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Contrapositive Approval of Non-Physicalism of Cloning and Deletion through the Notion of Incomparability

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Abstract. No Cloning (see [1]) and No Deleting (see [2, 3]) principals are the most fundamental concepts in quantum information theory. Many people have observed these laws in different manners and many new concepts of physics have already been revealed through these concepts. In this paper we try to make a different relation among these no-go principals and the incomparability of states under LOCC (see [4]). We use incomparability under LOCC as a platform and make a bridge between these no-go principals. The interesting fact observed is that these no-go principals are highly correlated to each other.

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1 Introduction

The quantum cloning [5] is one of the most vital devices in the diversified branch of quantum information theory. Through its application in, some fields have prospered in order to cope with the growing demand and trend for their fruitful applicability to diversified fields of knowledge. That is why this is not just an exercise for the encoding of quantum information in quantum systems, it is for the limitation of quantum coping without errors. This feature of quantum information is helpful to perform some information theoretic tasks and opens a new field of discussion. For encoding and carrying of quantum information [6], we always require quantum systems because any physical system obeys the laws of a quantum system. But in the classical world, we have an example which is seen in DNA. In DNA, carrying of quantum information is based on quantum system but the encoding is merely a classical system. The information is carried out by the molecules of DNA, the system of which is obviously quantum but it is encoded in the nature of molecules like adenine, thymine, cytosine, guanine, not in their state. This type of encoding is merely classical one because we cannot find superposition of adenine and thymine there. If the information is encoded in accordance with DNA, its replication can be done perfectly. We call this process cloning which is performed by nature.

Now, we look at the coping of quantum information encoded in a quantum system to another quantum system (see [6]). The process of replication or coping the state, written as $|\psi\rangle \longrightarrow |\psi\rangle \otimes |\psi\rangle$ called cloning can be done perfectly with probability 1, if and only if the basis in which $|\psi\rangle$ belongs is known. Otherwise, perfect cloning is impossible and it will produce either not perfect copies or they are perfect but sometimes the coping process simply gives no outcomes. This is the fundamental concept of NO-CLONING theorem of quantum information [1]. No-cloning theorem is not just a limitation in laboratory physics but it is as fundamental as Heisenberg's uncertainty relation. But how impossibility of coping of quantum information is helpful and provides illustration of its power? It is not possible to copy perfectly a state of quantum system for a proper encoding of information using a set of orthogonal states. Hence, if such a system arrives unperturbed at a receiver, then we will be sure that it has not been copied by any adversary. This gives us the idea of quantum cryptography [7] by deleting any eavesdropper on a communication channel which is impossible for classical information [1, 8]. If $|\psi\rangle$ be the input state, then the exact cloning operation can be described as

$$|\psi\rangle \otimes |\rangle \otimes |R\rangle \longrightarrow |\psi\rangle \otimes |\psi\rangle \otimes |R\rangle, \quad (1)$$

where $|\rangle$ is some suitable chosen blank state and $|R\rangle$ is the state of an auxiliary system. Now why the auxiliary system is necessary for performing quantum cloning? It is a fact that the most general evaluation of a quantum system is a trace preserving a complete positive map. Now, a well known theorem by Kraus says that any such map can be implemented by using an external helping system to the system under study, then after passing through an unitary evaluation, we have to trace out the helping system. Therefore, for two orthogonal states $|0\rangle$ and $|1\rangle$, we have

$$|0\rangle\otimes|\rangle\otimes|R\rangle \longrightarrow |0\rangle\otimes|0\rangle\otimes|R(0)\rangle \quad \text{and} \quad |1\rangle\otimes|\rangle\otimes|R\rangle \longrightarrow |1\rangle\otimes|1\rangle\otimes|R(1)\rangle.$$

Due to linearity, we have

$$(|0\rangle + |1\rangle) \otimes |\rangle \otimes |R\rangle \longrightarrow |0\rangle \otimes |0\rangle \otimes |R(0)\rangle + |1\rangle \otimes |1\rangle \otimes |R(1)\rangle.$$

That is, we obtain,

$$(|0\rangle + |1\rangle) \otimes |\rangle \otimes |R\rangle = |00\rangle|R(0)\rangle + |11\rangle|R(1)\rangle \quad (2)$$

which is impossible since

$$(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)|R(0+1)\rangle = (|00\rangle + |10\rangle + |01\rangle + |11\rangle)|R(0+1)\rangle.$$

Equation (2) may be valid only for an orthogonal basis but cannot hold for any arbitrary states. Hence, we arrive at one of the NO-GO theorems in quantum mechanics that is NO-CLONING theorem.

No-Cloning Theorem

Any arbitrary state can not be cloned perfectly.

Formal Definition of Cloning

Quantum cloning is a completely positive map; physically we call it interaction between two quantum systems A and B with an auxiliary system R, resulting a distribution of quantum information between all the subsystems. If we take $|\psi\rangle_A \otimes |\psi\rangle_B$ as the initial input state, then in the output state the information gathered in the state $|\psi\rangle_A$ will be distributed among A, B and ancillary. Due to the possibility of imperfect cloning, it can be found that there exists an unitary operation

$$|\psi\rangle_A \otimes |\rangle_B \otimes |R\rangle \longrightarrow |\Psi\rangle_{ABR}. \tag{3}$$

. Then the partial trace on the original qubit A and on the cloned qubit B satisfies

$$\rho_A = \rho_B = F|\psi\rangle\langle\psi| + (1-F)|\psi^\perp\rangle\langle\psi^\perp|$$

with an admissible fidelity $F = 5/6$ and is the same for any input state $|\psi\rangle$. Now the generalization of (3), that is, N copies to M copies, the cloning gives

$$U(|\psi\rangle^{\otimes N}) \otimes (|\rangle^{\otimes N-M}) \otimes |R\rangle \longrightarrow |\Psi\rangle, \tag{4}$$

where $|\psi\rangle$ is the input state of Hilbert space $\mathcal{H}$ and $|\rangle$ is some suitable chosen blank state and $|R\rangle$ is the state of an auxiliary system, N is the number of original state $|\psi\rangle$ to be copied.

The quantum cloning machine (QCM) is a trace preserving positive map or equivalently the pair $QCM = \{U, |R\rangle\}$.

Fidelity of QCM

The fidelity of QCM is defined as $F_j = \langle\psi|\rho_j|\psi\rangle$, $j = 1, 2, 3, \cdots, M$, where ρ_j is the partial state clone j in the state $|\Psi\rangle$.

Universal QCM

When the quality of identical copies at the output is independent of input state, that is, F_j is independent of $|\psi\rangle$, the machine is said to be state independent or universal.

Symmetric QCM

When the identical copies have the same fidelity, that is, if $F_j = F_{j'}$ for all $j, j' = 1, 2, 3, ..., M$, the machine is said to be symmetric.

Optical QCM

When the identical copies have the maximal fidelity for given fidelity of input states, then the machine is said to be optimal.

Optimal unitary transformation which implements the universal quantum cloning machine is given by

$$|0\rangle_A|\rangle_B|R\rangle \longrightarrow \sqrt{\frac{2}{3}}|00\rangle_{AB}|r\rangle + \sqrt{\frac{1}{3}}|+\rangle_{AB}|r_\perp\rangle \tag{5}$$

and

$$|1\rangle_A|\rangle_B|R\rangle \longrightarrow \sqrt{\frac{2}{3}}|11\rangle_{AB}|r_\perp\rangle + \sqrt{\frac{1}{3}}|+\rangle_{AB}|r\rangle. \tag{6}$$

Now, the linearity of quantum theory does not allow detection of one unknown quantum state against a copy in either a reversible or an irreversible manner.This is called quantum no-deletion theorem and it is a complementary to the quantum no-cloning theorem. Hence, like no-cloning, it is not possible to delete one copy using any quantum mechanical operation. Let us take two copies of an arbitrary, unknown quantum states as the input states and output blank state along with the original. Then the quantum deletion operation can be described as

$$|\psi\rangle_A|\psi\rangle_B|R\rangle \longrightarrow |\psi\rangle_A|\rangle_B|R'\rangle, \tag{7}$$

where $|\psi\rangle_A$ is the unknown quantum state, $|R\rangle$ is the initial state of deletion machine, $|\rangle_B$ is the blank state and $|R'\rangle$ is the final state of the machine. But by the linearity of quantum theory there is no any quantum operation that can perform the deletion operation for any arbitrary state $|\psi\rangle$. To present our work we need to define the condition for a pair of states to be incomparable with each other.

The notion of incomparability of a pair of bipartite pure states is a consequence of Nielsen's [9] majorization criterion. Suppose we want to convert the pure bipartite state $|\Psi\rangle$ to $|\Phi\rangle$ shared between two parties, say, Alice and Bob by deterministic LOCC. Consider the pair $(|\Psi\rangle, |\Phi\rangle)$ in their Schmidt bases $\{|i_A\rangle, |i_B\rangle\}$ with decreasing order of Schmidt coefficients

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\alpha_i}|i_A i_B\rangle, \qquad |\Phi\rangle = \sum_{i=1}^{d} \sqrt{\beta_i}|i_A i_B\rangle,$$

where $\alpha_i \geq \alpha_{i+1} \geq 0$ and $\beta_i \geq \beta_{i+1} \geq 0$ for $i = 1, 2, \ldots, d-1$, and $\sum_{i=1}^{d} \alpha_i = 1 = \sum_{i=1}^{d} \beta_i$. The Schmidt vectors corresponding to the states $|\Psi\rangle$

and $|\Phi\rangle$ are $\lambda_\Psi \equiv (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and $\lambda_\Phi \equiv (\beta_1, \beta_2, \ldots, \beta_d)$. Then Nielsen's criterion says $|\Psi\rangle \to |\Phi\rangle$ is possible with certainty under LOCC, if and only if λ_Ψ is majorized by λ_Φ, denoted by $\lambda_\Psi \prec \lambda_\Phi$ and described as

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i \quad \forall\ k = 1, 2, \ldots, d. \tag{8}$$

It is interesting to note that as a consequence of non-increase of entanglement by LOCC, if $|\Psi\rangle \to |\Phi\rangle$ is possible under LOCC with certainty, then $E(|\Psi\rangle) \geq E(|\Phi\rangle)$ [where $E(\cdot)$ denotes the von-Neumann entropy of the reduced density operator of any subsystem and known as the entropy of entanglement]. Now, in case of failure of the above criterion (1), it is usually denoted by $|\Psi\rangle \not\to |\Phi\rangle$. Also it may happen that $|\Phi\rangle \to |\Psi\rangle$ under LOCC. And if it happens that both $|\Psi\rangle \not\to |\Phi\rangle$ and $|\Phi\rangle \not\to |\Psi\rangle$, then we denote it as $|\Psi\rangle \not\leftrightarrow |\Phi\rangle$ and describe $(|\Psi\rangle, |\Phi\rangle)$ as a pair of incomparable states. One of the peculiar features of such incomparable pairs is that we are unable to say which state has a greater amount of entanglement content. For 2×2 systems there are no pair of incomparable pure entangled states as described above. Now we want to mention explicitly the criterion of incomparability for a pair of pure entangled states $|\Psi\rangle, |\Phi\rangle$ of $m \times n$ system, where $\min\{m, n\} = 3$. Suppose the Schmidt vectors corresponding to the two states are (a_1, a_2, a_3) and (b_1, b_2, b_3), respectively, where $a_1 > a_2 > a_3 > 0$, $b_1 > b_2 > b_3 > 0$, $a_1 + a_2 + a_3 = 1 = b_1 + b_2 + b_3$. Then it follows from Nielsen's criterion that $|\Psi\rangle$ and $|\Phi\rangle$ are incomparable if and only if, either of the pair of relations

$$\begin{array}{lcl} a_1 > b_1 & \text{and} & a_3 > b_3 \\ b_1 > a_1 & \text{and} & b_3 > a_3 \end{array} \tag{9}$$

will hold.

The most powerful usefulness of incomparability is that if a pair of states is incomparable, then we cannot compare the amount of entanglement of the pair. Now for our current discussion, we consider the following states:

$$|\psi\rangle_{AB} = N_1(\alpha_1|00\rangle + \alpha_2|11\rangle + \alpha_3|20\rangle) \tag{10}$$

and

$$|\phi\rangle_{AB} = N_2(\beta_1|00\rangle + \beta_2|11\rangle + \beta_3|21\rangle). \tag{11}$$

The eigenvalues corresponding to the states are $\alpha_1^2/(\alpha_1+\alpha_2+\alpha_3)$ and all others are zeros; $\beta_1^2/(\beta_1 + \beta_2 + \beta_3)$ and all others are zeros, respectively. So, the Schmidt vectors of $|\psi\rangle_{AB}$ are $\alpha_1/\sqrt{\alpha_1 + \alpha_2 + \alpha_3}$ and all others are zeros and those of $|\phi\rangle_{AB}$ are $\beta_1/\sqrt{\beta_1 + \beta_2 + \beta_3}$ and all others are zeros. After performing universal optimal cloning operation, we obtain the output states as

$$|\chi_\alpha^{OUT}\rangle = \frac{1}{\sqrt{\alpha_1+\alpha_2+\alpha_3}}\Big[\Big(\alpha_1\frac{2}{3}|0\rangle|00\rangle + \alpha_2\frac{1}{3}|1\rangle|+\rangle + \alpha_3\frac{2}{3}|2\rangle|00\rangle\Big)|r\rangle + \Big(\alpha_1\frac{1}{3}|0\rangle|+\rangle + \alpha_2\frac{2}{3}|1\rangle|11\rangle + \alpha_3\frac{1}{3}|2\rangle|+\rangle\Big)|r\perp\rangle\Big].$$

Taking trace over the side of machine state $|r\rangle$ and B's side, we can obtain the density matrices $\rho_{AB}^{|\psi\rangle}$ and $\rho_{AB}^{|\phi\rangle}$ as

$$\rho_{AB}^{|\psi\rangle} = \frac{1}{\alpha_1+\alpha_2+\alpha_3}\begin{pmatrix} \frac{5\alpha_1^2}{6} & 0 & 0 & \frac{2\alpha_1\alpha_2}{3} & \frac{5\alpha_3\alpha_1}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha_1^2}{6} & 0 & 0 & 0 & \frac{1\alpha_3\alpha_1}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_2^2}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2\alpha_1\alpha_2}{3} & 0 & 0 & \frac{5\alpha_2^2}{6} & \frac{2\alpha_3\alpha_2}{3} & 0 & 0 & 0 & 0 \\ \frac{5\alpha_3\alpha_1}{6} & 0 & 0 & \frac{2\alpha_3\alpha_2}{3} & \frac{5\alpha_3^2}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{1\alpha_3\alpha_1}{6} & 0 & 0 & 0 & \frac{\alpha_3^2}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_{AB}^{|\phi\rangle} = \frac{1}{\beta_1+\beta_2+\beta_3}\begin{pmatrix} \frac{5\beta_1^2}{6} & 0 & 0 & \frac{2\beta_1\beta_2}{3} & \frac{5\beta_3\beta_1}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_1^2}{6} & 0 & 0 & 0 & \frac{1\beta_3\beta_1}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_2^2}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2\beta_1\beta_2}{3} & 0 & 0 & \frac{5\beta_2^2}{6} & \frac{2\beta_3\beta_2}{3} & 0 & 0 & 0 & 0 \\ \frac{5\beta_3\beta_1}{6} & 0 & 0 & \frac{2\beta_3\beta_2}{3} & \frac{5\beta_3^2}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{1\beta_3\beta_1}{6} & 0 & 0 & 0 & \frac{\beta_3^2}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues of $\rho_{AB}^{|\phi\rangle}$ are

$$\frac{5\alpha_1^2}{6(\alpha_1+\alpha_2+\alpha_3)},\ \frac{\alpha_1^2}{6(\alpha_1+\alpha_2+\alpha_3)},\ \frac{\alpha_2^2}{6(\alpha_1+\alpha_2+\alpha_3)},\ \frac{3\alpha_2^2}{10(\alpha_1+\alpha_2+\alpha_3)},$$

and all others are zeros. So, the Schmidt vectors corresponding to the state $|\psi\rangle_{AB}^{\text{Clone}}$ are

$$\frac{\sqrt{5}\alpha_1}{\sqrt{6}\sqrt{(\alpha_1+\alpha_2+\alpha_3)}}, \quad \frac{\sqrt{1}\alpha_1}{\sqrt{6}\sqrt{(\alpha_1+\alpha_2+\alpha_3)}},$$
$$\frac{\sqrt{1}\alpha_2}{\sqrt{6}\sqrt{(\alpha_1+\alpha_2+\alpha_3)}}, \quad \frac{\sqrt{3}\alpha_2}{\sqrt{10}\sqrt{(\alpha_1+\alpha_2+\alpha_3)}},$$

and all others are zeros.

Similarly, the eigenvalues of $\rho_{AB}^{|\phi\rangle}$ are

$$\frac{5\beta_1^2}{6(\beta_1+\beta_2+\beta_3)}, \frac{1\beta_1^2}{6(\beta_1+\beta_2+\beta_3)}, \frac{1\beta_2^2}{6(\beta_1+\beta_2+\beta_3)}, \frac{3\beta_1^2}{10(\beta_1+\beta_2+\beta_3)},$$

and all others are zeros. So, the Schmidt vectors corresponding to the state $|\phi\rangle_{AB}^{\text{Clone}}$ are

$$\frac{\sqrt{5}\beta_1}{\sqrt{6}\sqrt{(\beta_1+\beta_2+\beta_3)}}, \quad \frac{\sqrt{1}\beta_1}{\sqrt{6}\sqrt{(\beta_1+\beta_2+\beta_3)}},$$
$$\frac{\sqrt{1}\beta_2}{\sqrt{6}\sqrt{(\beta_1+\beta_2+\beta_3)}}, \quad \frac{\sqrt{3}\beta_2}{\sqrt{10}\sqrt{(\beta_1+\beta_2+\beta_3)}},$$

and all others are zeros.

To reach our goal, we first assume for the initial states $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$ that $|\psi\rangle_{AB} \longrightarrow |\phi\rangle_{AB}$ which demands that

$$\sum_{i=1}^{3}\alpha_i \leq \sum_{i=1}^{3}\beta_i \Longrightarrow \alpha_i \leq \beta_i.$$

We consider the case here $\alpha_1 < \beta_1$ and $\alpha_2 \neq \beta_2$. So $|\psi\rangle_{AB} \longrightarrow |\phi\rangle_{AB}$ holds good. We also let $\alpha_2 > \beta_2$. Then for the cloned states $|\psi\rangle_{AB}^{\text{Clone}}$ and $|\phi\rangle_{AB}^{\text{Clone}}$, we have the following relation

$$\frac{\sqrt{5}\alpha_1}{\sqrt{6}\sqrt{(\alpha_1+\alpha_2+\alpha_3)}} + \frac{\sqrt{1}\alpha_1}{\sqrt{6}\sqrt{(\alpha_1+\alpha_2+\alpha_3)}} + \frac{\sqrt{1}\alpha_2}{\sqrt{6}\sqrt{(\alpha_1+\alpha_2+\alpha_3)}} + \frac{\sqrt{3}\alpha_2}{\sqrt{10}\sqrt{(\alpha_1+\alpha_2+\alpha_3)}} \not\leq \frac{5\beta_1^2}{6(\beta_1+\beta_2+\beta_3)} + \frac{1\beta_1^2}{6(\beta_1+\beta_2+\beta_3)} + \frac{1\beta_2^2}{6(\beta_1+\beta_2+\beta_3)} + \frac{3\beta_1^2}{10(\beta_1+\beta_2+\beta_3)}.$$

Now, we have $|\psi\rangle_{AB}^{\text{Clone}} \not\rightarrow |\phi\rangle_{AB}^{\text{Clone}}$.

Here we see that the cloning operation does not preserve the status of the relationship of the states which they have had before the operation. It is known and proved in many ways that the cloning is not a physical operation [10], that is the quantum cloning under LOCC is quite impossible. A physical operation must preserve the basic status of the state, so the cloning to be a physical operation as before cloning, the states which are not incomparable, must have to be not incomparable after the quantum cloning under LOCC. But the actual scenario is not such. Though the states before quantum cloning are not incomparable, they become incomparable after quantum cloning under LOCC. This phenomenon strongly approves quantum cloning as a non physical operation under LOCC.

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