Bulg. J. Phys. **36** (2009) 199–213

Talk on Anomalies in Quantum Field Theory and Cohomologies of Configuration Spaces*

N.M. Nikolov

Institute for Nuclear Research and Nuclear Energy,
Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria

Dedicated to ***Ivan Todorov*** *on the occasion of his 75th birthday*

Received *12 September 2009*

Abstract. In this work the deviation from commutativity of the renormalization and the action of all linear partial differential operators is investigated. This deviation is the source of the anomalies in quantum field theory, including the renormalization group action, and it is characterized by certain renormalization cocycles. Cohomological differential equations are obtained over the so called (ordered) configuration spaces, which determine the renormalization cocycles up to the renormalization freedom.

I would like to explain first how I came to this subject. About four years ago I became interested in the possibility of extending the algebraic methods of conformal field theory to more general quantum field theories and especially to perturbative quantum field theory. It is known that the models of conformal field theory admit a purely algebraic description. This happens not only in two space–time dimensions, where we have an infinite dimensional group of conformal transformations but also in higher dimensions. The possibility of a purely algebraic description of conformally invariant quantum fields is mainly due to the simple type of the singularities of the products of these fields. These singularities, at least in the presence of the most strong conformal invariance that is so called global conformal invariance†, are singularities on light–like distances of rational type. The (purely) algebraic structure, which describes a Globally Conformal Invariant QFT is called vertex algebra but it could be also called Operator

*Talks given at the Conference on Algebraic and Combinatorial Structures in Quantum Field Theory, Cargèse, March 31, 2009 and at the Conference "Algebraic Methods in Quantum Field Theory" 50 Years of Mathematical Physics in Bulgaria on the occasion of the 75th anniversary of Ivan Todorov, Sofia, May 15, 2009. Based on the paper: arXiv:0903.0187

†The notion of "global conformal invariance" was introduced almost 10 years ago in the paper "Rationality of conformally invariant local correlation functions on compactified Minkowski space" by N.M. Nikolov and I.T. Todorov (Commun. Math. Phys. **218** (2001) 417-436).

Product Expansion (OPE) Algebra because it just describes the OPE of all local fields in the theory in algebraic terms.

In such a way a question arises whether we can find more wide algebra of functions, which would be sufficient for describing the singularities of the products of quantum fields in a more general situation. We may look for this algebra as a certain differential–algebraic extension of the algebra of rational functions with light–cone singularities. We have somewhat analogous situation to the situation in abstract algebra, where one starts with the field of rational numbers and then introduces the algebraic numbers and even further, the "differential algebraic numbers" known as periods.

The problem of finding of appropriate function spaces for describing the correlation functions in quantum field theory I have investigated first in the direction of developing general theory of OPE algebras, their deformation theory and its relation to perturbative quantum field theory. Now, I am preparing the results of this my long research, which are still not complete. But at some point I decided to look at the problem from the point of view of perturbative quantum field theory and especially from the point of view of the Gell–Mann–Low renormalization group and then I found much easier and even nice picture. In this talk I will present this my research.

So, this talk will be devoted to renormalization theory in configuration spaces for Euclidean perturbative quantum field theory. Such an approach, named after H. Epstein and V. Glaser, was long ago developed on Minkowski space and even on pseudo–Riemann manifolds but it was not systematically considered on Euclidean space. I will make some introductory remarks to this subject: why renormalization in configuration spaces?

The renormalization theory is usually considered in momentum space, where there are general methods to deal with Feynman integrals. On the other hand, the renormalization on configuration space has a direct geometric interpretation allowing generalization even on manifolds. This geometric interpretation is completely lost in momentum space, or at least it becomes rather implicit. Another feature of the Epstein-Glaser renormalization is that it is done for the products of fields. This also facilitates the generalization of to perturbation theory on manifolds but still it has the disadvantage of being rather complicated technically, especially for concrete calculations.

One of the results I will present is an analog of the Epstein–Glaser approach, which is entirely stated in terms of renormalization of integrals of functions. It is an old idea that renormalization is a problem of extending distributions. In this work we axiomatize these extension maps and call them renormalization maps. This approach then has the additional advantage of being independent of concrete models of quantum fields like the φ^4-theory or quantum electrodynamics, *etc*. This is because we just consider the renormalization maps as acting on certain function spaces regardless of any model of perturbative QFT. In this way we get rid of the technical difficulties present in one or other models, in other words, we separate them from the renormalization problem. Furthermore, reformulating the renormalization problem as operations on spaces of functions, makes the geometric interpretation possible. This is because the function spaces on which we define these renormalization maps are the spaces of regular functions on certain domains. Hence, the geometric properties of these domains determine the ambiguity of the renormalization. Our main goal in this work is to use this geometric characterization of the renormalization ambiguity in order to derive an algebraic algorithm for determining the Gell–Mann–Low renormalization group action, *i.e.*, the action of the one parameter group $\mathbb{R}^+$ on the space of coupling constants, which is induced by the scaling transformations (in terms of formal diffeomorphisms). In particular, we are interested in algebraic algorithms for calculating the perturbative expansions of β–functions and anomalous dimensions.

So, the renormalization in the approach I am presenting is introduced by a system of linear maps each of them being applied on a certain algebra of smooth functions that are not globally defined and as a result producing everywhere defined distributions:

$$\left\{\begin{array}{c}\text{algebra } \mathscr{O}_n \text{ of}\\ \text{non globally defined}\\ \text{smooth functions}\end{array}\right\} \quad \xrightarrow{R_n} \quad \left\{\begin{array}{c}\text{space of}\\ \text{globally defined}\\ \text{distributions}\end{array}\right\} .$$

I begin with the definition of the algebras of regular functions on which we apply the renormalization maps. Shortly speaking, this is a sequence of algebras $\mathscr{O}_2$, $\mathscr{O}_3, \dots, \mathscr{O}_n, \dots$, *etc.*, where $\mathscr{O}_n$ is an algebra of translation invariant functions of n vector arguments and these functions one can think of as coming from Feynman diagrams. More precisely we assume that the algebra $\mathscr{O}_n$ is linearly spanned by finite linear combinations of products of the form

$$G \,=\, \prod_{1\,\leqslant\, j\,<\,k\,\leqslant\, n} G_{jk}\big(\mathrm{x}_j - \mathrm{x}_k\big), \quad \mathrm{x}_k \,\in\, \mathbf{E} \,\equiv\, \mathbb{R}^D \,,$$

where $\mathrm{x}_k = (x^1, \dots, x^D)$ are Euclidean vectors and the functions $G_{jk}(\mathrm{x})$ belong to the algebra $\mathscr{O}_2$, which is a subalgebra of the algebra $\mathcal{C}^{\infty}\big(\mathbf{E}\backslash\{\mathbf{0}\}\big)$ of the smooth functions outside the origin. One can think of the algebra $\mathscr{O}_2$ as an algebra containing the propagators of the theory. In this way the algebras $\mathscr{O}_n$ for $n > 2$ are entirely determined by the algebra $\mathscr{O}_2$.

One technical assumption for the algebra $\mathscr{O}_2$, and thus for all other $\mathscr{O}_n$, is that it is closed with respect to multiplication of its elements by polynomials as well as with respect to applying derivatives.

A main example is obtained by setting $\mathscr{O}_2$ to be the algebra of rational functions of the form polynomials over powers of the Euclidean square $\mathrm{x}^2 = (x^1)^2 + \cdots + (x^D)^2$,

$$\mathscr{O}_2 = \left\{ \frac{p(\mathrm{x})}{(\mathrm{x}^2)^N} : p(\mathrm{x}) - \text{polynomial}, N \in \mathbb{N} \right\} \qquad (\mathrm{x} = \mathrm{x}_1 - \mathrm{x}_2)\,.$$

In this case $\mathscr{O}_n$ is the algebra of rational functions that are ratios of translation invariant polynomials of n vectors and powers of the product of all Euclidean squares of the mutual differences

$$\mathscr{O}_n = \left\{ \frac{p(\mathrm{x}_1 - \mathrm{x}_n, \dots, \mathrm{x}_{n-1} - \mathrm{x}_n)}{\Big(\prod\limits_{j<k} (\mathrm{x}_j - \mathrm{x}_k)^2 \Big)^N} : p - \text{polynomial}, N \in \mathbb{N} \right\}.$$

This example would correspond to the case when we perturb free massless fields and consider the mass terms in the Lagrangian as perturbations. This is convenient if we wish to work algebraically as much as possible.

Another remark: the algebra $\mathscr{O}_n$ consists of translation invariant functions, which are regular on the so called configuration space

$$F_n = \big\{ (\mathrm{x}_1, \dots, \mathrm{x}_n) \in \mathbf{E}^n : \mathrm{x}_j \neq \mathrm{x}_k \ \ (\forall j \neq k) \big\}\,.$$

From the point of view of the algebraic geometry $\mathscr{O}_n$ is the ring of regular functions on the quasiaffine manifold that is the complement of union of quadrics

$$F_{n;\mathbb{C}} = \big\{ (\mathrm{x}_1, \dots, \mathrm{x}_n) \in \mathbb{C}^{Dn} : (\mathrm{x}_j - \mathrm{x}_k)^2 \neq 0 \ \ (\forall j \neq k) \big\}\,.$$

Now, the renormalization maps we shall define as linear maps

$$R_n : \mathscr{O}_n \to \mathscr{D}'\big(\mathbf{E}^{\times n}/\mathbf{E}\big)$$

without any requirement of continuity with respect of some nontrivial topology. These linear maps are supposed to fulfill the following axiomatic conditions $(r1)$–$(r4)$.

The first condition $(r1)$ is the permutation symmetry: $R_n\big(\sigma^* G\big) = \sigma^* R_n(G)$, $\forall \sigma \in \mathscr{S}_n$, where $\sigma^* F(\mathrm{x}_1, \dots, \mathrm{x}_n) := F(\mathrm{x}_{\sigma_1}, \dots, \mathrm{x}_{\sigma_n})$.

I shall use the convention that for every non-empty finite subset S of the set of natural numbers $\mathbb{N}$ we define an algebra $\mathscr{O}_S$ isomorphic to $\mathscr{O}_n$ if n is the number of elements of S, but $\mathscr{O}_S$ is spanned by products of 2–point functions of differences of vectors indexed by the elements of the set S (*i.e.*, $\prod_{j,\,k\,\in\,S,\ j\,<\,k} G_{jk}\big(\mathrm{x}_j - \mathrm{x}_k\big)$) Then we define also renormalization maps

$$R_S : \mathscr{O}_S \to \mathscr{D}'\big(\mathbf{E}^S/\mathbf{E}\big)$$

setting

$$R_S(G) = (\sigma^*)^{-1} R_n\big(\sigma^* G\big)\,,$$

where $\sigma : \{1, \dots, n\} \cong S$ and $\sigma^* G(\mathrm{x}_1, \dots, \mathrm{x}_n) := G(\mathrm{x}_{\sigma_1}, \dots, \mathrm{x}_{\sigma_n})$. I will often use these set–subscript notations because of their convenience for the related combinatorics and everywhere further S and S' will denote finite subsets of the set of positive integers.

The next condition $(r2)$ is preservation of certain filtrations defined by the notion of the scaling degree:

$$\text{Scaling degree of } R_n G \leqslant \text{ Scaling degree of } G\,.$$

The scaling degree gives the rate of the singularity for coinciding arguments. In particular, we require that all the elements of our algebras have finite scaling degrees.

The next condition $(r3)$ requires commutativity between the renormalization maps and the multiplication by polynomials

$$R_n\big(pG\big) \,=\, pR_nG, \quad p = p(\mathrm{x}_1 - \mathrm{x}_n, \dots, \mathrm{x}_{n-1} - \mathrm{x}_n) \text{ is a polynomial.}$$

I consider in this property only polynomials since I wish to work algebraically but if we work on manifolds then it is natural to require commutativity between the renormalization maps and multiplication by everywhere smooth functions. In the latter case the above property becomes very natural from geometric point of view since it allows us to make localization (*i.e.,* to use localization techniques like partition of unity). We shall also see in what follows that the above property $(r3)$ is crucial for the reduction of our cohomological analysis to de Rham cohomologies of configuration spaces.

The last requirement $(r4)$ is nonlinear and relates recursively R_n with the renormalization maps of lower order. To introduce it I shall introduce first some notations. Let $\mathfrak{P}$ be a partition of the set $S = \{j_1, \dots, j_n\}$

$$\Big(\underset{j_1}{\bullet}\,\underset{j_2}{\bullet}\,\underset{j_3}{\bullet}\Big)\cdots\Big(\cdots\underset{j_k}{\bullet}\cdots\Big)\cdots\Big(\cdots\underset{j_n}{\bullet}\Big)\,.$$

Then we shall identify $\mathfrak{P}$ also with the equivalence relation on S whose equivalence classes coincide with the elements of $\mathfrak{P}$. This equivalence relation I shall denote by $\sim_{\mathfrak{P}}$. Then I define the following open subsets of the Cartesian power $\mathbf{E}^S$, one for every S–partition $\mathfrak{P}$

$$F_{\mathfrak{P}} \,=\, \big\{(\mathrm{x}_{j_1}, \dots, \mathrm{x}_{j_n}) \in \mathbf{E}^S : \mathrm{x}_j \,\neq\, \mathrm{x}_k \;\; (\forall j \nsim_{\mathfrak{P}} k)\big\}\,.$$

Also for a completely multiplicative elements

$$G \,=\, \prod_{\substack{j,\,k\,\in\,S\\ j\,<\,k}} G_{jk}\big(\mathrm{x}_j - \mathrm{x}_k\big).$$

I introduce the decomposition

$$G_S \,=\, G_{\mathfrak{P}} \,\cdot\, \prod_{S'\,\in\,\mathfrak{P}} G_{S'}\,,$$

where $G_{\mathfrak{P}}$ contains all two–point functions between nonequivalent points

$$G_{\mathfrak{P}} \;=\; \prod_{\substack{j \,\not\sim_{\mathfrak{P}}\, k \\ j\,<\,k}} G_{jk}\big(\mathrm{x}_j - \mathrm{x}_k\big)$$

and the remaining two-point functions are combined into products

$$G_{S'} \;=\; \prod_{\substack{j,\,k\,\in\,S' \\ j\,<\,k}} G_{jk}\big(\mathrm{x}_j - \mathrm{x}_k\big)\,.$$

Under these conventions the forth requirement on the renormalization maps states that:

$$R_S G_S\Big|_{F_{\mathfrak{P}}} \;=\; G_{\mathfrak{P}}\,\cdot\,\prod_{S'\,\in\,\mathfrak{P}} R_{S'}G_{S'}\,.$$

Here $G_{\mathfrak{P}}$ is a multiplicator on $F_{\mathfrak{P}}$.

From an axiomatic point of view it is enough to impose the condition $(r4)$ only for partitions with two elements. By convention for elements S' of the partition $\mathfrak{P}$, which contain one element we set $G_{S'} = 1$. Similarly, if the partition $\mathfrak{P}$ consists of one element we set $G_{\mathfrak{P}} = 1$. Then as a particular case of ($r4$) we obtain that

$$R_n G\Big|_{F_n} \;=\; G\,,$$

i.e., the renormalization maps provide extensions of smooth functions on the configuration spaces F_n to everywhere defined distributions.

Let me summarize the axiomatic conditions on the renormalization maps:

($r1$) is the permutation symmetry;

($r2$) is the preservation of the filtrations;

($r3$) is commutativity of renormalization maps with the multiplication by polynomials;

($r4$) is the recursive relation

$$R_S G_S\Big|_{F_{\mathfrak{P}}} \;=\; G_{\mathfrak{P}}\,\cdot\,\prod_{S'\,\in\,\mathfrak{P}} R_{S'}G_{S'}\,.$$

Now, I shall show briefly how these maps can be combined within the Euclidean perturbative quantum field theory. There we need to define products of interactions

$$I_1(\mathrm{x}_1)\cdots I_n(\mathrm{x}_n),$$

as quadratic forms on the Euclidean Fock space. Here, every $I_k(\mathrm{x})$ is a Wick polynomial of the basic fields $\varphi(\mathrm{x})$ and its derivatives $\partial^{\mathrm{r}}\varphi(\mathrm{x})$

$$I_k(\mathrm{x}) \;=\, :\text{Polynomial}\big(\varphi(\mathrm{x}), \partial_{\mathrm{x}}\varphi(\mathrm{x}), \ldots\big):\;.$$

Note that $I_k(\mathrm{x})$ is well defined only as a quadratic form on the Euclidean Fock space *but it is not representable by operator if its degree as a polynomial is larger than 1.* The latter is in contrast to the situation on the Minkowski space, where we have well defined Wick monomials due to the possibility of multiplying Wightman distributions. Thus, in order to define the product $I_1(\mathrm{x}_1) \cdots I_n(\mathrm{x}_n)$ of Wick monomials of Euclidean fields we need renormalization. To this end we formally decompose $I_1(\mathrm{x}_1) \cdots I_n(\mathrm{x}_n)$ by the Wick theorem

$$I_1(\mathrm{x}_1)\cdots I_n(\mathrm{x}_n) \,=\, \sum_{A_1,\dots,A_n} G_{A_1,\dots,A_n}(\mathrm{x}_1,\dots,\mathrm{x}_n) :\!\Phi_{A_1}(\mathrm{x}_1)\cdots\Phi_{A_n}(\mathrm{x}_n)\!: ,$$

where the normal products are quadratic forms smoothly dependent on $\mathrm{x}_1, \dots, \mathrm{x}_n$ and $G_{A_1,\dots,A_n}$ are functions belonging to the algebra $\mathscr{O}_n$. The functions $G_{A_1,\dots,A_n}$ come from the Wick pairings and they belong to the algebra $\mathscr{O}_n$. Then the renormalized product of interactions is defined as

$$\big(I_1(\mathrm{x}_1)\cdots I_n(\mathrm{x}_n)\big)^{\mathrm{ren}} \,=\, \sum_{A_1,\dots,A_n} R_n\big(G_{A_1,\dots,A_n}\big)(\mathrm{x}_1,\dots,\mathrm{x}_n) \times \,:\!\Phi_{A_1}(\mathrm{x}_1)\cdots\Phi_{A_n}(\mathrm{x}_n)\!: \,.$$

In fact, we have a formula

$$I_1(\mathrm{x}_1)\cdots I_n(\mathrm{x}_n) \,=\, \prod_{1\leqslant j<k\leqslant n} \exp\left(\sum_{\mathrm{r,s}} C_{\mathrm{r,s}}\big(\mathrm{x}_j-\mathrm{x}_k\big)\frac{\partial}{\partial\varphi_{\mathrm{r}}(\mathrm{x}_j)}\frac{\partial}{\partial\varphi_{\mathrm{s}}(\mathrm{x}_k)}\right) \times \,:\!I_1(\mathrm{x}_1)\cdots I_n(\mathrm{x}_n)\!: \,.$$

$$\varphi_{\mathrm{r}}(\mathrm{x}) \,:=\, \partial_{\mathrm{x}_1}^{\mathrm{r}}\varphi(\mathrm{x}), \quad C_{\mathrm{r,s}}\big(\mathrm{x}_1-\mathrm{x}_2\big) \,:=\, \partial_{\mathrm{x}_1}^{\mathrm{r}}\partial_{\mathrm{x}_2}^{\mathrm{s}}\left\langle\,\varphi(\mathrm{x}_1)\varphi(\mathrm{x}_2)\,\right\rangle \,.$$

This formula is convenient since the prefactor has the multiplicative form used in the renormalization recursion.

Let us come back to the axiomatic requirements for the renormalization maps:

($r1$) permutation symmetry;

($r2$) preservation of the filtrations;

($r3$) commutativity of renormalization maps with the multiplication by polynomials;

($r4$) recursive relation

$$R_S G_S\Big|_{F_{\mathfrak{P}}} \,=\, G_{\mathfrak{P}}\,\cdot\,\prod_{S'\in\mathfrak{P}} R_{S'}G_{S'} \,.$$

In fact, I shall not consider these conditions as a definition of renormalization maps but I shall give a construction for these maps by means of simpler objects and these I shall consider as the exact definition of renormalization.

To this end let me point out that the open sets $F_{\mathfrak{P}}$ that I defined in the condition $(r4)$ form an open covering of the complement of the total diagonal Δ_S

$$\mathbf{E}^S \backslash \Delta_S \, = \bigcup_{\substack{\mathfrak{P} \text{ is a proper} \\ S\text{–partition}}} F_{\mathfrak{P}} \, .$$

Then we obtain a linear map

$$\overset{\bullet}{R}_S : \mathscr{O}_S \to \mathscr{D}'_{temp}\Big(\big(\mathbf{E}^S \big/ \mathbf{E}\big) \backslash \{\mathbf{0}\} \Big) \, ,$$

where the subscript "temp" means that we consider distributions with a finite scaling degree. (Note that $\big(\mathbf{E}^S \big/ \mathbf{E}\big) \backslash \{\mathbf{0}\} \cong \big(\mathbf{E}^S \backslash \Delta_S\big) \big/ \mathbf{E}$ and a distribution on $\big(\mathbf{E}^S \big/ \mathbf{E}\big) \backslash \{\mathbf{0}\}$ is the same as a translation invariant distribution on $\mathbf{E}^S$ defined outside the total diagonal Δ_S.) I shall call these linear maps $\overset{\bullet}{R}_S$ secondary renormalization maps and they are completely determined by $R_{S'}$ for sets S' containing less than n elements. Then to construct R_S we should compose $\overset{\bullet}{R}_S$ with a linear map

$$P_S : \mathscr{D}'_{temp}\Big(\big(\mathbf{E}^S \big/ \mathbf{E}\big) \backslash \{\mathbf{0}\} \Big) \to \mathscr{D}'\big(\mathbf{E}^S \big/ \mathbf{E}\big) \, ,$$

$$R_S \, = \, P_S \circ \overset{\bullet}{R}_S \, ,$$

$$\mathscr{O}_S \xrightarrow{\overset{\bullet}{R}_S} \mathscr{D}'_{temp}\Big(\big(\mathbf{E}^S \big/ \mathbf{E}\big) \backslash \{\mathbf{0}\} \Big) \xrightarrow{P_S} \mathscr{D}'\big(\mathbf{E}^S \big/ \mathbf{E}\big) \, .$$

These maps I shall call primary renormalization maps.

The axiomatic conditions on P_S are the following.

$(p1)$ $P_S u \Big|_{(\mathbf{E}^S/\mathbf{E}) \backslash \{\mathbf{0}\}} = u$, *i.e.* P_S makes extension of distributions.

$(p2)$ Preservation of the filtrations.

$(p3)$ Orthogonal invariance. (This is with respect to all Euclidean transformations of $\mathbf{E}^S \big/ \mathbf{E}$. It then will imply both, the permutation symmetry of R_n and their orthogonal invariance with respect to $\mathbf{E}$.)

$(p4)$ Commutativity with the multiplication by polynomials.

$(p5)$ If $u(\mathrm{x}) \in \mathscr{D}'\big(\mathbf{E}^{S \backslash S'}\big)$ is a distribution supported at zero and $v(\mathrm{y})$ is a distribution belonging to $\mathscr{D}'_{temp}\Big(\big(\mathbf{E}^{S'} \big/ \mathbf{E}\big) \backslash \{\mathbf{0}\} \Big)$ then

$$P_S(u \otimes v) \, = \, u \otimes P_{S'} v \, .$$

I shall not consider here the construction of these renormalization maps but it can be found in my paper (Section 2.5).

Before I proceed to the announced topic on anomalies and cohomologies I will consider briefly the problem of changing the renormalization. The following statement holds:

Theorem. *Let $\{P_n\}_{n=2}^{\infty}$ and $\{P'_n\}_{n=2}^{\infty}$ be two systems of primary renormalization maps, which define the systems $\{R_n\}_{n=2}^{\infty}$ and $\{R'_n\}_{n=2}^{\infty}$ of renormalization maps, respectively. Then for every finite $S \subset \mathbb{N}$ and a multiplicative element $G_S \in \mathscr{O}_S$ the following formula holds:*

$$R'_S\, G_S \;=\; \sum_{\substack{\mathfrak{P} \text{ is a} \\ S\text{–partition}}} \Big(R_{S/\mathfrak{P}} \otimes \mathrm{id}_{\mathscr{D}'_{\mathfrak{P},0}}\Big) \circ \textit{n.f.}_{\mathfrak{P}} \Big(G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} Q_{S'}\, G_{S'}\Big)\,.$$

Here $Q_S = \big(P'_S - P_S\big) \circ \overset{\bullet}{R}{}'_S$ are linear maps $\mathscr{O}_S \to \mathscr{D}'_{S,0} := \mathscr{D}'\big[\mathbf{0} \in \mathbf{E}^S/\mathbf{E}\big]$ (distributions supported at the origin) if $|S| > 1$, otherwise, $Q_{S'}\, G_{S'} = 1$; $S/\mathfrak{P} := \{\max\, S' : S' \in \mathfrak{P}\}$; $\textit{n.f.}_{\mathfrak{P}}$ is the linear map $\textit{n.f.}_{\mathfrak{P}} : \mathscr{O}_{\mathfrak{P}} \otimes \mathscr{D}'_{\mathfrak{P},0} \to \mathscr{O}_{S/\mathfrak{P}} \otimes \mathscr{D}'_{\mathfrak{P},0}$.

The details on this theorem can be found in my paper (Section 2.6) and here I would like to point out that the change of the renormalization is characterized by a sequence of linear maps $Q_n : \mathscr{O}_n \to \mathscr{D}'_{n,0}$, $Q_1 = 1$, satisfying the properties

($c1$) permutation symmetry;

($c2$) preservation of the filtrations;

($c3$) commutativity with multiplications by polynomials.

The set of all such systems of linear maps form a group with a multiplication

$$Q''_S\, G_S \;=\; \sum_{\substack{\mathfrak{P} \text{ is a} \\ S\text{–partition}}} \Big(Q'_{S/\mathfrak{P}} \otimes \mathrm{id}_{\mathscr{D}'_{\mathfrak{P},0}}\Big) \circ \textit{n.f.}_{\mathfrak{P}} \Big(G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} Q_{S'}\, G_{S'}\Big)\,.$$

This group I called "universal renormalization group" and it corresponds to Stueckelberg renormalization group in our formalism.

When we apply the renormalization maps to a perturbative quantum field theory with some fixed initial field content we obtain a representation of the above universal renormalization group in the group of all formal diffeomorphisms of the coupling constants that parameterize all possible interactions for the fixed set of initial fields. A key role in the derivation of this representation play the formulas

$$R'_S\, G_S \;=\; \sum_{\substack{\mathfrak{P} \text{ is a} \\ S\text{–partition}}} \Big(R_{S/\mathfrak{P}} \otimes \mathrm{id}_{\mathscr{D}'_{\mathfrak{P},0}}\Big) \circ \textit{n.f.}_{\mathfrak{P}} \Big(G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} Q_{S'}\, G_{S'}\Big)$$

and

$$I_1(\mathrm{x}_1)\cdots I_n(\mathrm{x}_n) \,=\, \prod_{1\leqslant j<k\leqslant n} \exp\left(\sum_{\mathrm{r,s}} C_{\mathrm{r,s}}\big(\mathrm{x}_j-\mathrm{x}_k\big)\frac{\partial}{\partial\varphi_{\mathrm{r}}(\mathrm{x}_j)}\frac{\partial}{\partial\varphi_{\mathrm{s}}(\mathrm{x}_k)}\right) \times\, :I_1(\mathrm{x}_1)\cdots I_n(\mathrm{x}_n):\,,$$

which I have already presented. It is important here the multiplicative form of the prefactor and by a preliminary investigations of mine, for the description of the action of the universal renormalization group by formal diffeomorphisms it would be convenient to introduce a certain coalgebraic structure in the space of all interactions.

I proceed to the anomalies in perturbative quantum field theory. We speak about anomalies when a symmetry of the unrenormalized, or bare, Feynman integrals is broken after the renormalization. It is clear that for this a main role plays the absence of commutativity between the renormalization maps and the action of linear partial differential operators.

By construction the renormalization maps commute with the multiplication by polynomials and then what remains as a source for the anomalies are the commutators

$$\big[\partial_{x_k^\mu}, R_n\big] \,=\, \big[\partial_{x^\xi}, R_n\big], \quad \xi \,=\, (k,\mu)\,.$$

Let us denote $\omega_{n;\,\xi} := \big[\partial_{x^\xi}, R_n\big]$ and apply the main formula

$$R_n \,=\, P_n \circ \dot{R}_n\,.$$

We obtain a decomposition,

$$\begin{aligned} \omega_{n;\,\xi} &= \gamma_{n;\,\xi} + \dot{\omega}_{n;\,\xi} \qquad (n>2), \qquad \omega_{2;\,\xi} \,\equiv\, \gamma_{2;\,\xi}\,, \\ \gamma_{n;\,\xi} &:= \big[\partial_{x^\xi}, P_n\big]\circ \dot{R}_n \qquad (n>2)\,, \\ \dot{\omega}_{n;\,\xi} &:= P_n \circ \big[\partial_{x^\xi}, \dot{R}_n\big] \qquad (n>2)\,, \end{aligned}$$

where the important point is that $\dot{\omega}_{n;\,\xi}$ are recursively determined due to the commutator with the secondary renormalization maps: it will produce at least one delta function or its derivatives supported on some partial diagonal and after that we have a renormalization on less than n variables.

On the other hand, $\gamma_{n;\,\xi}$ are linear maps $\mathscr{O}_n \to \mathscr{D}'_{n,0} = \mathscr{D}'\big[\mathbf{0}\in\mathbf{E}^n\big/\mathbf{E}\big]$, *i.e.* they produce distributions supported at zero, or, in other words, total delta functions and derivatives. Thus, $\gamma_{n;\,\xi}$ are simpler linear maps than $\omega_{n;\,\xi}$. Our idea is to characterize $\gamma_{n;\,\xi}$ by cohomological equations such that the ambiguity in their solutions to be exactly corresponding to the renormalization ambiguity. Such a system of equations is the following:

$$\begin{aligned} &\big[\partial_{x^\xi}, \gamma_{2;\,\eta}\big] - \big[\partial_{x^\eta}, \gamma_{2;\,\xi}\big] \,=\, 0\,, \\ &\big[\partial_{x^\xi}, \gamma_{n;\,\eta}\big] - \big[\partial_{x^\eta}, \gamma_{n;\,\xi}\big] \\ &\qquad = \,-\big[\partial_{x^\xi}, P_n\big] \circ \big[\partial_{x^\eta}, \overset{\bullet}{R}_n\big] \,+\, \big[\partial_{x^\eta}, P_n\big] \circ \big[\partial_{x^\xi}, \overset{\bullet}{R}_n\big] \quad (n > 2)\,. \end{aligned}$$

It can be verified by straightforward computations. We see that the right hand side looks like a differential of one–form. Later we shall reduce this expression to such a differential. For the right hand side it is important to note that it is determined by the renormalization recursion. The reason is the same as above: the second commutators in both terms contains a secondary renormalization map and then they produce at least one delta function or its derivatives supported at some partial diagonal; this reduces the number of points and thus also the order of the renormalization. The main result here is

Theorem. *Let $n > 2$; given a system of primary renormalization maps P_2, P_3, $\dots$, P_n (which therefore determine renormalization maps R_2, R_3, $\dots$, R_n), let $\{\gamma_{n;\,\xi}\}_\xi$ be defined by P_2, P_3, $\dots$, P_n and $\{\gamma'_{n;\,\xi}\}_\xi$ be a solution of the cohomological equations, which differs from $\{\gamma_{n;\,\xi}\}_\xi$ by an* exact solution, i.e.*, the difference $\gamma'_{n;\,\xi} - \gamma_{n;\,\xi}$ is of a form*

$$\gamma'_{n;\,\xi} - \gamma_{n;\,\xi} \,=\, \big[\partial_{x^\xi}, Q_n\big]$$

($\xi = 1, \dots, D(n-1)$), for some linear map

$$Q_n : \mathscr{O}_n \to \mathscr{D}'_{n,0}\,.$$

Then there exists a primary renormalization map P'_n, which together with P_2, $\dots$, P_{n-1} determines a system of renormalization maps $R_2, \dots, R_{n-1}$ and R'_n and a primary renormalization cocycle coinciding with $\{\gamma'_{n;\,\xi}\}_\xi$.

In this way the set of all renormalizations determines a cohomology class and the above equations fix this cohomology class. Another important corollary is that if the cohomologies related to the differential in the right hand side are zero, then every solution of these equations would correspond to some renormalization map and there will be no need to find what is exactly this renormalization map. Unfortunately, if one starts with the algebras of rational functions, which we have set in the beginning then these cohomologies are nonzero, although they are finite dimensional since they correspond to de Rham cohomologies of the so called Bernstein D-modules. So, this is exactly the reason why we need to introduce transcendental extensions in order to describe the anomalies in perturbative quantum field theory and in particular, for calculating beta functions.

I shall now give a reduction of the objects, which we have to determine by the cohomological equations to simpler objects. These were the maps $\gamma_{n;\,\xi}$ and

I will call them (primary) renormalization cocycles. An important feature of these maps $\gamma_{n;\,\xi}$ is that they commute with the multiplication by polynomials. In particular, $\left[x^{\eta}, \gamma_{n;\,\xi}\right] = 0$, *i.e.*, $x^{\eta}\gamma_{n;\,\xi}G - \gamma_{n;\,\xi}(x^{\eta}G) = 0$. It follows then that $\gamma_{n;\,\xi}$ has the following form:

$$\gamma_{n;\,\xi}(G) \,=\, \sum_{\mathbf{r}\,\in\,\mathbb{N}_0^N} \frac{(-1)^{|\mathbf{r}|}}{\mathbf{r}!}\,\Gamma_{n;\,\xi}\big(\mathbf{x}^{\mathbf{r}}\,G\big)\,\delta^{(\mathbf{r})}(\mathbf{x})\,,$$

where $\Gamma_{n;\,\xi}$ are linear functionals on $\mathscr{O}_n$ belonging to the subspace*

$$\mathscr{O}_n^{\vee} \,:=\, \big\{\Gamma \in \mathscr{O}_n' : \exists N \text{ s.t. } \Gamma\big(G\big) \,=\, 0 \text{ if Sc.d.}\, G \,\leqslant\, N\big\}.$$

Because of the condition on the linear functionals to belong to the subspace $\mathscr{O}_n^{\vee}$ the sum in the above formula is finite. Moreover, the correspondence $\gamma_{n;\,\xi} \leftrightarrow \Gamma_{n;\,\xi}$ is one–to–one and the commutator $\left[\partial_{x^{\eta}}, \gamma_{n;\,\xi}\right]$ is transformed to the dual derivative $-\Gamma_{n;\,\xi}\circ\partial_{x^{\eta}}$ of the linear functional $\Gamma_{n;\,\xi}$. In this way we reduce the description of every map $\gamma_{n;\,\xi}$ to one linear functional $\Gamma_{n;\,\xi}$ (defined on the same domain as $\gamma_{n;\,\xi}$). This is the point where the condition $(r3)$ plays a crucial role, as I mentioned before. Let us organize $\Gamma_{n;\,\xi}$ into a 1-form

$$\mathbf{\Gamma}_n \,=\, \sum_{\xi\,=\,1}^{N} \Gamma_{n;\,\xi}\,dx^{\xi} \,\in\, \Omega^1\big(\mathscr{O}_n^{\vee}\big) \,:=\, \mathscr{O}_n^{\vee}\otimes\Lambda^1\mathbf{E}\,,$$

with coefficients in the differential module $\mathscr{O}_n^{\vee}$. Then the cohomological equations can be written in terms of these linear functionals as

$$\begin{aligned} & d\mathbf{\Gamma}_2 = 0, \\ & d\mathbf{\Gamma}_n = \sum_{m\,=\,2}^{n-1} \mathbf{\Gamma}_{n-m+1} \overset{\circ}{\wedge} \mathbf{\Gamma}_m \qquad (n>2)\,, \end{aligned}$$

where $\overset{\circ}{\wedge}$ is a *non–associative* and *non–graded–commutative* bilinear operation defined in the following way:

$$\begin{aligned} & \big(\mathbf{\Gamma}_{n-m+1} \overset{\circ}{\wedge} \mathbf{\Gamma}_m\big)\big(G_S\big) \\ & \qquad = \sum_{\substack{S'\,\subsetneq\, S \\ |S'|\,=\,m}} \sum_{\mathbf{r}'\,\in\,\mathbb{N}_0^{N'}} \frac{1}{\mathbf{r}'!}\,\mathbf{\Gamma}_{S/S'}\Big(\boldsymbol{\partial}_{\mathbf{x}'}^{\mathbf{r}'}\,G_{\mathfrak{P}(S')}\Big|_{\mathbf{x}'\,=\,0}\Big) \wedge \mathbf{\Gamma}_{S'}\big(\mathbf{x}'^{\mathbf{r}'}\,G_{S'}\big). \end{aligned}$$

Still one can show that the latter operation is a graded *pre–Lie* operation. If we further organize the sequence $\{\mathbf{\Gamma}_n\}$ into a single object the cohomological equations read even more compactly,

$$d\underline{\mathbf{\Gamma}} \,=\, \underline{\mathbf{\Gamma}} \overset{\circ}{\wedge} \underline{\mathbf{\Gamma}}\,,$$

*We changed the notation $\mathscr{O}_n^{\bullet}$ used in the paper arXiv:0903.0187 to $\mathscr{O}_n^{\vee}$.

and this shows their integrability since this is the form of Maurer-Cartan equations in a graded differential Lie algebra.

Now, I shall consider the problem of determining the cohomology spaces $H^1\big(\mathscr{O}_n^{\vee}\big)$ of the complex $\Omega^k\big(\mathscr{O}_n^{\vee}\big) := \mathscr{O}_n^{\vee} \otimes \Lambda^k \mathbf{E}$, which control the ambiguity of the solutions of the cohomological equations. I have shown in my paper (Theorem 3.4 of Section 3.2) that we have the following natural duality:

$$H^1\big(\mathscr{O}_n^{\vee}\big) \,\cong\, \Big(H^{D(n-1)-1}\big(\mathscr{O}_n\big)\Big)',$$

where $D(n-1)$ is the top degree. Let me remind you that the algebra $\mathscr{O}_n$ coincides with the algebra of regular functions on the complement of the union of quadrics

$$F_{n;\mathbb{C}} \,=\, \big\{(\mathrm{x}_1,\dots,\mathrm{x}_n) \in \mathbb{C}^{Dn} : (\mathrm{x}_j - \mathrm{x}_k)^2 \neq 0 \;\; (\forall j \neq k)\big\}$$

modulo translations. And so,

$$H^1\big(\mathscr{O}_n^{\vee}\big) \,\cong\, \Big(H_{DR}^{D(n-1)-1}\big(F_{n;\mathbb{C}}/\mathbb{C}^D\big)\Big)'.$$

If we work instead of with the algebras $\mathscr{O}_n$ with (translation invariant) $\mathcal{C}^{\infty}$–functions on

$$F_n \,=\, \big\{(\mathrm{x}_1,\dots,\mathrm{x}_n) \in \mathbf{E}^n : \mathrm{x}_j \,\neq\, \mathrm{x}_k \;\; (\forall j \neq k)\big\}\,,$$

then we would obtain the de Rham cohomologies of the space $F_n\big/\mathbf{E}$ of codegree 1. It is known that they are zero for $n \geqslant 3$ (see, e.g., *E.R. Fadell and S.Y. Husseini, Geometry and Topology of Configuration Spaces, Springer*). In this way we arrive to the idea to look for an intermediate differential–algebraic extension,

$$\mathscr{O}_n \,\subseteq\, \widetilde{\mathscr{O}}_n \,\subseteq\, \mathcal{C}_{temp}^{\infty}\big(\mathbf{F}_n\big),$$

which would trivialize the cohomologies

$$H^{D(n-1)-1}\big(\widetilde{\mathscr{O}}_n\big) \,=\, 0\,.$$

Let me mention a strategy for solving the cohomological equations:

$$d\boldsymbol{\Gamma}_2 \,=\, 0,$$

$$d\boldsymbol{\Gamma}_n \,=\, \mathcal{F}_n\big[\boldsymbol{\Gamma}_1,\dots,\boldsymbol{\Gamma}_{n-1}\big] \,=\, \sum_{m\,=\,2}^{n-1} \boldsymbol{\Gamma}_{n-m+1} \overset{\circ}{\wedge} \boldsymbol{\Gamma}_m \qquad (n > 2)\,,$$

for $\boldsymbol{\Gamma}_n \,=\, \sum_{\xi}\, \Gamma_{n;\,\xi}\, dx^{\xi}$, where $\Gamma_{n;\,\xi} \,:\, \mathscr{O}_n \,\to\, \mathbb{R}$, which then, let me remind you, would determine the renormalization cocycles

$$\gamma_{n;\,\xi} \,:=\, \big[\partial_{x^{\xi}}, P_n\big] \circ \overset{\bullet}{R}_n \,=\, \sum_{\mathbf{r}} \frac{(-1)^{|\mathbf{r}|}}{\mathbf{r}!}\, \big(\Gamma_{n;\,\xi} \circ \mathbf{x}^{\mathbf{r}}\big)\, \delta^{(\mathbf{r})}(\mathbf{x})\,.$$

Assume

$$\exists\ K_n\ :\ \Omega^{D(n-1)-1}\big(\widetilde{\mathscr{O}}_n\big) \to \Omega^{D(n-1)-2}\big(\widetilde{\mathscr{O}}_n\big)\,, \quad K_n\,\circ\,d + d\,\circ\,K_n\ =\ \mathrm{id}\,.$$

Then a solution of the cohomological equations is

$$\mathbf{\Gamma}_n\ =\ \mathcal{F}_n\,\circ\,K_n\,.$$

Let me briefly demonstrate how all the above ideas look in two space–time dimensions. There the quadrics are reducible and introducing the "Euclidean light–cone coordinates"

$$(x^1, x^2)\ \leftrightarrow\ (z, w):\quad z\ :=\ x^1 + i\,x^2 \quad \text{and} \quad w\ :=\ x^1 - i\,x^2$$

we obtain decompositions for the algebras $\mathscr{O}_n$

$$\mathscr{O}_n\ \cong \mathbb{Q}\big[z_1, \dots, z_{n-1}\big]\Big[\Big(\prod_j z_j\Big)^{-1}\Big(\prod_{j\,<\,k}(z_j - z_k)\Big)^{-1}\Big]$$
$$\otimes\,\mathbb{Q}\big[w_1, \dots, w_{n-1}\big]\Big[\Big(\prod_j w_j\Big)^{-1}\Big(\prod_{j\,<\,k}(w_j - w_k)\Big)^{-1}\Big]\,.$$

There is a result (*F.C.S. Brown, Multiple zeta values and periods of moduli spaces* $\mathfrak{M}_{0,n}$*, math/0606419*) stating that the algebra of the so called multiple polylogarithms provides a differential extension

$$\mathbb{Q}\big[z_1, \dots, z_{n-1}\big]\Big[\Big(\prod_j z_j\Big)^{-1}\Big(\prod_{j\,<\,k}(z_j - z_k)\Big)^{-1}\Big]$$
$$\subset\ \text{Multiple Polylogs}\big(z_1, \dots, z_{n-1}\big)\,,$$

which trivializes all the de Rham cohomologies. Thus, we set

$$\widetilde{\mathscr{O}}_n\ =$$
$$\Big(\text{Multiple Polylogs}\big(z_1, \dots, z_{n-1}\big) \otimes \text{Multiple Polylogs}\big(w_1, \dots, w_{n-1}\big)\Big)^{\substack{\text{Monod–}\\ \text{romy}}},$$

which in fact, requires an additional extension of the scalars:

$$\mathbb{Q}\ \subset\ \text{Ring of multiple zeta values.}$$

In this way the linear functionals that determine the renormalization cocycles would take values in the ring of multiple zeta values. Since these maps are algebraically related to the Gell–Mann–Low renormalization group action, and in particular, to the series of the beta functions and anomalous dimensions, it follows that the coefficients of the latter series will be multiple zeta values for any theory in two space–time dimensions.

Now the problem in higher dimensions looks like to find an higher dimensional analog of the multiple polylogarithms

$$\mathbb{Q}\big[\mathrm{x}_1,\dots,\mathrm{x}_{n-1}\big]\Big[\Big(\prod_{j}\mathrm{x}_j^2\Big)^{-1}\Big(\prod_{j\,<\,k}(\mathrm{x}_j-\mathrm{x}_k)^2\Big)^{-1}\Big] \overset{?}{\subset} \widetilde{\mathscr{O}}_n\,.$$

I have recently found a candidate for a such an extension related to the problem of inverting the Laplace operator on rational functions. One needs such an extension for developing a rigorous notion of perturbative operator product expansion algebras and solving there field equations. The extension $\widetilde{\mathscr{O}}_n$ I obtained requires again only the extension of the scalars from the field of rational numbers to the ring of multiple zeta values. So, it seems from this point of view that such a transcendental extension is sufficient for the purposes of the perturbative quantum field theory.

As a conclusion I would like to mention that the renormalization in configuration spaces provides a geometric insight to the problem what are the transcendental extensions, which we need for the function spaces that would be appropriate for the description of the correlation functions in perturbative quantum field theory. It is important also to stress that we play with the renormalization ambiguity in order to find an algorithm for the Gell–Mann–Low renormalization group action, on the space of all possible interactions. Of course, it would be trivial to play with the full renormalization ambiguity in order to fix the series of the beta function in a particular theory. With our method we intend to fix simultaneously the structure of infinite number of formal power series including not only the beta functions for all theories but also the series of the anomalous dimensions.

Based on the additional results on perturbative operator product expansion algebras, which I have mentioned the above, I would conjecture also that: the coefficients of the beta functions in any perturbative quantum field theory on even space–time dimensions are multiple zeta values.

Acknowledgments

I am grateful to Professor R. Stora for his critical remarks to this work. I am also grateful to Professor I. Todorov for his comments. I am grateful to the organizers of the Conference on Algebraic and Combinatorial Structures in Quantum Field Theory in Cargèse and to the Institut d'Études Scientifiques de Cargèse for the support and the hospitality. This work was partially supported by Bulgarian NSF grant DO 02–257 and French–Bulgarian project Rila under the contract Egide – Rila N112.