

## Exact S-Wave Solution of Schrödinger Green's Function Equation by Transformation Method

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**Abstract.** We have applied the extended transformation method to a non-power law potential to generate a set of exactly solved quantum system in any chosen dimensional spaces. Exact analytic solutions of the Schrödinger Green's function equations are obtained which are generated from the Morse potential with steeper potential surface. The bound state quantized energy spectra and the corresponding wave functions of the generated potentials are obtained.

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### 1 Introduction

Potential governing a quantum system (QS) very often does not yield exact solution of the Schrödinger equation, which compels us to rely on various approximation schemes. However, for accuracy in analysis within the framework of the approximation schemes it is necessary that an exactly solvable potential (ESP) may differ from the given potential by as little as possible. This leads us to acquire as many ESPs as possible. Considerable effort has been made in recent years towards obtaining exact solution of the Schrödinger equation for potentials of physical interest [1-7]. To increase the set of ESPs, one of the present authors has developed a mapping procedure [8,9], called extended transformation method (ET) which includes a coordinate transformation followed by a functional transformation and a set of plausible ansätze.

In this paper we apply this ET method to the exactly solved Morse-like potential [10], to generate new QSS, within the framework of Green's function (GF) technique, in  $D$ -dimensional spaces. Each equipped with normalized exact analytic solution (EAS)s and associated energy eigenvalues. In the case of non-power law potential ET may be applied repeatedly by selecting the working potential (WP) differently from the multi-term potential to generate a variety of new QSS, except for one which reverts back to the parent QS.

The organization of the paper is as follows: in Section 2, we give the formalism of the ET method. In Sections 3 and 4 application of the ET method to the Morse system along with generation of a Sturmian type of QS is given. The method of regrouping the set of energy-dependent QSs to a normal/physical QS is also discussed in the same section. The second order application of the ET method on the Morse QS is given in Section 3.2. Discussion and conclusion of our investigations are given in Section 5.

## 2 Formalism

We Consider a  $D$ -dimensional QS with spherically symmetric potential  $V(r)$  where the propagator  $\Omega(\vec{r}t, \vec{r}_0t_0)$  of a particle moving from an initial position  $\vec{r}_0$  at time  $t_0$  to a final position  $\vec{r}$  at time  $t$  satisfies the differential equation ( $\hbar = 1 = 2m$ )

$$\left[ \frac{\partial}{\partial t} + \nabla^2 - V(r) \right] \Omega(\vec{r}t, \vec{r}_0t_0) = \frac{i\delta(\vec{r}, \vec{r}_0)\delta(t, t_0)}{r_0^{D-1}}. \quad (1)$$

It is convenient for our purpose to deal with energy and angular momentum dependent GF,  $G(\vec{r}, \vec{r}_0; E_n)$ . For this we Fourier transform  $\Omega(\vec{r}t, \vec{r}_0t_0)$  as

$$G(\vec{r}, \vec{r}_0; E) = i \int_0^\infty e^{iE(t-t_0)} \Omega(\vec{r}t, \vec{r}_0t_0, V(r)) d(t - t_0), \quad (2)$$

which satisfies the Schrödinger GF equation

$$[E + \nabla^2 - V(r)]G(\vec{r}, \vec{r}_0; E) = \frac{\delta(r - r_0)}{r_0^{D-1}}. \quad (3)$$

It is followed by a partial wave expansion

$$G(\vec{r}, \vec{r}_0; E) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{rr_0} G_l(r, r_0, E) Y_l^m(\theta, \Phi) Y_l^{*m}(\theta_0, \Phi_0), \quad (4)$$

In non-power law potentials exact solutions are available only for S-wave ( $l = 0$ ). We now determine the radial GF  $G(r, r_0; E)$ , as we shall be dealing with Central potential, the radial GF is found to satisfy the radial Schrödinger equation

$$\left( \frac{\partial^2}{\partial r^2} + \frac{D_A - 1}{r} \frac{\partial}{\partial r} + \epsilon_n^A - V_A(r) \right) G_A(r, r_0; E_A) = \frac{\delta(r - r_0)}{r_0^{D_A-1}}. \quad (5)$$

henceforth called our A-QS in  $D_A$ -dimensional space. The corresponding integral equation is

$$\varphi_A(r) = \int G_A(r, r_0; E_A) (\epsilon_n^A - V_A(r)) \varphi_A(r_0) r_0^{D_A-1} dr_0,$$

where the GF and energy eigenvalues  $\epsilon_n^A$  are known for the given potential  $V_A(r)$ .

The completeness of the set of energy eigenfunctions allows us to have eigenfunctions expansion of the energy GF as

$$G_A(r, r_0; E_A) = \sum_{n=0}^{\infty} \frac{\varphi_A(r)\varphi_A^*(r_0)}{E_A - \epsilon_n^A - i\epsilon}, \quad (6)$$

from which we read off the analytic form of the wave function of the solved quantum system.

We now consider changing the variables in Eq. (5) in such a way that the resulting equation is of the same form but with new parameters. For this purpose we apply a transformation method called extended transformation (ET) [8] which comprises of a coordinate transformation  $r \rightarrow g_B(r)$ ,  $r_0 \rightarrow g_B(r_0)$ , followed by a functional transformation

$$G_B(r, r_0; E_B) = f_B^{-1}(r)G_A(g_B(r), g_B(r_0); g_B'^2 E_A)f_B^{-1}(r_0). \quad (7)$$

That the resulting equation is of the same form as (5) but with new potential and eigenvalues and leads to the following equation:

$$\left[ \frac{\partial^2}{\partial r^2} \left( \frac{d}{dr} \ln \frac{f_B^2(r)g_B^{D_A-1}(r)}{g_B'(r)} \right) \frac{\partial}{\partial r} + \left( \frac{d}{dr} \ln f_B(r) \right) \left( \frac{d}{dr} \ln \frac{f_B' g_B^{D_A-1}(r)}{g_B'(r)} \right) + g_B'^2 (\epsilon_n^A - V_A(g_B(r))) \right] G_B(r, r_0; E_B) = \frac{\delta(r - r_0)}{r_0^{D_B-1}}, \quad (8)$$

where  $g_B(r)$  and  $g_B(r_0)$  are the transformation function, which are continuous and at least three times differentiable function, and  $f_B(r)$  and  $f_B(r_0)$  are the  $r$ -dependent modulated amplitude to be determined. Equation (8) has to be re-caste into a form, which is similar to Eq. (5). The dimension  $D_B$  of the Euclidean space of the transformed QS, can now be chosen arbitrarily. For this purpose we require to put the coefficient  $\frac{\partial}{\partial r}$  of Eq. (8) as

$$\frac{d}{dr} \ln \frac{f_B^2(r)g_B^{D_A-1}(r)}{g_B'(r)} = \frac{D_B - 1}{r}, \quad (9)$$

which fixes  $f_B(r)$  as a function of  $g_B(r)$  and its derivative.

Integrating Eq. (9), we obtain

$$f_B^{-1}(r) = N_B g_B'^{1/2} g_B^{(D_A-1)/2}(r) r^{-(D_B-1)/2}, \quad (10)$$

where  $N_B$  is the normalization constant.

Therefore, Eqs. (7) and (10) lead to

$$G_B(r, r_0; E_B) = g_B'^{-1/2} g_B^{(D_A-1)/2}(r) r^{-(D_B-1)/2} \times G_A(g_B(r), g_B(r_0); g_B'^2 E_A) g_B'^{-1/2}(r_0) g_B^{(D_A-1)/2}(r_0) r_0^{-(D_B-1)/2}. \quad (11)$$

The corresponding  $D_B$ -dimensional Schrödinger GF equation for B-QS can be written as

$$\left[ \frac{\partial^2}{r^2} + \frac{D_B-1}{r} \frac{\partial}{\partial r} + \frac{1}{2} \frac{g_B'''}{g_B'} - \frac{3}{4} \left( \frac{g_B''}{g_B'} \right)^2 + g_B'^2(r) (\epsilon_n^A - V_A(g_B(r))) - \frac{D_A-1}{2} \frac{D_A-3}{2} \left( \frac{g_B'}{g_B} \right)^2 + \frac{D_B-1}{2} \frac{D_B-3}{2} \frac{1}{r^2} \right] G_B(r, r_0; E_B) = \frac{\delta(r-r_0)}{r_0^{D_B-1}}. \quad (12)$$

The prime denotes the differentiation of the function with respect to the variable  $r$ .

In order to mould Eq. (12) to the standard Schrödinger GF equation form given by Eq. (5), the following plausible ansatz has to be carried out, which is an integral part of the transformation method. In Eq. (12)  $V_A(g_B(r))$  is termed as the working potential (WP), which specifies  $g_B(r)$  and is assumed to be monoterm. In the case where the system A has  $q$  number of terms, the WP can be chosen in principle  $2^q - 1$  different ways and the WP is designated by  $V_A^{(W)}(g_B(r))$ . The ansatz is

$$g_B'^2 V_A^{(W)}(g_B(r)) = -\epsilon_n^B, \quad (13)$$

$$g_B'^2 E_n^A = -V_B^{(1)}(r), \quad (14)$$

$$g_B'^2 (V_A(g_B(r)) - V_A^{(W)}(g_B(r))) = V_B^{(2)}(r), \quad (15)$$

$$\begin{aligned} -\frac{1}{2} \frac{g_B'''}{g_B'} + \frac{3}{4} \left( \frac{g_B''}{g_B'} \right)^2 + \frac{D_A-1}{2} \frac{D_A-3}{2} \left( \frac{g_B'}{g_B} \right)^2 \\ - \frac{D_B-1}{2} \frac{D_B-3}{2} \frac{1}{r^2} = V_B^{(3)}(r), \end{aligned} \quad (16)$$

where  $V_B^{(1)}(r) + V_B^{(2)}(r) + V_B^{(3)}(r) = V_B(r)$ .

Then the familiar Schrödinger GF equation form of B-QS comes out to be

$$\left[ \partial_r^2 + \frac{D_B-1}{2} \partial_r + \epsilon_n^B - V_B(r) \right] G_B(r, r_0; E_B) = \frac{\delta(r, r_0)}{r_0^{D_B-1}}. \quad (17)$$

From Eqs. (6), (7) and (11), the eigenfunction expansion of B-QS Green's function is

$$G_B(r, r_0; E_B) = \sum_{n=0}^{\infty} \frac{\left( \frac{g_B^{D_A-1}(r)}{g_B'(r)r^{D_B-1}} \right)^{\frac{1}{2}} \varphi_A(g_B(r)) \varphi_A^*(g_B(r_0)) \left( \frac{g_B^{D_A-1}(r_0)}{g_B'(r_0)r_0^{D_B-1}} \right)^{\frac{1}{2}}}{E_B - \epsilon_n^B - i\epsilon}. \quad (18)$$

The B-QS eigenfunctions  $\varphi_B(r)$  can be read off from Eq. (18)

$$\varphi_B(r) = N_B \left( \frac{g_B^{D_A-1}(r)}{g_B'(r)r^{D_B-1}} \right)^{\frac{1}{2}} \varphi_A(g_B(r)), \quad (19)$$

and it is known, as  $\varphi_A(r)$  and  $g_B(r)$  are known. Equations (18) and (19) hold good for any parent (A-QS) and daughter (B-QS) quantum system.

In non-power law cases in general  $F(\epsilon_n^B)$  cannot be factored out from  $V_B^{(1)}(r)$ . The generated QS would be a Sturmian (energy dependent). In order to make it normal/ physical QS we require a case specific regrouping technique, whenever it is possible.

An important property of the transformation method is that the wavefunctions of the generated Qs are almost always normalizable. The normalizability condition for  $D_B$ -dimensional B-QS eigenfunctions is

$$\int_{N=0}^{\infty} \varphi_B^2(r) r^{D_B-1} dr = |N_B|^{-2} = \text{finite.}$$

The normalization constant is given by

$$N_B = \left[ \frac{-\epsilon_n^B}{\langle V_A^W(g_B(r)) \rangle} \right]^{1/2}.$$

The expectation value of ESP is always finite and so a part of it is also finite.

Hence all the  $\varphi_B(r)$  are normalizable for which  $E_B \neq 0$ , unless  $g_B(r)$  is not badly behaved, so far as its local and asymptotic properties are concerned, the transformation method carries over the normalizability property of the parent QS to the daughter (generated) QS.

### 3 Generation of Exact S-Wave Solution from Morse-like Potential

We have consider the Morse-like QS [10] as A-QS potential and it is written as  $D_A = 3$ )

$$V_A(r) = 4\alpha^2\beta^2 \tanh^2 \beta r - \frac{1}{4} \tanh^2 \beta r + \frac{3}{4}(\sinh \beta r)^{-2}, \quad (20)$$

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where  $\alpha$  and  $\beta$  are the parameters of the potential and  $r$  is a dimensionless spatial coordinate. The exact  $S$ -state energy eigenfunctions are given as

$$\varphi_A(r) = \frac{N_A}{r} \frac{1}{(\tanh \beta r)^{1/2}} \operatorname{sech}^{2(\alpha-n)} \beta r \times {}_2F_1(-n, 2\alpha - n, 2\alpha - 2n + 1; \operatorname{sech}^2 \beta r), \quad (21)$$

where the normalization constant is

$$N_A = \frac{2\Gamma(2\alpha - n)}{\Gamma(2\alpha - 2n + 1)\Gamma(n + 1)} [n(2\alpha - n)(2\alpha - n)]^{1/2}.$$

The corresponding energy eigenvalues are

$$\epsilon_n^A = -\beta^2 \left[ 4(\alpha - n)^2 - \left( 4\alpha^2 - \frac{1}{4} \right) \right] \quad (22)$$

and the energy level spacing

$$\Delta \epsilon_n^A = 8\beta^2 [\alpha - (n + 1/2)]. \quad (23)$$

The quantum numbers are restricted as  $\alpha > n + \frac{1}{2}$ ,  $n = 1, 2, 3, \dots$  and it has a finite number of bound states.

#### 3.1 First Order Transformation

In principle, in this quantum system (A-QS) allows seven ( $2^3 - 1$ ) choices for selection of WP as it is a three-term potential, as mentioned earlier.

Choosing  $V_A^{(W)} = -\frac{1}{4}\beta^2 \tanh^2 \beta r$  as WP and utilizing Eq. (13), we have found the transformation function

$$g_B(r) = \frac{1}{\beta} \cosh^{-1}(e^{\beta \xi_n r}), \quad (24)$$

where

$$\xi_n = \sqrt{\frac{4\epsilon_n^B}{\beta^2}}. \quad (25)$$

The integration constant is put equal to zero, *i.e.*  $g_B(r)$  has the desired local property  $g_B(0) = 0$ .

Now Eqs. (14) and (24) lead to

$$V_B^{(1)}(r) = -\frac{1}{2} C_B^2 e^{\beta \xi_n r} \operatorname{csch}(\beta \xi_n r). \quad (26)$$

$C_B^2$  is the characteristic constant of B-QS and is

$$C_B^2 = \frac{\xi_n^2}{\beta^2} \epsilon_n^A. \quad (27)$$

Equations (15) and (24) lead to

$$V_B^{(2)}(r) = 4\alpha^2 \beta^2 \xi_n^2 + \frac{3}{4} \beta^2 \xi_n^2 \frac{e^{2\beta \xi_n r}}{(2\beta \xi_n r - 1)^2}, \quad (28)$$

and Eqs. (16) and (24) will give

$$V_B^{(3)} = \frac{\beta^2 \xi_n^2}{4(2\beta \xi_n r - 1)^2} - \frac{\beta^2 \xi_n^2 e^{2\beta \xi_n r}}{(2\beta \xi_n r - 1)^2} + \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2}. \quad (29)$$

The multi term potential of the B-QS is

$$V_B(r) = 4\alpha^2 \beta^2 \xi_n^2 - C_B^2 \frac{e^{2\beta \xi_n r}}{2\beta \xi_n r - 1} - \frac{\beta^2 \xi_n^2}{4} \frac{1}{(2\beta \xi_n r - 1)^2} + \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2}. \quad (30)$$

A major complication that always arises in dealing with non-power law potential is that always the potential  $V_B(r)$  is of a Sturmian type, which is characterized by  $n$ -dependent potential and hence  $n$  loses its quantum number character and becomes a system index. The Sturmian form of B-QS can be converted to a normal/physical QS by a case specific regrouping technique, where we have to redefine the A-QS parameters.

To make  $\xi_n$   $n$ -independent, we make the A-QS parameter  $\beta \rightarrow \beta_n$ , so that the combination  $\xi_n \beta_n = s$ ,  $n$ -independent, where a scale factor  $s$  is introduced. This makes the potential  $V_B(r)$   $n$ -independent and is

$$V_B(r) = 4\alpha^2 s^2 - C_B^2 \frac{e^{2sr}}{2sr - 1} - \frac{s^2}{4} \frac{1}{(2sr - 1)^2} + \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2}. \quad (31)$$

The expression (27) is utilized to find the energy eigenvalues of B-QS, and is

$$\epsilon_n^B = - \frac{C_B^2}{4 \left[ 4(\alpha - n)^2 - \left( 4\alpha^2 - \frac{1}{4} \right) \right]}. \quad (32)$$

The energy level spacing of the B-QS is

$$\Delta \epsilon_n^B = \frac{C_B^2 (1 + 2n - 2\alpha)}{\left[ 4(\alpha - n - 1)^2 - \left( 4\alpha^2 - \frac{1}{4} \right) \right] \left[ 4(\alpha - n)^2 - \left( 4\alpha^2 - \frac{1}{4} \right) \right]}. \quad (33)$$

Utilizing Eqs. (6) and (7) we obtain the eigenfunction expansion of B-QS as

$$G_B(r, r_0; E_B) = \sum_{n=0}^{\infty} \frac{\frac{g_B(r)}{(g'_B(r)r^{D_B-1})^{1/2}} \varphi_A(g_B(r)) \varphi_A^*(g_B(r_0)) \frac{g_B(r_0)}{(g'_B(r_0)r_0^{D_B-1})^{1/2}}}{E_B - \frac{C_B^2}{4\left[4(\alpha-n)^2 - \left(4\alpha^2 - \frac{1}{4}\right)\right]} - i\epsilon}, \quad (34)$$

which allows us to obtain the bound state eigenfunction  $\varphi_B(r)$  as

$$\varphi_B(r) = \frac{N_B}{r^{(D_B-1)/2}} e^{-2(\alpha-n)sr} {}_2F_1(-n, 2\alpha-n, 2\alpha-2n+1; e^{-2sr}). \quad (35)$$

$N_B$  is the normalization constant.

### 3.2 Second Order Transformation

Application of ET on B-QS comprising Eq. (30), we can generate another new-Sturmian QS, say, the C-QS, given by

$$\left[ \frac{\partial^2}{\partial r^2} + \frac{D_C-1}{r} \frac{\partial}{\partial r} + \frac{1}{2} \frac{g_C''(r)}{g_C'(r)} - \frac{3}{4} \left( \frac{g_C''(r)}{g_C'(r)} \right)^2 + g_C'^2(r) (\epsilon_n^B - V_B(g_C(r))) - \frac{D_B-1}{2} \frac{D_B-3}{2} \left( \frac{g_C'(r)}{g_C(r)} \right)^2 + \frac{D_C-1}{2} \frac{D_C-3}{3} \frac{1}{r^2} \right] G_C(r, r_0; E_C) = \frac{\delta(r-r_0)}{r_0^{D_C-1}}. \quad (36)$$

From the multiterm B-QS, now we have chosen  $-\frac{\beta^2 \xi_n^2}{4} \frac{1}{(2\beta \xi_n r - 1)^2}$  as the WP so that

$$g_C'^2 \left( -\frac{\beta^2 \xi_n^2}{4} \frac{1}{(e^{2\beta \xi_n r} - 1)^2} \right) = -\epsilon_n^C, \quad (37)$$

$$\begin{aligned} & \frac{1}{2} \frac{g_C'''}{g_C'} - \frac{3}{4} \left( \frac{g_C''}{g_C'} \right)^2 + g_C'^2(r) (\epsilon_n^B - V_B(g_C(r))) \\ & - \frac{D_B-1}{2} \frac{D_B-3}{2} \left( \frac{g_C'}{g_C} \right)^2 + \frac{D_C-1}{2} \frac{D_C-3}{2} \frac{1}{r^2} \\ & = \epsilon_n^C - V_C(r). \end{aligned} \quad (38)$$

The transformation function is found by integration Eq. (37) as

$$g_C(r) = \frac{1}{\gamma} \ln \sec \gamma \tau_n r, \quad (39)$$



where we put  $\gamma = \beta_n \xi_n$  and  $\tau_n = \sqrt{4\epsilon_n^C / \gamma^2}$ .

The newly generated C-QS potential comes out to be ( $D_B = 3$ )

$$V_C(r) = 3\gamma^2 \tau_n^2 \csc^2 2\gamma \tau_n r - \tau_n^2 (C_B^2 + \gamma^2) \sec^2 \gamma \tau_n r + (4\alpha^2 \gamma^2 \tau_n^2 - C_C^2) \tan^2 \gamma \tau_n r - \frac{D_C - 1}{2} \frac{D_C - 3}{2} \frac{1}{r^2}. \quad (40)$$

To make the potential normal we use the same procedure as B-QS, where we convert B-QS parameter  $\xi \rightarrow \xi_n$ , i.e.,  $\beta \xi \rightarrow \beta \xi_n \rightarrow \gamma_n$ , other words to make the C-QS potential normal, we should come from Sturmian B-QS. As a consequence the combination  $\gamma_n \tau_n = a$ , where  $a$  is a scale factor and  $\tau_n^2 C_B^2 = \frac{\tau_n^2 \xi_n^2}{\beta^2} \epsilon_n^A = b^2$ ,

$$V_C(r) = 3a^2 \csc^2 2ar - (a^2 + b^2) \sec^2 ar + (4\alpha^2 a^2 - C_C^2) \tan^2 ar - \frac{D_C - 1}{2} \frac{D_C - 3}{2} \frac{1}{r^2}. \quad (41)$$

The generated C-QS potential is identified as the family of the Poschl-Teller potential [15]. The energy eigenvalues of the  $V_C(r)$  is

$$\epsilon_n^C = -\frac{\alpha^2 C_C^2}{C_B^2} \left[ 4(\alpha - n)^2 - \left( 4\alpha^2 - \frac{1}{4} \right) \right]. \quad (42)$$

The energy level spacing of the normal C-QS is

$$\Delta \epsilon_n^C = \frac{4\gamma^2 C_C^2}{C_B^2} [2(\alpha - n) - 1]. \quad (43)$$

The eigenfunction expanded to of GF, is now

$$G_C(r, r_0; E_C) = \sum_{n=0}^{\infty} \frac{\frac{g_C(r)}{(g'_C(r)r^{D_C-1})^{1/2}} \varphi_B(g_C(r)) \varphi_B^*(g_C(r_0)) \frac{g_C(r_0)}{(g'_C(r_0)r_0^{D_C-1})^{1/2}}}{E_B + \frac{a^2 C_C^2}{C_B^2} \left[ 4(\alpha - n)^2 - \left( 4\alpha^2 - \frac{1}{4} \right) \right] - i\epsilon}. \quad (44)$$

From Eq. (44) the radial wave function is read off, which is

$$\varphi_C(r) = N_C r^{-(D_C-1)/2} \cot^{1/2} ar \cos^{2(\alpha-n)} ar \times {}_2F_1(-n, 2\alpha - n, 2\alpha - 2n + 1; \cos^2 ar). \quad (45)$$

#### 4 Generation of Exact S-Wave Solution from Sturmian-like Potential

Next we choose  $\frac{3}{4}\beta^2(\sinh \beta r)^{-2}$  from multi-term A-QS potential (20) as WP.

#### 4.1 First Order Transformation

Utilizing ansatz (13) we have found the transformation function

$$g_B(r) = \frac{2}{\beta} \tanh^{-1}(Ae^{\eta_n r}), \quad g_B(0) = g_0 \neq 0, \quad (46)$$

where

$$\eta_n = \left( -\frac{4}{3} \epsilon_n^B \right)^{1/2} \quad (47)$$

and  $\epsilon_n^B$  is the new energy of the B-QS to be determined and

$$A = \tanh^{-1} \frac{\beta g_0}{2}. \quad (48)$$

Equations (14) and (46) yield

$$V_B^{(1)}(r) = -C_B^2 \frac{e^{2\eta_n r}}{(1 - A^2 e^{2\eta_n r})^2}. \quad (49)$$

$C_B^2$  is the characteristic constant of the B-QS and is

$$C_B^2 = 4A^2 \eta_n^2 \epsilon_n^A / \beta^2. \quad (50)$$

Equation (50) subsequently provides us B-system energy eigenvalues.

Equations (15) and (46) lead to

$$V_B^{(2)}(r) = 4A^4 \eta_n^2 (16\alpha^2 - 1) \frac{e^{4\eta_n r}}{(1 - A^4 e^{4\eta_n r})^2}, \quad (51)$$

and Eqs. (16) and (46) give

$$v_B^{(3)}(r) = \frac{\eta_n^2 \left[ \frac{1}{4} - \frac{3}{2} A^2 e^{2\eta_n r} + \frac{1}{4} A^4 e^{4\eta_n r} \right]}{(1 - A^2 e^{2\eta_n r})^2} - \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2} \quad (52)$$

The multiterm potential of the B-QS which will be written as

$$V_B(r) = -C_B^2 \frac{e^{2\eta_n r}}{(1 - A^2 e^{2\eta_n r})^2} + 4A^4 \eta_n^2 (16\alpha^2 - 1) \frac{e^{4\eta_n r}}{(1 - A^4 e^{4\eta_n r})^2} + \frac{\eta_n^2 \left[ \frac{1}{4} - \frac{3}{2} A^2 e^{2\eta_n r} + \frac{1}{4} A^4 e^{4\eta_n r} \right]}{(1 - A^2 e^{2\eta_n r})^2} - \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2}. \quad (53)$$

By any means, we cannot convert this Sturmian potential to normal ( $n$ -independent) potential.

The expression (50) is utilized to find the energy eigenvalues, and is

$$\epsilon_n^B = -\frac{3C_B^2}{4[16(\alpha - n)^2 - (16\alpha^2 - 1)]}. \quad (54)$$

The energy level spacing of B-QS is

$$\Delta\epsilon_n^B = \frac{48C_B^2(1 + 2n - 2\alpha)}{[16(\alpha - n - 1)^2 - (16\alpha^2 - 1)][15(\alpha - n)^2 - (15\alpha^2 - 1)]}. \quad (55)$$

The eigenfunction expansion of B-QS Green's function is given by Eq. (18). The corresponding B-QS exact eigenfunctions  $\varphi_B(r)$  are found as

$$\begin{aligned} \varphi_B(r) = N_B r^{-(D_B-1)/2} \frac{(2 - A^4 e^{4\eta_n r})^{1/2}}{e^{\eta_n r}} \left( \frac{1 - A^2 e^{2\eta_n r}}{1 + A^2 e^{2\eta_n r}} \right)^{2(\alpha-n)} \\ \times {}_2F_1 \left( -n, 2\alpha - n, 2\alpha - 2n + 1; \left( \frac{1 - A^2 e^{2\eta_n r}}{1 + A^2 e^{2\eta_n r}} \right)^2 \right). \end{aligned} \quad (56)$$

Proceeding in the similar manner, by the application of ET on the B-QS as obtained from Eq. (53), we can generate second-generation QS.

## 5 Discussions and Conclusions

This paper is concerned with the generation of the new completely solved bound state system in the non-relativistic regime, by a transformation method in any arbitrary  $D$ -dimensional space. Here we have used the GF technique, in conjunction with ET to solve the problem of finding new QSs. For quantum multiterm potentials it is possible to generate a finite number of different exactly solved QSs by selecting differently the WPs, as mentioned earlier. We however restrict ourselves to considering one term WP. Two or multiterm WPs are not considered as they offer some practical difficulties that the integral specifying the transformation function  $g(r)$  cannot be extracted analytically in the most cases. Again if such integrals are found they are of the form  $F(g(r)) = r + C$ , and the analytical inverse function  $F^{-1}(g)$  cannot be found.

Here we have reported Morse potential with steeper potential surface for transformation. The generated QSs, in general are Sturmian potentials. There are no general procedures that can be applied to convert them into normal potential. However in some instances, case specific procedures can be applied by redefining certain quantities, so as to convert a set of the sturmian QS to normal/physical QS. The first order application of ET, choosing  $-\frac{1}{4}\beta^2 \tanh^2 \beta r$  as WP leads to the B-QS potential

$$V_B(r) = 4\alpha^2 s^2 - C_B^2 \frac{e^{2sr}}{2sr - 1} - \frac{s^2}{4} \frac{1}{(2sr - 1)^2} + \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2}.$$

In the case of non-power law potential the transformed QS always comes out as the Sturmian QS. It is evident that The ET may be applied successively any number of times to generate new QSs, when we are considering non-power law potentials. But an important point to be noted is that, ET does not have the group composition law as the transformation depends on the working potential of the parent QS. Therefore the ET does not have a group structure in general. The inverse transformation is obtained when WP is chosen as the term containing the characteristic constant, coming from the energy term of the parent QS.

The second order application of ET to B-QS, taking  $-\frac{1}{4}\beta^2\xi_n^2\frac{1}{(2\beta\xi_nr-1)^2}$  (Sturmian form) generates a potential similar to the Poschl-Teller potential

$$V_C(r) = 3a^2\csc^2 2ar - (a^2 + b^2)\sec^2 ar + (4\alpha^2 a^2 - C_C^2)\tan^2 ar - \frac{D_C - 1}{2} \frac{D_C - 3}{2} \frac{1}{r^2},$$

with a finite number of eigenstates.

Choosing the potential  $\frac{3}{4}(\sinh \beta r)^{-2}$  from the multiterm potential of A-QS as WP, leads to the B-QS potential

$$V_B(r) = -C_B^2 \frac{e^{2\eta_n r}}{(1 - A^2 e^{2\eta_n r})^2} + 4A^4 \eta_n^2 (16\alpha^2 - 1) \frac{e^{4\eta_n r}}{(1 - A^4 e^{4\eta_n r})^2} + \frac{\eta_n^2 \left[ \frac{1}{4} - \frac{3}{2} A^2 e^{2\eta_n r} + \frac{1}{4} A^4 e^{4\eta_n r} \right]}{(1 - A^2 e^{2\eta_n r})^2} - \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2},$$

which is a stubborn Sturmian type and cannot be made normal QS.

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