Rotational Perturbations of Friedmann Universe Endowed with a Slight Rotation

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Abstract. Presenting some new interesting Friedmann cosmological solutions of the Einstein field equations incorporated the dynamics, behaviour and phenomena of rotating viscous fluid models are investigated and their physical and geometrical properties are studied in order to substantiate the possibility of the existence of such astrophysical bodies in this universe. The nature and role of the metric rotation $\Omega(r,t)$ as well as that of matter rotation $\omega(r,t)$ are studied for uniform and non-uniform motion. The field equations impose restrictions on the matter rotation $\omega(r,t)$ and some of the solutions for $\Omega(r,t)$, which is related to the local dragging of inertial frames, are expressed in terms of hypergeometric functions. It is found that the rotational perturbation declines for expanding models where R(t) is an increasing function of time and the cosmic time is increasing as well due to the involvement of bulk viscosity. Also it is found that the metric rotation function $\Omega(r,t)$ becomes very large for small values of "r" at which it must occur higher order term in the perturbed metric.

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1 Introduction

The Friedmann cosmological solutions of the Einstein field equations have incorporated the observed large scale expansion, homogeneity and isotropic characteristics of the universe. Since there has been observational evidence as to the rotation of the universe in some sense, and as also it is well known that almost all the astrophysical objects in this universe exhibit some form of rotation, whether differential or uniform, and are in a state of rotation about their axes, it is worthy to think that the universe itself is endowed with some rotation.

In the recent few years, various researchers have been studying the dynamics of rotating objects in a large scale as it plays an important role in understanding the structure and equilibrium configurations of the astrophysical objects. Some of them discussed the general properties of density, distortion and rotational perturbations of Friedmann cosmologies. In order to incorporate the possibility that the universe is endowed with a slight rotation, the rotational perturbation of Friedmann models have been discussed here. To consider the slow rotation of the universe the models are homogeneous: every rest observer in the substratum Friedmann cosmology sees himself as the centre of the same distribution of small rotation. The problem of the universal rotation of nearby matter about an observer moving with matter relative to an inertial frame. From recent observations it is believed that the universe may be rotating at the rate of $\leqslant 10^{-3}$ rad s^{-1}. The existence of such a small rotation, when taken into consideration during the early stages of the universe, would play a prominent role in the dynamics of the universe as well as the processes that involve the formation of galaxies and other cosmological objects. As the rotation plays an important role in the structure and equilibrium configuration of elementary particles as well as that of the astrophysical objects, the equilibrium configuration of rotating fluid can be considered as a small perturbation on a non-rotating configuration.

That is why during the last few years there has been considerable effort in introducing rotation in the General theory of relativity so that it can be applied to the realistic astrophysical situation. Thereafter some physicists investigated on the rotational motion of cosmological objects. Lense and Thirring (1918) were the first to attempt the study of gravitational field due to a rotating body.Thereafter some physicists investigated on the rotational motion of cosmological objects. Gödel [1, 2], Das *et al*. [3], Hartle and Sharp [4, 5], Ellis [6], Cohen and Brill [7], Hawking [8], Silk and Wright [9], Chandrashekar and Friedmann [10–12], Adams *et al*. [13, 14], Fennelly [15], Kaminisini *et al*. [16], Sanz [17], Whitman and Pizzo [18], Bayin and Cooperstock [19], Bayin [20, 21], Krori *et al*. [22], Whitman [23], Islam [24], Van den Bergh and Wil [25], Tiwari *et al*. [26], Maniharsingh and Bhamra [27], Tarachand and Singh [28], Maniharsingh [29–36] have studied the rotating fluid distributions under different conditions in trying to understand the structure and equilibrium and nature and role of rotating astrophysical objects in the Universe. It has been observed that all neutron stars are found to satisfy the condition of slow rotation thereby showing that slowly rotating perfect fluid solutions can be treated as the mathematical models for neutron stars.

Even though various authors have considered some general properties of density, distortion and rotational perturbation of Friedmann cosmologies. However according to Sanz [17] there have been no exact analytic solutions of the perturbation equations published apart from his own with respect to distortion. But in this paper, rotational perturbations of Friedmann models are considered in detail in order to incorporate the possibility that the universe is endowed with a slight rotation and several analytic solutions are presented. Due to the extreme difficulty in obtaining the exact solutions of the highly nonlinear partial differential equations for a rotating perfect fluid we have to study the slowly rotating solutions by taking approximations to the first order in the metric rotation function $\Omega(r,t)$ and have studied the problem of slowly rotating cosmological fluid spheres. The general solutions for $\Omega(r,t)$ have been obtained for cosmological models by imposing restrictions on the matter rotation $\Omega(r,t)$ related to the dragging of inertial frames and uniform rotation. In this problem we take up the rotational motivated problem to present a more general solution to the slowly rotating field equations. It is well known that no real astrophysical object is composed of perfect fluid. It is, therefore, interesting to consider viscous-fluid cosmological models for the present problem. Our aim is to present a more general solution to the slowly rotating field equations for viscous fluid under different physical restrictions by considering rotational perturbations of the Robertson-Walker metric to incorporate the possibility of the rotating universe. By imposing restriction on the matter rotation $\omega(r,t)$ the field equation with viscous fluid as the source have been solved and some of the solutions for $\Omega(r,t)$ which is related to the local dragging of the inertial frame have been obtained. Analytical solutions are presented for cosmological model by taking the energy-momentum tensor for viscous fluid and non-vanishing bulk viscosity co-efficient.

In Section 2, we have presented the mathematical formulation of the problem and in Section 3, the surviving field equations taking approximation to the first order in $\Omega(r,t)$ are presented and the exact solutions for $\Omega(r,t)$ corresponding to several physical assumptions are obtained. In Section 4, a general discussion of the results obtained is presented.

2 Formulation of the Problem

The cosmological metric form in which the whole Universe will be perturbed on the analogy of Lands berg's is considered as

$$ds^2 = dt^2 - R^2(t)\left[(1-Kr^2)^{-1}dr^2 + r^2\left\{d\theta^2 + \sin^2\theta(d\varphi - \Omega dt^2\right\}\right], \quad (1)$$

where $\Omega(r,t)$ represents the angular velocity of the inertial frame along the axis of rotation.

Owing to the slow rotation it is sufficient to calculate up to the first order of an angular velocity ω.

Hence, the perturbed metric can be taken in the form as

$$ds^2 = dt^2 - R^2(t)\left\{(1-Kr^2)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right\} + 2\Omega(r,t)R^2r^2\sin^2\theta d\varphi dt. \quad (2)$$

The Einstein's field equations are given by

$$G_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = -8\pi T_{ij}, \quad (3)$$

where the energy momentum tensor T_{ij} of the viscous fluid is given by

$$T_{ij} = (\rho + p - \xi\Theta)\, u_i u_j - (p - \xi\Theta) g_{ij} - 2\eta\sigma_{ij}, \tag{4}$$

where p is the isotropic pressure; ρ – the matter density; u^i – the four velocity vector; η and ζ are the coefficient of shear and bulk viscosity respectively.

Moreover, $H_{ij} = g_{ij} - u_i u_j$ is the projection tensor and σ_{ij} is the shear tensor defined by

$$\sigma_{ij} = \frac{1}{2}\left(u_{i;\alpha} H^{\alpha}_{j} + u_{j;\alpha} H^{\alpha}_{i}\right) - \frac{1}{3}\Theta H_{ij}, \tag{5}$$

where $\Theta = u^{\alpha}_{;\alpha} = \nabla.u$ is the expansion factor and all other symbols have their usual meanings.

The components of u^i are given by

$$\begin{aligned} u^4 &= \frac{dt}{ds}, \\ u^3 &= \left(\frac{d\varphi}{dt}\right)\left(\frac{dt}{ds}\right) = u^4.\omega, \\ u^1 = u^2 &= 0 \qquad \text{and} \qquad u^i u_i = 1, \end{aligned} \tag{6}$$

where ω is the angular velocity of the fluid distribution in the units of coordinate time.

3 Surviving Field Equations

The surviving field equations by taking approximation to the first order in Ω(r, t) can be reduced as

$$\begin{aligned} G_{11} &\equiv (1 - Kr^2)^{-1}(2R\ddot{R} + \dot{R}^2 + K - \Lambda R^2) \\ &= -8\pi R^2 (1 - Kr^2)^{-1}(p - \zeta\Theta), \end{aligned} \tag{7}$$

$$G_{22} \equiv r^2(2R\ddot{R} + \dot{R}^2 + K - \Lambda R^2) = -8\pi R^2 r^2 (p - \zeta\Theta), \tag{8}$$

$$G_{33} \equiv r^2 \sin^2\theta(2R\ddot{R} + \dot{R}^2 + K - \Lambda R^2) = -8\pi R^2 r^2 \sin^2\theta(p - \zeta\Theta), \tag{9}$$

$$G_{44} \equiv -\frac{3(\dot{R}^2 + K)}{R^2} + \Lambda = -8\pi\rho, \tag{10}$$

$$G_{13} = G_{31} \equiv -\frac{1}{2} r^2 R^2 \sin^2\theta \left(\dot{\Omega}' + \frac{3\dot{R}}{R}\Omega'\right) = +8\pi r^2 R^2 \eta\omega' \sin^2\theta \tag{11}$$

and

$$\begin{aligned} G_{43} = G_{34} &\equiv -r^2\sin^2\theta \Big\{(2R\ddot{R} + \dot{R}^2 + K)\Omega + \frac{1}{2}(1 - Kr^2)\Omega'' \\ &\qquad + \left(\frac{2}{r}\right)\left(1 - \frac{5}{4}Kr^2\right)\Omega' - R^2\Lambda\Omega\Big\} \\ &= -8\pi\Big\{(\rho + p - \zeta\Theta) r^2 R^2 \sin^2\theta(\Omega - \omega) - (p - \zeta\Theta) r^2 R^2 \sin^2\theta\,\Omega\Big\}. \end{aligned} \tag{12}$$

From Eqs. (7) to (9), we obtain

$$8\pi(p-\zeta\Theta) = -\left(\frac{2\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}+\frac{K}{R^2}\right)+\Lambda, \tag{13}$$

$$8\pi\rho = \frac{3(\dot{R}^2+K)}{R^2}-\Lambda \tag{14}$$

$$\dot{\Omega}'+\frac{3\dot{R}}{R}\Omega' = -16\pi\eta\omega'. \tag{15}$$

Using Eqs. (13) and (14) in (12), we obtain

$$\left(1-Kr^2\right)\Omega''+\left(\frac{4}{r}-5Kr\right)\Omega' = 4\left(\dot{R}^2+K-R\ddot{R}\right)(\Omega-\omega) \tag{16}$$

Equations (13) and (14) are the same as the unperturbed equations and determine the pressure p and the density ρ.

From Eq. (15), we obtain

$$\frac{3\dot{R}}{R}\Omega+\dot{\Omega} = -16\pi\eta\omega+c_1(t). \tag{17}$$

where c_1(t) is an arbitrary function of time which is set equal to zero for our problem.

As Eq. (16) cannot be solved in general for an arbitrary distribution of matter so we take a physical assumption that ω varies directly as Ω.

Taking $\omega = \psi\Omega$ in Eq. (17), we will find

$$\Omega(r,t) = A(r)e^{-16\pi n\psi t}R^{-3}, \tag{18}$$

where $A(r)$ is an arbitrary function of r and ψ is a constant.

Using equation (18) in (16), we obtain

$$\frac{A''}{A}(1-Kr^2)+\left(\frac{4}{r}-Kr^2\right)\frac{A'}{A} = 4(\dot{R}^2-R\dot{R}+K)(1-\psi). \tag{19}$$

Since the left hand side of the equation is a function of 'r' alone, the right hand side is a function of 't' alone.

Now Eq. (19) separates into

$$\left(1-Kr^2\right)\frac{A''}{A}+\left(\frac{4}{r}-Kr^2\right)\frac{A'}{A} = S \tag{20}$$

and

$$4\left(\dot{R}^2-R\ddot{R}+K\right)(1-\psi) = S. \tag{21}$$

where S is a separation constant.

For a cosmological background $R(t)$ is known so equation (20) determines $A(r)$.

By defining a new variable $x = Kr^2$, ($K \neq 0$), Eq. (20) reduces to

$$x(1-x)A_{xx} + \left(\frac{5}{2} - 3x\right) A_x - \frac{S}{4K} A = 0 \quad , \tag{22}$$

First we consider $x \leqslant 1$ which is the case of Friedman closed models.

This is a homogeneous hypergeometric differential equation

$$x(1-x)\frac{d^2A}{dr^2} + \{\gamma - (1+\alpha+\beta\}\,\frac{dA}{dr} - \alpha\beta A = 0, \tag{23}$$

where

$$1+\alpha+\beta = 3, \qquad \alpha\beta = \frac{S}{4K}, \qquad (K \neq 0) \qquad \text{and} \qquad \gamma = \frac{5}{2}. \tag{24}$$

The general solution of Eq. (22) is

$$A(r) = A_0F(\alpha,\beta,\gamma,x) + B_0x^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma;\alpha-\gamma,x), \tag{25}$$

where A_0, B_0 are constants and F is the hypergeometric function.

Solving these expressions for α, β we find for closed models $(K = 1)$

$$\alpha = \frac{1}{2}(2+\sqrt{4-S}), \beta = \frac{1}{2}(2-\sqrt{4-S}\,)\,. \tag{26}$$

The general solution in terms of the original coordinates

$$\Omega(r,t) = A(r)e^{-16\pi n\psi t}R^{-3}, \tag{27}$$

where $A(r) = A_0F(\alpha,\beta,\gamma,r^2) + B_0(r^2)^{1-\gamma}F(\alpha+1-\gamma),\beta+1-\gamma;2-\gamma,r^2)$.

The constants A_0 and B_0 are so chosen that Ω and its first derivatives are continuous and non-singular. Since the value of the hypergeometric functions is unity at the origin, Ω will be non-singular there provided $1-\gamma > 0$.

But since $1-\gamma < 0$ we must take $B_0 = 0$ for Ω to be finite everywhere.

The solution then is

$$\Omega(r,t) = A_0F(\alpha,\beta,\gamma,r^2)e^{-16\pi n\psi t}R^{-3}(t), \tag{28}$$

where $F(\alpha,\beta,\gamma,r^2)$ converges for all positive values of r^2 and since $\alpha+\beta < 2$ it diverges if $r^2 = 1$ and does not terminate.

In particular, we discuss some explicit solutions

(a) $F(2, 0, {}^5/_2, r^2) = 1, (S = 0)$

$$\Omega(r,t) = A_0 e^{-16n\pi\psi t} R^{-3}(t),$$

which is a function of 't' alone.

This does not correspond to a physical rotation.

(b) $F(-{}^1/_2, {}^5/_2, {}^5/_2, r^2) = (1-r^2)^{1/2} (S = -5)$

$$\Omega(r,t) = A_0(1-r^2) e^{-16\pi n\psi t} R^{-3}.$$

For open models ($K = -1$), the solutions have given in terms of hypergeometric functions satisfying the field equations by replacing r^2 by $-r^2$. But the solutions are converges only when $r^2 < 1$.

For flat models ($K = 0$). The Eq. (23) becomes

$$\frac{A''}{A} + \frac{4}{r}\frac{A'}{A} = S. \tag{29}$$

This can be solved to give

$$A(r) = \left\{ \frac{\sqrt{S}(A_1 e^{\sqrt{s}r} - B_1 e^{-\sqrt{s}r})}{r^2} - \frac{A_1 e^{\sqrt{s}r} + B_1 e^{\sqrt{s}r}}{r^3} \right\}, \tag{30}$$

where S, A_1, B_1 are constants.

$$\therefore A(r) = \left\{ \frac{\sqrt{S}(A_1 e^{\sqrt{s}r} - B_1 e^{-\sqrt{s}r})}{r^2} - \frac{A_1 e^{\sqrt{s}r} + B_1 e^{\sqrt{s}r}}{r^3} \right\} e^{-16n\pi\psi t} R^{-3}(t).$$

4 Discussion of the Results

After analyzing the perturbation in the form of differential rotations of Friedmann cosmologies and imposing the restriction which the field equations impose upon the angular velocity of matter varying directly to metric rotational function Ω, the rotation cannot be intrinsic. To the first order in the metric rotation function the field equations reduce to the unperturbed equations for pressure and density as in Eqs. (19) and (20).Thus rotational perturbations decline for expanding models where R(t) is an increasing function of time and the cosmic time is increasing as well due to the involvement of bulk viscosity.

Even though Ω plays a role in the dragging of local inertial frames except when it coincides with the angular velocity of matter ω for a particular case, it will not be the angular velocity of the inertial frames and the rotation is still essential.

For all the solutions the arbitrary constants in $A(r)$ should be chosen so that $\Omega(r,t)$ may play precisely the dragging of the local inertial frame. In equation (30) the metric rotation function $\Omega(r,t)$ becomes very large for small 'r' at which point it must occur higher order terms in the perturbed metric.

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