{\alpha/b}\,. \quad (A.2)$$

• The polynomials (15) read explicitly - for $2n$ - even, $n \neq 0$

$$\tilde{B}(\sigma_2, \sigma_1)^{(2n;p(2m))} = ((-1)^{2m}\tilde{c}_2 - \tilde{c}_1) \\ \times \prod_{k=1}^{n} (\tilde{c}_1^2 + \tilde{c}_2^2 - \tilde{c}_1\tilde{c}_2(-1)^{2m} 2\cos\frac{2k\pi}{b^2} - (2\sin\frac{2k\pi}{b^2})^2)\,, \quad \text{(A.3)}$$

while for $2n$ – odd

$$\tilde{B}(\sigma_2, \sigma_1)^{(2n;p(2m))} = \prod_{k=0}^{n-\frac{1}{2}} (\tilde{c}_1^2 + \tilde{c}_2^2 - \tilde{c}_1\tilde{c}_2(-1)^{2m} \\ \times 2\cos\frac{(2k+1)\pi}{b^2} - \Big(2\sin\frac{(2k+1)\pi}{b^2})^2\Big).$$

• Recall the prefactor in (7) from [12]:

$$\Pi'_L(\beta_3, \beta_2, \beta_1) = \frac{(-1)^{(m_{123}+1)n_{123}}\, S_b^2\Big(\frac{1}{b}\Big)}{\prod_i S_b\Big(\frac{n_{123}+1-2n_i}{b}\Big) S_b\Big(\frac{n_{123}+2}{b}\Big)} \\ \times\ b^{e_0(Q-\beta_{123})} \frac{\Gamma_b(2Q-\beta_{123})}{\Gamma_b(Q)} \prod_i \frac{\Gamma_b(Q-\beta_{123}+2\beta_i)}{\Gamma_b(Q-2\beta_i)}\,. \quad \text{(A.4)}$$

The analytic continuation preserves the first line in (A.4). Write each ratio of Gamma's with non-negative m, n in terms of products of ordinary Gamma functions using the functional relations

$$\frac{\Gamma_b(x+b^\epsilon)}{\Gamma_b(x)} = \frac{\sqrt{2\pi}\, b^{\epsilon b^\epsilon x - \frac{1}{2}\epsilon}}{\Gamma(x b^\epsilon)}\,, \quad \epsilon = \pm 1\,.$$

This can be done in two different ways, *e.g.*,

$$\frac{\Gamma_b(Q - mb + \frac{n}{b})}{\Gamma_b(Q)} \\ = b^{\frac{1}{2}[-n-m-2mn+m(m-1)b^2 - \frac{n(n+1)}{b^2}]} \frac{\prod_{j=0}^{m-1}\Gamma(1+n-jb^2)}{(2\pi)^{\frac{m-n}{2}}\prod_{l=1}^{n}\Gamma\big(1+\frac{l}{b^2}\big)}\,. \quad \text{(A.5)}$$

The powers of b from each such ratio are collected separately to an overall power which combined with the power of b in (A.4) gives

$$b(b^2)^{-3-m_{123}} \to ib\,(-1)^{m_{123}+1}(b^2)^{-3-m_{123}} \quad \text{(A.6)}$$

Each ratio of Γ's is continued and expressed again in terms of $\Gamma_b(x)$

$$\frac{\prod_{j=0}^{m-1}\Gamma(1+n-jb^2)}{\prod_{l=1}^{n}\Gamma(1+\frac{l}{b^2})} \to \frac{\prod_{j=0}^{m-1}\Gamma(1+n+jb^2)}{\prod_{l=1}^{n}\Gamma(1-\frac{l}{b^2})}$$
$$= b^{\frac{1}{2}[m-n+2mn+m(m-1)b^2-\frac{n(n+1)}{b^2}]}\,\frac{(2\pi)^{\frac{1}{2}(m-n)}\Gamma_b\left(\frac{1}{b}\right)S_b\left(\frac{n+1}{b}\right)}{\Gamma_b\left(\frac{1}{b}+mb+\frac{n}{b}\right)S_b\left(\frac{1}{b}\right)}\,. \quad \text{(A.7)}$$

Combining all such terms with m, n taking the values in (A.4) we get in particular an overall power

$$(b^2)^{Q(e_{123}-e_0)+3+m_{123}},$$

which has to be multiplied by (A.6) in the final expression. For the analytic continuation of the prefactor (A.4) we finally get the expression in (25). Choosing the chiralities as $(+-+)$ the product of the matter (25) and the (Coulomb gas) Liouville (20) prefactors reads

$$\Pi^{(M)}(e_3,e_2,e_1)\Pi^{(LCG)}(\beta_3,\beta_2,\beta_1)(ib)\prod_{i=1,3}\Gamma(b(Q-2\beta_i)\Gamma(\frac{1}{b}(Q-2\beta_2)$$
$$= \frac{2\pi(-1)^{(m_{123}+1)(n_{123}+1)+1+(m_{123}+1)2n_2+n_{123}2m_2}S_b(Q-\frac{2n_2+2}{b})}{\prod_{i=1,3}S_b(\frac{n_{123}+1-3n_i}{b})S_b((m_{123}+1-2m_i)b)S_b(Q-(2m_i+1)b)}\,. \quad \text{(A.8)}$$

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