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Exact S-wave Solution of Schrödinger Equation for Quantum Bound State Non-Power Law Systems

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Abstract. We have applied the extended transformation method to a non-powerlaw potential to generate a set of exactly solved quantum systems (EQS) in any chosen dimensional Euclidean space. The bound state S- wave solution of the Schrödinger equation of the generated quantum systems are reported. The generated quantum systems are generally in sturmian form. We also report a case-specific regrouping technique to convert a sturmian quantum system (QS) to a normal/physical exactly solved quantum system. The normalizability of the generated quantum systems is given.

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1 Introduction

The exactly solvable Schrödinger equation for a given physical quantum potential has been the object of investigation for quite a long period. It is well known that in quantum mechanics, the family of Schrödinger solvable potentials is very restricted, also that the exact solvability is very dedicated property. However, the same handful exactly solvable potentials infinite well, the harmonic and anharmonic oscillators, Morse, Hulthen and Posch-Teller potentials have been invariably repeated by many authors. Some recent papers in this field have been given in [1,2]. In general, there are a few main methods to study exact solutions of quantum mechanical systems. It is well known in quantum mechanics that a total wavefunction provides implicitly all relevant information about the behaviour of a physical system. Hence if it is exactly solvable for a given potential, the wavefunction can describe such a system completely. Many exactly solvable potentials are polynomial hyperbolic function of spatial coordinate. This apparent shortage of exact solution has led some researchers into considering other forms. In this paper, the extended transformation [3] method is applied with which new exactly solvable potentials can be generated from already known non-relativistic exactly solved quantum system. The method is based on transformation called the extended transformation (ET) that includes a coordinate transformation (CT) followed by a functional transformation (FT) and a set of plausible ansatz to restore the transformed equation to the standard Schrödinger equation form. In case of non power law potentials, ET may be applied repeatedly by selecting the "working potential" (WP) differently from the multiterm potential to generate a variety of new quantum systems (QSs) except for one which reverts back to the parent QS. In case of k-term potential, the working potential can be chosen in $2^k - 1$ different ways. In fact, we have a number of choices to select the working potential and the least being a single term. That the wavefunctions of the generated QSs are almost always normalizable is a positive feature of the transformation procedure. Another superiority of ET is that through FT it is possible to pre-assign the dimension of the transformed QS. In this paper, we have applied the extended transformation method to generate a class of new QSs from an exactly solved non-power law quantum system [4] whose potential has a barrier. Our objective is to generate new exactly solved quantum systems, since the exactly solved quantum potentials facilitate physical applications in different branches of Physics. In dealing with non-power law potentials a major complication arises with because the generated quantum systems are always a sturmian type. The method of regrouping the set of energy dependent potentials to physical quantum systems is also reported in this paper.

2 Formalism

The radial part of the dimensionless Schrödinger equation for an exactly solved multiterm quantum mechanical system with central non-power law potential denoted by $U(r)$, which we have termed as A-quantum system (A-QS) in one-dimensional Euclidean spaces $[\hbar = 1 = 2m]$, is

$$\Psi_A''(r) + [E_n^A - V_A(r)]\Psi_A(r) = 0. \tag{1}$$

Here r is a dimensionless spatial coordinate.

The quantum system is described by the following potential [4]:

$$V_A(r) = v\Big(q_2 \tanh^2 r + q_1 \frac{\tanh(r)}{\cosh(r)} + q_0\Big), \tag{2}$$

where $v = d^2 V_0$ is the dimensionless potential depth, d is the length scale parameter and $E_n^A = d^2 E$ is the dimensionless eigenvalue.

The q_j's are the parameters of the potential, which can be chosen at will. The potential contains the solved symmetric well for $q_1 = 0$ as a special case.

The quantized energy eigenvalues of the potential Eq. (2) of $A - QS$ are given in [4] by

$$E_n^A = v(q_2+q_0) - \left[n+\frac{1}{2} - \left\{\frac{1}{2}\left[\left(\frac{1}{4}+vq_2\right)^2+v^2q_1^2\right]^{\frac{1}{2}}+\frac{1}{2}\left(\frac{1}{4}+vq_2\right)\right\}^{\frac{1}{2}}\right]^2. \tag{3}$$

The energy spectrum is a finite one and the upper bound of the quantum numbers n is given according to [4] by

$$n \leq \left\{\frac{1}{2}\left[\left(\frac{1}{4}+vq_2\right)^2+v^2q_1^2\right]^{\frac{1}{2}}+\frac{1}{2}\left(\frac{1}{4}+vq_2\right)\right\}^{\frac{1}{2}} - \frac{1}{2}. \tag{4}$$

Unlike power law potentials, for non-power law potentials exact solutions are available only for s-wave, whose $l=0$.

The energy eigenfunction is expressed in Jacobi Polynomials form as given in [4] by

$$\Phi_A(r) = N_A(-1)^{4\lambda_1+2\lambda_2+1}u^{-\lambda_1+\frac{3}{4}}(1-u)^{-\lambda_2+\frac{3}{4}} \times p_n^{(-2\lambda_1+1;-2\lambda_2+1)}(-i\sinh(r)), \tag{5}$$

where

$$u = \frac{1}{2}\left(1+i\sinh\,(r)\right) \tag{6}$$

and

$$\lambda_1^A = \frac{1}{2}+\frac{1}{2}\left[\frac{1}{4}+v\left(q_2-iq_1\right)\right]^{\frac{1}{2}}, \tag{7}$$

$$\lambda_2^B = \frac{1}{2}+\frac{1}{2}\left[\frac{1}{4}+v\left(q_2+iq_1\right)\right]^{\frac{1}{2}}. \tag{8}$$

We now turn our attention to generate a new quantum system taking as "seed" the already analytically solved quantum problem. The method we use, the extended transformation (ET) [3], rests on a coordinate transformation followed by a functional transformation.

Applying the extended transformation method to Eq. (1), which comprises of

$$r \to g_B(r) \tag{9}$$

and

$$\Psi_B(r) = f_B^{-1}(r)\,\Phi_A\left(g_B(r)\right), \tag{10}$$

where $\Phi_A(r)$ is the eigenfunction of A-QS and is known, where $\Psi_B(r)$ is the eigenfunction of transformed quantum system, henceforth called $B-$quantum system B-QS. Here, $g_B(r)$ is the transformation function which is at least three times differentiable $f_B(r)$ is a modulated amplitude which is non-singular function of r.

Application of ET on the A-QS yields

$$\Psi_B''(r) + \Big(\frac{d}{dr}\ln\frac{f_B^2}{g_B'}\Big)\Psi_B'(r) + \Bigg[\Big(\frac{d}{dr}\ln\, f_B\Big)\Big(\frac{d}{dr}\ln\frac{f_B'}{g_B'}\Big) + {g_B'}^2\Big(E_n^A - v\Big(q_2\tanh^2 r + q_1\frac{\tanh(g(r))}{\cosh(g(r))}\Big)\Big)\Bigg]\Psi_B(r) = 0. \quad (11)$$

The prime denotes the differentiation of the function with respect to the variable r.

The dimension of the Euclidean spaces of the transformed quantum system, henceforth called the B-quantum system (B-QS) can be chosen arbitrarily, let it be denoted by D_B.

Then

$$\frac{d}{dr}\,\ln\frac{f_B^2}{g_B'} = \frac{D_B - 1}{r} = \frac{d}{dr}\ln\, r^{D_B - 1}, \quad (12)$$

which fixes $f_B(r)$ as a function of $g_B(r)$ and it derivative.

Integrating Eq. (12):

$$f_B(r) = N{g_B'}^{\frac{1}{2}}r^{\frac{D_B-1}{2}} - 2\ln\, N, \quad (13)$$

where N is the normalization constant.

Therefore, Eqs. (10) and (13) lead to:

$$\Psi_B(r) = {g_B'}^{-\frac{1}{2}}(r)r^{-\frac{D_B-1}{2}}\Phi_A(g_B(r)). \quad (14)$$

The corresponding D_B dimensional Schrödinger equation for B-QS with $[\hbar = 2m = 1]$ can be written as:

$$\Psi_B''(r) + \frac{D_B-1}{r}\Psi_B'(r) + \left[\frac{1}{2}\{g_B, r\} + {g_B'}^2\left(E_n^A - V_A(g_B(r))\right)\right]\Psi_B(r) = 0, \quad (15)$$

where

$$\{g_B,\, r\} = \frac{g_B'''(r)}{g_B'(r)} - \frac{3}{2}\left(\frac{g_B''(r)}{g_B'(r)}\right)^2. \quad (16)$$

is the Schwartzian derivative symbol .

In order to mold Eq. (15) to the standard equation form some plausible ansatz have to be carried out, which are an integral part of the transformation method. The system A has a two term potential, the working potential (WP) can be chosen in $2^2 - 1$, *i.e.* three different ways and the WP is designated by $V_A^W\,(g_B(r))$.

In our present case, we have considered

$$V_A^W\,(g_B\,(r)) = vq_2\,\tanh^2\,(g_B\,(r))\,. \quad (17)$$

To implement ET on the A-QS, the following ansatz is carried out:

$$g_B'^2 V_A^W(g_B(r)) = -E_n^B, \tag{18}$$

where E_n^B is the energy eigenvalue of B-QS.

$$g_B'^2 E_n^A = -V_B^1(r), \tag{19}$$

$$-g_B'^2 \left[V(g_B(r)) - V_A^W(g_B(r))\right] = -V_B^2(r), \tag{20}$$

$$-\frac{1}{2}\{g_B, r\} = V_B^3(r). \tag{21}$$

The functional form of the transformation function $g_B(r)$ obtained from Eq. (18) by simple integration is

$$g_B(r) = \text{arccosh}\left(\exp(\eta_n r)\right), \tag{22}$$

where

$$\eta_n = \pm\left(-\frac{E_n^B}{vq_2}\right)^{\frac{1}{2}}. \tag{23}$$

The transformation function $g_B(r)$ has the desirable local property $g_B(0) = 0$ by putting the integration constant equal to zero. This local property is desirable from the point of view of normalizability of the wavefunction of the generated QS.

Now Eqs. (19) and (22) lead to

$$V_B^1(r) = \frac{C_B^2}{1 - \exp(-2\eta_n r)}, \tag{24}$$

where C_B^2 is the characteristic constant of the B-sturmian quantum system (n-dependent potential) denoted by B-SQS and is

$$C_B^2 = \eta_n^2\left(-E_n^A\right). \tag{25}$$

The characteristic constant C_B^2 plays the same role on $-Ze^2$ in case of Coulomb and $\frac{1}{2}m\omega^2$ in case of H.O. system.

Eqs. (20) and (22) yield

$$V_B^2(r) = \frac{\eta_n^2 q_1 v}{\sqrt{\exp(2\eta_n r) - 1}}, \tag{26}$$

and Eqs. (21) and (22) yield

$$V_B^3(r) = \frac{\eta_n^2}{2(1 - \exp(-2\eta_n r))} - \frac{3}{4}\frac{\eta_n^2 \exp(4\eta_n r)}{(\exp(2\eta_n r) - 1)^2} + \frac{\eta^2}{4}. \tag{27}$$

The multiterm potential of the B-SQS which can be written as

$$V_B(r) = V_B^1(r) + V_B^2(r) + V_B^3(r) \tag{28}$$

becomes

$$V_B\left(r\right) = \frac{C_B^2 + \frac{\eta_n^2}{2}}{1 - \exp\left(-2\eta_n r\right)} + \frac{\eta_n^2 q_1 v}{\sqrt{\exp\left(2\eta_n z\right) - 1}} - \frac{3}{4}\frac{\eta_n^2 \exp\left(4\eta_n r\right)}{\left(\exp\left(2\eta_n r\right) - 1\right)^2} + \frac{\eta^2}{4}. \tag{29}$$

The potential $V_B(r)$ is n-dependent. This special type of energy dependent potential is equipped with only a single normalized eigenstate. The B-SQS comprises of a finite set of quantum systems. The B-SQS can be converted to a normal quantum system by a case specific regrouping technique when we have redefined the parameters of the A-QS preserving the type of constraint equations. Considering the expression Eq. (23), the form of $V_B(r)$ given by Eq. (29) suggest that it can be made normal if η_n becomes n-independent. This is achieved by intending q_2 proportional to E_n^B and writing

$$q_2 = S_1(-E_n^B),$$

where S_1 is a scale factor. This makes $\eta_n \to \eta$.

Therefore the normal physical non-sturmian form of the newly generated B-EQS potential comes out to be

$$V_B(r) = P\frac{\exp(2\eta r)}{\exp(2\eta r) - 1} + \frac{Q}{\sqrt{\exp(2\eta r) - 1}} - R\frac{\exp(4\eta r)}{(\exp(2\eta r) - 1)^2}, \tag{30}$$

writing

$$P = C_B^2 + \frac{\eta^2}{2}, \qquad Q = \eta^2 q_1 v \qquad \text{and} \qquad R = \frac{3\eta^2}{4}\,.$$

The quantized energy eigenvalues of the normal B-QS from Eq. (25) come out to be

$$E_n^B = -\left[\frac{\eta^2}{2}\sqrt[3]{-4(2\Lambda^3 - 3\Lambda\Theta + \Omega) + 4\sqrt{4(\Theta - \Lambda^2)^3 + (2\Lambda^3 - 3\Lambda\Theta + \Omega)^2}} - \frac{2\eta^2(\Theta - \Lambda^2)}{\sqrt[3]{-4(2\Lambda^3 - 3\Lambda\Theta + \Omega) + 4\sqrt{4(\Theta - \Lambda^2)^3 + (2\Lambda^3 - 3\Lambda\Theta + \Omega)^2}}}\right], \tag{31}$$

where

$$\Lambda = \frac{5}{12}\Big(\frac{2C_B^2}{\eta^2} - \frac{1}{2}\Big) - \frac{3}{4}\Big(n + \frac{1}{2}\Big)^2 - \frac{\Big(\frac{2C_B^2}{\eta^2} - \frac{1}{2}\Big)^2}{48\Big(n + \frac{1}{2}\Big)^2}, \tag{32}$$

$$\Theta = \frac{1}{48\Big(n + \frac{1}{2}\Big)}\Bigg[\Bigg(10\Big(n + \frac{1}{2}\Big)^2 + \frac{2C_B^2}{\eta^2} - \frac{1}{2}\Bigg)\frac{Q^2}{\eta^4} + \Bigg(2\Big(n + \frac{1}{2}\Big)^2 - \frac{2C_B^2}{\eta^2} + \frac{1}{2}\Bigg)^2\Bigg(6\Big(n + \frac{1}{2}\Big) - \frac{2C_B^2}{\eta^2} + \frac{1}{2}\Bigg)\Bigg], \tag{33}$$

$$\Omega = \frac{1}{16\Big(n + \frac{1}{2}\Big)^2}\Bigg[\Big(\frac{Q^2}{\eta^4} - \frac{1}{4}\Big)\Bigg(2\Big(n + \frac{1}{2}\Big)^2 + \frac{2C_B^2}{\eta^2} - \frac{1}{2}\Bigg)^2 + 2\Big(n + \frac{1}{2}\Big)^2\Big(\frac{2C_B^2}{\eta^2} - \frac{1}{2}\Big)^2 - \frac{Q^2}{4\eta^4}\Bigg]. \tag{34}$$

The energy eigenvalue spectrum is finite and the upper bound of the quantum number n is found as:

$$n \le \left\{\frac{1}{2}\left[\left(\frac{1}{4} + \frac{(-E_n^B)}{\eta^2}\right)^2 + \frac{Q^2}{\eta^4}\right]^{\frac{1}{2}} + \frac{1}{2}\left(\frac{1}{4} + \frac{(-E_n^B)}{\eta^2}\right)\right\}^{\frac{1}{2}} - \frac{1}{2}.$$

The familiar Schrödinger equation for normal form of B-QS for D-dimensional Euclidean space considering the natural units $(\hbar = 1 = 2m)$ comes out to be

$$\Psi_B{''}(r) + \frac{D-1}{r}\Psi_B{'}(r) + [E_n^B - V_B(r)]\Psi_B(r) = 0. \tag{35}$$

The corresponding quantized energy eigenfunction of the normal B-QS follows from Eq. (14) as

$$\Psi_B(r) = N_B\left[\frac{\exp(\eta r)}{\sqrt{\exp(2\eta r) - 1}}\right]^{-\frac{1}{2}} r^{-\left(\frac{D-1}{2}\right)}\left[1 + i\sqrt{\exp(2\eta r) - 1}\right]^{\frac{2B_\alpha + 1}{4}} \times \left[1 - i\sqrt{\exp(2\eta r) - 1}\right]^{\frac{2B_\beta + 1}{4}} P_n^{(B_\alpha, B_\beta)}\left[-i\sqrt{\exp(2\eta r) - 1}\right], \tag{36}$$

where

$$B_\alpha = -2\lambda_1^B + 1 = -\Big[\frac{1}{4} + \frac{(-E_n^B)}{\eta^2} - \frac{iQ}{\eta^2}\Big]^{\frac{1}{2}}, \tag{37}$$

$$B_\beta = -2\lambda_2^B + 1 = -\Big[\frac{1}{4} + \frac{\left(-E_n^B\right)}{\eta^2} + \frac{iQ}{\eta^2}\Big]^{\frac{1}{2}}. \tag{38}$$

and N_B is the normalization constant of the generated B-QS.

The first type of transformation is termed as the first order transformation.

3 Normalizability of the Generated Quantum System

An important property of the transformation method is that the wavefunctions of the generated quantum systems are almost always normalizable.

The normalization constant is given by

$$N_B = \left[\frac{-E_n^B}{< V_A^W\left(g_B\left(r\right)\right) >}\right]^{\frac{1}{2}}.$$

The expectation value of a potential of exactly solved quantum system (ESQS) is always finite and so a part of it is also finite.

The A-QS eigenfunction $\Phi_A(r)$ is the normalized wavefunction of a genuine quantum mechanical system. Its existence also implies that $\Psi_B(r)$ are also normalizable for $E_n^B \neq 0$, since the behaviour of the transformation function $g_B(r)$ is smooth so far local and asymptotic behaviour are concerned and it is three times differentiable. The transformation method carries over the normalizability of the parent quantum system to the generated quantum system.

4 Second Order Transformation

Application of extended transformation on the B-SQS Eq. (29) can generate another new sturmian quantum system denoted by C-SQS. Since B-QS potential is multiterm potential, we have a number of choices to select the working potential. The working potential can be chosen in $2^3 - 1$, *i.e.* seven different ways to generate C-QS. But we consider the single term only for simplicity.

We choose the following working potential:

$$V_B^W(g_C(r)) = -\frac{3}{4}\eta^2 \frac{\exp(4\eta g_C(r))}{[\exp(2\eta g_C(r)) - 1]^2}\,. \tag{39}$$

The transformation function $g_C(r)$ is obtained by integration and is

$$g_C(r) = \frac{1}{2\eta}\ln(1 + \exp(2\eta\xi_n r)), \tag{40}$$

where

$$\xi_n = \pm\Big(\frac{4E_n^C}{3\eta^2}\Big)^{\frac{1}{2}} \tag{41}$$

with the local property $g_C(0) = 0$.

The sturmian form of C-QS comes out as

$$V_C(r) = (C_C^2 - \eta^2\xi_n^2)\frac{\exp(4\eta\xi_n\, r)}{(\exp(2\rho_n r)+1)^2} + \frac{\eta^2 q_1 v\xi_n^2\, \exp(3\eta\xi_n\, r)}{(\exp(2\eta\xi_n\, r)+1)^2} + \left(C_B^2\xi_n^2 + \frac{\eta^2\xi_n^2}{2}\right)\frac{\exp\left(2\eta\xi_n\, r\right)}{\exp\left(2\eta\xi_n\, r\right)+1}. \quad (42)$$

The potential can be converted to a normal one if we make ξ_n, n-independent as seen from Eq. (41).

We set

$$\frac{3\eta^2}{4} = S_2\left(-E_n^C\right), \quad (43)$$

where a scale factor S_2 is introduced.

The normal form of C-QS is found as

$$V_C(r) = P_1\frac{\exp(4\rho r)}{(1+\exp(2\rho))^2} + Q_1\frac{\exp(3\rho r)}{(1+\exp(2\rho r))^2} + R_1\frac{\exp(2\rho r)}{(1+\exp(2b\rho r))}, \quad (44)$$

where

$$P_1 = C_D^2 - \eta^2\xi^2, \quad Q_1 = \eta^2 q_1 v\xi_n^2, \quad R_1 = C_B^2\xi^2 + \frac{\eta^2\xi^2}{2} \quad \text{and} \quad \rho = \eta\xi.$$

The corresponding normalized energy eigenfunction of the newly constructed C-QS is found from Eq. (14) and is

$$\Psi_C(r) = N\exp(-\rho r)(1+\exp(2\rho r))^{\frac{1}{2}}(1+i\exp(\rho r))^{\frac{2C_\alpha+1}{4}} \times (1-i\exp(\rho r))^{\frac{2C_\beta+1}{4}} P_n^{(C_\alpha,C_\beta)}\left[-i\,\exp(\rho r)\right],$$

where

$$C_\alpha = -2\lambda_1^C + 1, \qquad C_\beta = -2\lambda_2^C + 1$$

and

$$\lambda_1^C = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{4} + \frac{C_C^2}{\rho^2} - \frac{iQ_1}{\rho^2}\right]^{\frac{1}{2}}, \quad (45)$$

$$\lambda_2^C = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{4} + \frac{C_C^2}{\rho^2} + \frac{iQ_1}{\rho^2}\right]^{\frac{1}{2}}. \quad (46)$$

The quantized energy eigenvalues of C-QS are found as

$$E_n^C = -\frac{3}{4}\frac{\rho^2 C_C^2}{\omega} + \frac{3}{4}\frac{\rho^4}{\omega}\Bigg[n + \frac{1}{2} - \left\{\frac{1}{2}\Big[\Big(\frac{1}{4}+\frac{C_C^2}{\rho^2}\Big)^2 + \frac{Q_1^2}{\rho^4}\Big]^{\frac{1}{2}} + \frac{1}{2}\Big(\frac{1}{4}+\frac{C_C^2}{\rho^2}\Big)\right\}^{\frac{1}{2}}\Bigg]^2, \quad (47)$$

where $\omega = C_B^2 \xi^2$.

The quantized energy eigenvalues spectrum is finite and the upper bound of the quantum number n is given by

$$n \le \frac{1}{2}\left\{\left[\left(\frac{1}{4}+\frac{C_C^2}{\rho^2}\right)^2 + \frac{Q_1^2}{\rho^4}\right]^{\frac{1}{2}} + \frac{1}{2}\left(\frac{1}{4}+\frac{C_C^2}{\rho^2}\right)\right\}^{\frac{1}{2}} - \frac{1}{2} \qquad (48)$$

5 Conclusion

We have generated a class of exactly solved quantum systems in non relativistic quantum mechanics using the extended transformation (ET) method in any arbitrary number of spatial D-dimensional Euclidean space. The Extended transformation method can be applied in principle to generate innumerable exactly solved non relativistic QSs starting from exactly solved non power law potentials. First order application of the ET to the potential Eq. (2) choosing $vq_2 \tanh^2(g_B(r))$ as working potential (WP) leads to the B-EQS

$$V_B(r) = P\frac{\exp(2\eta r)}{\exp(2\eta r)-1} + \frac{Q}{\sqrt{\exp(2\eta_n z)-1}} - R\frac{\exp(4\eta r)}{(\exp(2\eta r)-1)^2}.$$

In case of non-power law potential the transformed quantum system is always sturmian, the transformation function is non factorizable unlike the power law type potential.

The second order application of the ET method to the B-QS when

$$V_B^W(g_C(r)) = -\frac{3}{4}\eta^2\frac{\exp(4\eta g_C(r))}{[\exp(2\eta g_C(r))-1]^2}$$

is taken as the working potential produces another new exactly solved quantum system, denoted by C-EQS whose normal form is given by

$$V_C(r) = P_1\frac{\exp(4\rho r)}{(1+\exp(2\rho r))^2} + Q_1\frac{\exp(3\rho r)}{(1+\exp(2\rho r))^2} + R_1\frac{\exp(2\rho r)}{(1+\exp(2\rho r))}.$$

The term $vq_1 \dfrac{\tanh(r)}{\cosh(r)}$ cannot be used as working potential because after integration r, is a complicated function g is found which cannot be inverted g_B is a function of r. Although we can generate different QSs by choosing different terms of the multiterm potential as the WP, out of there one choices revert back the generated QS to the parent QS. But the the important point to note is that to generate C-QS starting from A-QS we have to pass through B-QS. Extended transformation therefore does not form a transformation group in the usual sense.

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