Bulg. J. Phys. **38** (2011) 43–71

*This paper is dedicated to the memory of* ***Matey Mateev***

# Strings, Deformed Backgrounds and String/Gauge Theory Correspondence

**R.C. Rashkov**[1,2]

[1]Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstr. 8-10, 1040 Vienna, Austria
[2]Department of Physics, University of Sofia, 1164 Sofia, Bulgaria*

Received *20 March 2011*

**Abstract.** In this article we review the deformations of the string background and its implications to the string gauge theory duality. We present the solution generating technique suggested by Lunin and Maldacena as well as some generalizations. The presentation is accompanied with examples and solutions of problems related to string/gauge theory duality.

## 1 Introduction

In the recent years the AdS/CFT correspondence [1] has become the major analytical tool for studying gauge theories at strong coupling. Thus it is of great interest to extend this duality to less supersymmetric theories which are phenomenologically more adequate. Recently Lunin and Maldacena [2] have found the supergravity dual of the so called $\beta$-deformed $\mathcal{N} = 4$ SYM [5] by performing an $SL(3, R)$ transformation on the well known $AdS_5 \times S^5$ background. This provides new possibilities for quantitative checks of the AdS/CFT correspondence. When the deformation parameter $\beta$ is real (the real part is denoted by $\gamma$, thus called $\gamma$-deformations) we have a continuous parameter which controls the deformation.

The technique advocated in [2] showed explicitly how to generate new solutions and also provided the interpretation of this new solutions in the dual field theory. To summarize their approach, the authors have used the global symmetries and chain of dualities to construct new solutions. More precisely, the construction consists of the following steps. The first is to construct a torus with two $U(1)$s being symmetries of the old solution of the background. In the second step one

*Permanent address.

 

combines these $U(1)$s with the $SL(2,R)$ symmetry of IIB supergravity producing as a net result an $SL(3,R)$ symmetry. Since the resulting symmetry acts as a duality transformation mapping solutions to solutions, one can apply $SL(3,R)$ on the known solutions to generate new ones. As mentioned in [2] that the above procedure can break all supersymmetry if at least one of the $U(1)$ coincides with the $U(1)_R$, *i.e.* if one constructs a torus by taking one of directions along the $R$-symmetry direction. In this case the new solution is a non-supersymmetric solution. This procedure is the analogue of Leigh–Strassler [3] deformations on the gravity side [2] of string/gauge theory duality. The marginal deformation of $N = 4$ field theories has been studied in [29]. On the field theory side of string/gauge theory duality there is also a $SL(3,R)$ transformations whose action corresponds to multiplying fields in a different way [2]. More explicitly, if $\Phi_i$ and $\Phi_j$ are two chiral super fields with $U(1)$ charges denoted as $Q_i$ and $Q_j$, then

$$\Phi_i.\Phi_j \rightarrow \Phi_i \star \Phi_j. \tag{1}$$

This way of deforming the standard product of the fields is almost the same as is done in the non-commutative field theories [4] derived from open strings. It is almost because here the $B$ field is not necessarily a constant. The two $U(1)$s are associated with the two sides of a torus and the role of modular and Kähler parameters of the torus is played by the component of the $B$ field along the torus and the volume of the torus. Consider a geometry which asymptotes to AdS space-time times a compact Sasaki–Einstein manifold. If the torus constructed on above stay inside the AdS space, then in the field theory it corresponds to a non-commutative field theory with momentum playing the role of the charges under the U(1)'s. Whereas when the torus stay completely inside the Sasaki–Einstein manifold, then the corresponding fields in the field theory are multiplied by the star product (1). If the torus stay both in the AdS and in the Sasaki–Einstein space by sharing one of its direction then the corresponding field theory is called a dipole deformation of the field theory [16,18]. There are some recent advancement on the study of Lunin-Maldacena background, see for instance [19–29].

In fact, the deformations can be seen in a unified framework as emerging from deformations of ordinary Yang-Mills theory by higher-dimensional (but not necessarily irrelevant) gauge-invariant operators. Gauge theories with deformed products of fields and their dual string theories, obtained with or without the knowledge of the $SL(2,R)$ transformation, have been thoroughly studied in the literature. The purpose of this article is to present introductory review of the TsT transformation of any type II background, including the transformation of the string worldsheet coordinate fields and of the corresponding open string boundary conditions, as well as to provide some new results on the subject. We will make a try to give some general perspective on how D-brane probes behave under the transformation, which is particularly relevant as D-brane probes often provide an invaluable bridge between the gauge and string sides of the string/gauge theories correspondence.

The solution generating technique can be applied to wide range of cases. The most well-known example is probably the one of exactly marginal deformations of (super)conformal gauge theories, such as the deformations of $\mathcal{N} = 4$ Super Yang-Mills theory considered in [3]. Its gravity dual (for the so-called $\beta$-deformation) was derived in [2]. Another example, as we have said in the beginning, is provided by gravity duals of non-commutative gauge theories, such as the ones in [7, 8]. Another often studied case is the one of "dipole" theories [16]– [18], which are non-local theories living in an ordinary commutative space. Besides being interesting by themselves, dipole deformations have been shown to be useful also in the context of ordinary confining $\mathcal{N} = 1$ gauge theories realized on D-branes wrapped on supersymmetric cycles of Calabi-Yau manifolds, where the deformation helps with disentangling gauge theory effect from spurious effects due to the Kaluza-Klein modes on the cycle [21]– [22].

This paper is organized as follows. We start in the next Section by reviewing the scheme of solution generating technique discussed above. After the short review of the dualities used later, we present the solutions generating techniques and its application on gauge theory and string theory sides. We discuss the introduction of the deformed star product in gauge theory Lagrangians and its consequences on the string theory side of the duality. In Section 3 we present general formulae for the so called TsT transformation of type II closed string backgrounds, world-sheet coordinate fields and the corresponding open string boundary conditions, then we proceed in Section 4 by systematically studying the applications to the string/gauge theory duality.

## 2 Solution Generating Technique: General Remarks and Examples

### 2.1 Dualities and Solution Generating Symmetries

Perhaps the best way to study the solution generating transformations of [2] is to look at the U-duality group of the eight dimensional Type II string theory: $SL(2,\mathbb{R}) \times SL(3,\mathbb{R})$. Obviously, we are not being rigorous by calling this solution generating group the U-duality group, as in fact the U-duality group is the discrete $SL(2,Z) \times SL(3,Z)$ [9]. However, for ease of language, we will refer to the above group as the U-duality group just as we will, throughout the text, refer to the solution generating $O(d,d,\mathbb{R})$ as the T-duality group. Let us first examine the subgroup $SO(2,2,\mathbb{R}) \simeq SL(2,\mathbb{R})_\tau \times SL(2,\mathbb{R})_\rho$, which is the "trivial" part of the T-duality group $O(2,2,\mathbb{R})$ acting within IIA or IIB [1]. Here the subindices $\tau$ and $\rho$ refer to the fact that the first $SL(2,\mathbb{R})$ acts on the complex structure modulus $\tau$ of the internal 2-torus, whereas the second one acts on its Kähler modulus $\rho$. $SL(2,\mathbb{R})_\tau$ is the geometric part of the T-

[1] Obviously, $O(2,2,Z)$ also has matrices with determinant -1, including the transformation which exchanges the Kähler and the complex structure moduli. Sometimes only this transformation is referred to as being the T-duality transformation (as in [5]), and sometimes it is referred to as the non-trivial part of the T-duality group. For an extensive review of T-duality, see [10].

duality associated with invariance under large diffeomorphisms of the two torus on which the ten dimensional type IIB string theory has been compactified. Its action on the complex structure modulus $\tau = \tau_1 + i\tau_2$ and the background matrix $E = g + B$ is

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad E \longrightarrow AEA^t. \tag{2}$$

Here

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an $SL(2,\mathbb{R})$ matrix embedded in $O(2,2,\mathbb{R})$ as

$$\begin{pmatrix} A & 0_2 \\ 0_2 & (A^t)^{-1} \end{pmatrix}, \tag{3}$$

so that the background matrix $E$ transforms as in (2).

On the other hand, the $SL(2,\mathbb{R})_\rho$ acts on the Kähler modulus $\rho = \rho_1 + i\rho_2 = B_{12} + i\sqrt{g}$ as

$$\rho \longrightarrow \frac{a\rho + b}{c\rho + d} \tag{4}$$

with $ad - bc = 1$. It is important to know how $SL(2,\mathbb{R})_\rho$ is embedded into $O(2,2,\mathbb{R})$.

One can embed the $SL(2,\mathbb{R})$ that acts on the Kähler modulus into the T–duality group $O(2,2,\mathbb{R})$ and thus consider the action of the latter on the background matrix $E = g + B$. This provides a considerably simpler way of obtaining the new solutions. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ the generic element of $SL(2,\mathbb{R})$ the appropriate embedding is the following:

$$\mathcal{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -c & d & 0 \\ c & 0 & 0 & d \end{pmatrix}. \tag{5}$$

It is then easy to see [10] that $\mathcal{T}$ transforms the original background matrix $E_0$ as

$$E_0 \to E = (\mathbf{A}E_0 + \mathbf{B})(\mathbf{C}E_0 + \mathbf{D})^{-1} \equiv \frac{\mathbf{A}E_0 + \mathbf{B}}{\mathbf{C}E_0 + \mathbf{D}}, \tag{6}$$

where the $2 \times 2$ matrices A,B,C,D are defined through (5). Then $\rho_1 = B_{12}$ and $\rho_2 = \sqrt{g}$ transform as

$$\rho_1 \longrightarrow \frac{ac(\rho_1^2 + \rho_2^2) + (ad + bc)\rho_1 + bd}{c^2(\rho_1^2 + \rho_2^2) + 2dc\rho_1 + d^2}$$
$$\rho_2 \longrightarrow \frac{\rho_2}{c^2(\rho_1^2 + \rho_2^2) + 2dc\rho_1 + d^2}$$

which is equivalent to (4).

To make connection with the solution generating technique of [2], we should not consider an arbitrary $SL(2,\mathbb{R})$ element but the precise one with $a = d = 1$, $b = 0$ and $c = \gamma$. In this case (5) reads

$$\mathcal{T} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{\Gamma} & \mathbf{1} \end{pmatrix} \quad \text{with} \quad \mathbf{\Gamma} = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}, \tag{7}$$

where $\mathbf{1}$ and $\mathbf{0}$ represent the $2 \times 2$ identity and zero matrices respectively. Following now the T–duality rules in [10], we can write the NS–NS fields of the new solution in terms of $E_0$ and $\Gamma$ as follows:

$$E = \frac{1}{E_0^{-1} + \mathbf{\Gamma}}\,, \qquad e^{2\Phi} = \det(1 + E_0\mathbf{\Gamma})e^{2\Phi_0}\,. \tag{8}$$

The RR-fields of the background can be obtained in a similar fashion using the transformation rules of [11–15].

Yet another $SL(2,\mathbb{R})$ symmetry of the IIB string theory is the S-duality, which already exists in ten dimensions and whose $Z_2$ part acts on the string coupling as a strong-weak coupling duality. This group, which we will denote by $SL(2,\mathbb{R})_s$ combines with the non-geometric $SL(2,\mathbb{R})_\rho$ to form the $SL(3,\mathbb{R})$ part of the U-duality group. From the point of view of type IIA string theory, this $SL(3,\mathbb{R})$ is the geometric part of the U-duality group, associated with the large diffeomorphisms of the three torus on which the eleven dimensional M-theory has been compactified. On the other hand, the geometric $SL(2,\mathbb{R})_\tau$ of IIB is mapped under T-duality to a non-geometric $SL(2,\mathbb{R})$ which acts on the Kähler modulus of Type IIA.

## 2.2 TsT Transformations

An important step towards a deeper understanding of AdS/CFT correspondence in its less supersymmetric sector has been recently given by Lunin and Maldacena [2]. From gauge theory point of view the possible deformations of $\mathcal{N} = 4$ SYM gauge theory that break the supersymmetry have been studied by Leigh and Strassler [3]. In [2] Lunin and Maldacena have found the gravity duals to the $\beta$-deformations of $\mathcal{N} = 4$ SYM theory studied in [3]. They have demonstrated that a certain deformation of the most supersymmetric string background corresponds to a gauge theory with less supersymmetry classified in [3]. This deformation of the string background can be obtained applying two T-dualities accompanied by certain shift parametrized by $\beta$ (TsT transformation). Next, for real values of $\beta$, Frolov has obtained the Lax operator for the deformed background which proves the integrability at classical level [5].

### Field theory side

In this paragraph we give very briefly an idea of the procedure of the $\beta$-deformation of SYM theory considered in [3].

Let us consider the $\mathcal{N} = 4$ SYM gauge theory in terms of $\mathcal{N} = 1$ SUSY. The theory contains a vector multiplet $V$ and three chiral multiplets $\Phi^i$. The superpotential is given by the expression

$$W = g' Tr \left[ [\Phi^1, \Phi^2] \Phi^3 \right]. \tag{9}$$

The action then can be written as

$$\begin{aligned} S = & \mathrm{Tr} \left\{ \int d^4 x d^4 \theta e^{-gV} \bar{\Phi}_i e^{gV} \Phi^i + \frac{1}{2g^2} \left[ \int d^4 x d^2 \theta W^\alpha W_\alpha + c.c. \right] \right. \\ & \left. \frac{g'}{3!} \left[ \int d^4 d^2 \theta \varepsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + c.c. \right] \right\}. \end{aligned} \tag{10}$$

We note that the $\mathcal{N} = 4$ theory is conformal at any value of the complex coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \tag{11}$$

and the deformations that change this value are exactly marginal.

In [3] Leigh and Strassler have considered deformations of the superpotential of the form

$$\begin{aligned} W &= h \, \mathrm{Tr} \left[ e^{i\pi\beta} \Phi_1 \Phi_2 \Phi_3 - e^{-i\pi\beta} \Phi_1 \Phi_3 \Phi_2 \right] + h' \, \mathrm{Tr} \left[ \Phi_1^3 + \Phi_2^3 + \Phi_3^3 \right] \\ &\equiv h \, \mathrm{Tr} \left( \Phi_1 [\Phi_2, \Phi_3]_\beta \right) + h' \, \mathrm{Tr} \left[ \Phi_1^3 + \Phi_2^3 + \Phi_3^3 \right], \end{aligned} \tag{12}$$

where the $\beta$-deformed commutator is defined as

$$[\Phi_2, \Phi_3]_\beta = e^{i\pi\beta} \Phi_2 \Phi_3 - e^{-i\pi\beta} \Phi_3 \Phi_2.$$

Let us focus on $h' = 0$ case. The symmetries are: one $U(1)$ R-symmetry group and two global $U(1) \times U(1)$ groups acting as follows:

$$\begin{aligned} U(1)_1 : &\quad (\Phi_1, \Phi_2, \Phi_3) \to (\Phi_1, e^{i\varphi_1} \Phi_2, e^{-i\varphi_1} \Phi_3) \\ U(1)_2 : &\quad (\Phi_1, \Phi_2, \Phi_3) \to (e^{-i\varphi_2} \Phi_1, e^{i\varphi_2} \Phi_2, \Phi_3). \end{aligned} \tag{13}$$

Since the theory is periodic in $\beta$ one can think of $\beta$ as living on a torus with complex structure $\tau_s$ and the $Sl(2, Z)$ duality group acts on it and $\beta$ as follows:

$$\begin{aligned} \tau_s \to \frac{a\tau_s + b}{c\tau_s + d}; &\quad \beta \to \frac{\beta}{c\tau_s + d}; \\ \beta \sim \beta + 1 &\sim \beta + \tau_s. \end{aligned} \tag{14}$$

As a result of all this we end up with a $\mathcal{N} = 1$ SCFT theory.

The classical vacuum structure is determined by the F- and D-flatness conditions

$$[\Phi_1, \Phi_2]_\beta = [\Phi_2, \Phi_3]_\beta = [\Phi_3, \Phi_1]_\beta = 0, \tag{15}$$

and

$$\sum_{i=1}^{3}[\Phi_i, \Phi_i^\dagger] = 0, \tag{16}$$

respectively. When $\beta = 0$, the above conditions are easily solved by diagonalizing each of the three complex scalars,

$$\langle \Phi_i \rangle = \mathrm{diag}(x_1^{(i)} x_2^{(i)}, \cdots, x_N^{(i)}).$$

Note that the $3N^2$ comlex eigenvalues $x_a^{(i)}$ are unconstrained and after taking into account the Weyl group which permutes the eigenvalues, we recover the familiar Coulomb branch of the $\mathcal{N} = 4$ theory.

For generic $\beta$ the F-flatness conditions are no longer solved by arbitrary diagonal matrices (15). For each value of the Cartan index $a \in 1, 2, \cdots, N$, at most one of the three eigenvalues, $x_a^{(1)}, x_a^{(2)}$ and $x_a^{(3)}$ , for each $a$, can be non-zero.

The Lagrangian corresponding to the superpotential (12) has a $U(1)^3 \simeq U(1)_R^{(1)} \times U(1)_R^{(2)} \times U(1)_R^{(3)}$ R-symmetry, where each complex scalar field $\Phi_i$ is charged under $U(1)_R^{(i)}$ and neutral under the other factors. At a generic point on the Coulomb branch the complete R-symmetry group is spontaneously broken and the gauge group is spontaneously broken to $U(1)^N$. For further discussion we refer to the original literature.

## String theory side

The gravity dual for real $\beta$ can be obtained in three steps [5]. Consider a background with a subspace possessing $U^2(1)$ symmetry. For instance, one can think of it as a torus compactification of the initial background. In the first step we perform a T-duality with respect to one of the $U(1)$ isometries parametrized by the angle $\varphi_1$. The second step consists in performing a shift $\varphi_2 \to \varphi_2 + \gamma\varphi_1$, where $\varphi_2$ parametrizes another $U(1)$ isometry of the background and $\gamma$ is a real parameter. In the last step we T-dualize back on $\varphi_1$. The resulting geometry is described by

$$ds^2_{str} = R^2 \left[ ds^2_{AdS_5} + \sum (dr_i^2 + G r_i^2 d\phi_i^2) + \tilde{\gamma}^2 r_1^2 r_2^2 r_3^2 (\sum d\phi_i)^2 \right], \tag{17}$$

where

$$G^{-1} = 1 + \gamma^2 (r_1^2 r_2^2 + r_2^2 r_3^2 + r_1^2 r_3^2);\ \tilde{\gamma} = R^2 \gamma. \tag{18}$$

The other fields are correspondingly (See for details [2, 5].)

$$e^{2\phi} = e^{2\phi_0} G \tag{19}$$

$$B^{NS} = \tilde{\gamma}^2 R^2 G (r_1^2 r_2^2 d\phi_1 d\phi_2 + r_2^2 r_3^2 d\phi_2 d\phi_3 + r_3^2 r_1^2 d\phi_3 d\phi_1) \tag{20}$$

$$C_2 = -3\gamma (16\pi N) w_1 d\psi \tag{21}$$

$$C_4 = (16\pi N) w_4 + G w_1 d\phi_1 d\phi 2 d\phi_3) \tag{22}$$

$$F_5 = (16\pi N)(w_{AdS_5} + G w_{S^5}). \tag{23}$$

Using the fact that the currents $J_\alpha$ before the TsT-transformations are equal to the currents $\tilde{J}_\alpha$ after the transformations, one can obtain the Lax operator for the deformed geometry, thus proving the integrability at classical level.

## $U(1)$ Currents and TsT Transformation

As mentioned in the previous Section, based on the observation that the string $U(1)$ currents before and after TsT-transformation are equal, one can obtain the Lax operator of the theory in the deformed background. Moreover, one can also conjecture that the equality of the currents holds for any two backgrounds related by TsT-transformation. Below we prove the following:

**Proposition:** The $U(1)$ currents of strings in any two backgrounds related by TsT transformation are equal.

We start with the general action

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}\right]. \tag{24}$$

We will assume that $G_{\mu\nu}$ and $B_{\mu\nu}$ do not depend on $X^1$ and $X^2$ allowing to perform TsT transformation.

In what follows we use the notations $\mu = 1, \cdots, d$, $i = 2, \cdots, d$, $a = 3, \cdots, d$. We will prove the statement in several steps.

Step 1: T-duality on $X^1$.

For completeness we write again the T-duality rules and relations

$$\begin{aligned}
\tilde{G}_{11} = \frac{1}{G_{11}}, \quad \tilde{G}_{ij} = G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}}, \quad \tilde{G}_{1i} = \frac{B_{1i}}{G_{11}}, \\
\tilde{B}_{ij} = B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}, \quad \tilde{B}_{1i} = \frac{G_{1i}}{G_{11}}, \qquad (25)\\
\epsilon^{\alpha\beta}\partial_\beta \tilde{X}^1 = \gamma^{\alpha\beta}\partial_\beta X^M G_{1M} - \epsilon^{\alpha\beta}\partial_\beta X^M B_{1M}, \qquad (26)\\
\partial_\alpha \tilde{X}^1 = \gamma_{\alpha\sigma}\epsilon^{\sigma\rho}\partial_\rho X^\mu G_{1\mu} - \partial_\alpha X^\mu B_{1\mu}, \qquad (27)\\
\partial_\alpha X^1 = \gamma_{\alpha\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{X}^\mu \tilde{G}_{1\mu} - \partial_\alpha \tilde{X}^\mu \tilde{B}_{1\mu}, \qquad (28)\\
\tilde{X}^i = X^i. \qquad (29)
\end{aligned}$$

The T-dual action has the same form but with transformed background fields

$$\tilde{S} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \tilde{G}_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \tilde{B}_{\mu\nu}\right]. \tag{30}$$

Step 2 consists in shift of $\tilde{X}^2$

$$\begin{aligned} &\tilde{X}^2 = \tilde{x}^2 + \hat{\gamma}\tilde{x}^1, \\ &\tilde{X}^1 = \tilde{x}^1, \qquad \tilde{X}^a = \tilde{x}^a. \end{aligned} \tag{31}$$

Note that the background remains independent of $\tilde{X}^1$ and $\tilde{X}^2$.

The shift described above produces the following transformations of the metric:

$$\begin{aligned} &\tilde{g}_{11} = \tilde{G}_{11} + 2\hat{\gamma}\tilde{G}_{12} + \hat{\gamma}^2\tilde{G}_{22}, \\ &\tilde{g}_{1i} = \tilde{G}_{1i} + \hat{\gamma}\tilde{G}_{2i}, \\ &\tilde{g}_{ij} = \tilde{G}_{ij}, \end{aligned} \tag{32}$$

and for the $\tilde{B}_{\mu\nu}$ we get

$$\begin{aligned} &\tilde{b}_{ij} = \tilde{B}_{ij}, \\ &\tilde{b}_{1i} \to \tilde{B}_{1i} + \hat{\gamma}\tilde{B}_{2i}. \end{aligned} \tag{33}$$

The relations (26-28) are also changed, for instance (28) becomes

$$\partial_\alpha X^1 = \gamma_{\alpha\sigma}\epsilon^{\sigma\rho}\partial_\rho\tilde{x}^\mu\tilde{G}_{1\mu} - \partial_\alpha\tilde{x}^\mu\tilde{B}_{1\mu} + \hat{\gamma}\gamma_{\alpha\sigma}\epsilon^{\sigma\rho}\partial_\rho\tilde{x}^1\tilde{G}_{12} - \hat{\gamma}\partial_\alpha\tilde{x}^1\tilde{B}_{12}. \tag{34}$$

Note that it is crucial that the background is independent of $X^1$ and $X^2$, otherwise we cannot perform a T-duality back on $\tilde{x}_1$.

In the new variables the action is given by

$$\tilde{\bar{S}} = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\partial_\alpha\tilde{x}^\mu\partial_\beta\tilde{x}^\nu\tilde{g}_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha\tilde{x}^\mu\partial_\beta\tilde{x}^\nu\tilde{b}_{\mu\nu}\right]. \tag{35}$$

In step 3 we T-dualize back on $\tilde{x}^1$.

The action again has the standard form

$$\tilde{\bar{\bar{S}}} = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}\left[\gamma^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu g_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu b_{\mu\nu}\right], \tag{36}$$

where $g_{\mu\nu}$ and $b_{\mu\nu}$ are obtained from $\tilde{g}_{\mu\nu}$ and $\tilde{b}_{\mu\nu}$ by making use of the standard rules Eqs. (25-28).

Now we will prove that the currents $J^\alpha_\mu$ and $j^\alpha_\mu$ obtained from (24) and (36) respectively are equal, *i.e.*,

$$J^\alpha_\mu = j^\alpha_\mu, \tag{37}$$

where

$$j^\alpha_\mu = -\sqrt{\lambda}\gamma^{\alpha\beta}\partial_\beta x^\nu g_{\mu\nu} + \sqrt{\lambda}\epsilon^{\alpha\beta}\partial_\beta x^\nu b_{\mu\nu}, \tag{38}$$

$$J^\alpha_\mu = -\sqrt{\lambda}\gamma^{\alpha\beta}\partial_\beta x^\nu G_{\mu\nu} + \sqrt{\lambda}\epsilon^{\alpha\beta}\partial_\beta x^\nu B_{\mu\nu}. \tag{39}$$

We will prove the statement directly, but in two steps.

a) First we will prove the equality (37) for $J_1^\alpha$ and $j_1^\alpha$ and then for $J_i^\alpha$ and $j_i^\alpha$

$$
\begin{aligned}
\frac{j_1^\alpha}{-\sqrt{\lambda}} &= \gamma^{\alpha\beta}\partial_\beta x^1 g_{11} + \gamma^{\alpha\beta}\partial_\beta x^i g_{1i} - \epsilon^{\alpha\beta}\partial_\beta x^i b_{1i} \\
&= \gamma^{\alpha\beta}\partial_\beta x^1 g_{11} + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^i g_{1i} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^i b_{1i} \\
&= \frac{\gamma^{\alpha\beta}}{\tilde{g}_{11}}\left(\gamma_{\beta\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{x}^\mu \tilde{g}_{1\mu} - \partial_\beta \tilde{x}^\mu \tilde{b}_{1\mu}\right) + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^i \frac{\tilde{b}_{1i}}{\tilde{g}_{11}} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^i \frac{\tilde{g}_{1i}}{\tilde{g}_{11}} \\
&= \gamma^{\alpha\beta}\gamma_{\beta\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{x}^\mu \frac{\tilde{g}_{1\mu}}{\tilde{g}_{11}} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^i \frac{\tilde{g}_{1i}}{\tilde{g}_{11}} \\
&= \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^1.
\end{aligned} \tag{40}
$$

Now we use (26) and find

$$
\frac{j_1^\alpha}{-\sqrt{\lambda}} = \gamma^{\alpha\beta}\partial_\beta X^\mu G_{1\mu} - \epsilon^{\alpha\beta}\partial_\beta X^\mu B_{1\mu} = \frac{J_1^\alpha}{-\sqrt{\lambda}}. \tag{41}
$$

b) We turn now to the case of $J_i^\alpha$ and $j_i^\alpha$ $(i = 2, \cdots, d)$. In this case there are more transformations to be performed but all of them are based on (25-28)

$$
\begin{aligned}
\frac{j_i^\alpha}{-\sqrt{\lambda}} &= \gamma^{\alpha\beta}\partial_\beta x^\mu g_{i\mu} - \epsilon^{\alpha\beta}\partial_\beta x^\mu b_{i\mu} \\
&= \gamma^{\alpha\beta}\partial_\beta x^1 g_{i1} + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^j g_{ij} - \epsilon^{\alpha\beta}\partial_\beta x^1 b_{1i} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^j b_{ij} \\
&= .\gamma^{\alpha\beta}\partial_\beta \tilde{x}^1 \tilde{g}_{i1} + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^j \tilde{g}_{ij} + \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^1 \tilde{b}_{1i} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^j \tilde{b}_{ij}.
\end{aligned} \tag{42}
$$

Now we go to the $\tilde{X}^\mu$ variables by making the inverse shift

$$
\frac{j_i^\alpha}{-\sqrt{\lambda}} = .\gamma^{\alpha\beta}\partial_\beta \tilde{X}^\mu \tilde{G}_{i\mu} - \epsilon^{\alpha\beta}\partial_\beta \tilde{X}^\mu \tilde{B}_{i\mu}. \tag{43}
$$

Since $\tilde{X}^i = X^i$, we separate $\tilde{X}^1$ and $\tilde{X}^i$ dependent parts and find

$$
\begin{aligned}
\frac{j_i^\alpha}{-\sqrt{\lambda}} &= \gamma^{\alpha\beta}\partial_\beta \tilde{X}^1 \tilde{G}_{i1} - \epsilon^{\alpha\beta}\partial_\beta \tilde{X}^1 \tilde{B}_{i1} + \gamma^{\alpha\beta}\partial_\beta \tilde{X}^j \tilde{G}_{ij} - \epsilon^{\alpha\beta}\partial_\beta \tilde{X}^j \tilde{B}_{ij} \\
&= \gamma^{\alpha\beta}\partial_\beta X^1 G_{i1} - \epsilon^{\alpha\beta}\partial_\beta X^1 B_{i1} + \gamma^{\alpha\beta}\partial_\beta X^j G_{ij} - \epsilon^{\alpha\beta}\partial_\beta X^j B_{ij}.
\end{aligned} \tag{44}
$$

Therefore

$$
\frac{j_i^\alpha}{-\sqrt{\lambda}} = \gamma^{\alpha\beta}\partial_\beta X^\mu G_{i\mu} - \epsilon^{\alpha\beta}\partial_\beta X^\mu B_{i\mu} = \frac{J_i^\alpha}{-\sqrt{\lambda}}, \tag{45}
$$

which proves the statement (37).

### $\boldsymbol{T s_1 \cdots s_d T}$ transformations

Obviously we can make a simple generalization of the TsT-transformation. We proceed as follows. First we make a T-duality on $X^1$ after which the original action

$$S = -\frac{\sqrt{\lambda}}{2} \int d\tau \frac{d\sigma}{2\pi} \left[ \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} - \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} \right] \tag{46}$$

becomes

$$S = -\frac{\sqrt{\lambda}}{2} \int d\tau \frac{d\sigma}{2\pi} \left[ \gamma^{\alpha\beta} \partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \tilde{G}_{\mu\nu} - \epsilon^{\alpha\beta} \partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \tilde{B}_{\mu\nu} \right], \tag{47}$$

where the tilde variables are defined in (25), with the relations (27) and (29) satisfied.

Second step consists in applying multi-shifts along the $U(1)$ isometries unaffected by the T-duality in the previous step. This slightly generalizes the Maldacena-Lunin procedure described in the previous Section

$$\begin{aligned} \tilde{X}^i &= \tilde{x}^i + \gamma^i \tilde{x}^1, \\ \tilde{X}^1 &= \tilde{x}^1, \end{aligned} \tag{48}$$

or $\tilde{\boldsymbol{X}} = \boldsymbol{A} \tilde{\boldsymbol{x}}$, where

$$\tilde{\boldsymbol{X}} = \begin{pmatrix} \tilde{X}^1 \\ \vdots \\ \tilde{X}^N \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ \gamma^2 & 1 & & & \vdots \\ \vdots & & \ddots & & 0 \\ \gamma^N & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{49}$$

Under these multi-shifts the background fields take the form

$$\begin{aligned} \tilde{g}_{11} &= \tilde{G}_{11} + 2\gamma^i \tilde{G}_{1i} + \gamma^i \gamma^j \tilde{G}_{ij}, \\ \tilde{g}_{1i} &= \tilde{G}_{1i} + \gamma^j \tilde{G}_{ij}, \\ \tilde{g}_{ij} &= \tilde{G}_{ij}, \\ \tilde{b}_{1i} &= \tilde{B}_{1i} + \gamma^j \tilde{B}_{ij}, \\ \tilde{b}_{ij} &= \tilde{B}_{ij}. \end{aligned} \tag{50}$$

The last step consists in T-dualization back on $\tilde{x}^1$. The resulting action is

$$S = -\frac{\sqrt{\lambda}}{2} \int d\tau \frac{d\sigma}{2\pi} \left[ \gamma^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu} - \epsilon^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu b_{\mu\nu} \right]. \tag{51}$$

As in the case of TsT-transformation, for the generalization described above we prove that

$$j_i^\alpha = J_i^\alpha. \tag{52}$$

The boundary condition for $x^1$ can be obtained from the equation

$$\partial_1 x^1 = \partial_1 X^1 - \gamma^i J_i^0, \quad i = 2, \cdots, N. \tag{53}$$

The boundary conditions for the other coordinates are easily obtained from

$$\partial_\alpha x^i = \partial_\alpha \tilde{x}^i - \gamma^i \partial_\alpha \tilde{x}^1. \tag{54}$$

Using the relation (34) and (48), we get

$$\partial_1 x^i = \partial_1 X^i + \gamma^i \left(\partial_0 x^\mu G_{1\mu} + \partial_1 x^j B_{1j}\right) = \partial_1 x^i + \gamma^i J_1^0. \tag{55}$$

Therefore, the boundary conditions for the fields in the deformed background are twisted as follows

$$\partial_1 x^1 = \partial_1 X^1 - \gamma^i J_i^0, \tag{56}$$

$$\partial_1 x^i = \partial_1 X^i + \gamma^i J_1^0. \tag{57}$$

Integrating over $\sigma$, we find

$$x^1(2\pi) - x^1(0) = 2\pi\left(n_1 - \gamma^i J_i\right), \tag{58}$$

$$x^i(2\pi) - x^i(0) = 2\pi\left(n_i + \gamma^i J_1\right), \tag{59}$$

where

$$X^\mu(2\pi) - X^\mu(0) = 2\pi n_\mu, \tag{60}$$

and the current

$$J_\mu = \int \frac{d\sigma}{2\pi} J_\mu^0. \tag{61}$$

One can use these results to generate new string background solutions. One obvious generalization of these results is to apply the TsT procedure to various isometry directions with different deformation parameters. Since this generalization is straightforward, we will not describe it here but refer to the literature.

### 2.3 Remarks on the Integrable Structures

An important issue in AdS/CFT correspondence is the problem of existence of integrable structures. The existence of the latter is crucial because we need as much as possible quantum charges associated with conserved quantities. Then, it is important to see what happens with the integrability when applying TsT transformations. Let us review the most symmetric, and probably the simplest case of $S^5$, where the Lax pair is well known.

Our starting point will be the parametrization of $S^5$ by unitary skew-symmetric $SU(4)$ matrices

$$g = \begin{pmatrix} 0 & X_3 & X_1 & X_2 \\ -X_3 & 0 & X_2^* & -X_1^* \\ -X_1 & -X_2^* & 0 & X_3^* \\ -X_2 & X_1^* & -X_3^* & 0 \end{pmatrix}$$
$$\equiv \begin{pmatrix} 0 & r_3 e^{i\tilde{\phi}_3} & r_1 e^{i\tilde{\phi}_1} & r_2 e^{i\tilde{\phi}_2} \\ -r_3 e^{i\tilde{\phi}_3} & 0 & r_2 e^{-i\tilde{\phi}_2} & -r_1 e^{-i\tilde{\phi}_1} \\ -r_1 e^{i\tilde{\phi}_1} & -r_2 e^{-i\tilde{\phi}_2} & 0 & r_3 e^{-i\tilde{\phi}_3} \\ -r_2 e^{i\tilde{\phi}_2} & r_1 e^{-i\tilde{\phi}_1} & -r_3 e^{-i\tilde{\phi}_3} & 0 \end{pmatrix}, \qquad (62)$$

where
$$r_1^2 + r_2^2 + r_3^2 = 1. \qquad (63)$$

The action takes the familiar form

$$S = \int d\tau d\sigma \gamma^{\alpha\beta} \mathrm{tr} J_\alpha J_\beta, \qquad (64)$$

and the equations of motion in terms of the right currents $J_\alpha$ are

$$\partial_\alpha \left(\gamma^{\alpha\beta} J_\beta\right) = 0, \quad J_\alpha = g^{-1}\partial_\alpha g. \qquad (65)$$

Next step is to represent the equations of motion as zero curvature condition

$$[\, D_\alpha, D_\beta \,] = 0, \qquad (66)$$

where the Lax operator is defined as

$$D_\alpha = \partial_\alpha - \frac{J_\alpha^+}{2(x-1)} + \frac{J_\alpha^-}{2(x+1)} \equiv \partial_\alpha - \mathcal{A}_\alpha. \qquad (67)$$

The self- and anti-selfdual parts of $J_\alpha$ are defined by

$$J_\alpha^\pm = \left(P^\pm\right)_\alpha^\beta J_\beta, \quad \left(P^\pm\right)_\alpha^\beta = \delta_\alpha{}^\beta - \mp \gamma_{\alpha\rho}\epsilon^{\rho\beta}. \qquad (68)$$

One can similarly define another Lax operator associated with the Left currents $L_\alpha = \partial_\alpha g . g^{-1}$.

After *TsT* transformations the action will contain in general explicit dependence on the angles $\tilde{\phi}_i$. However, one can easily see that the dependence on the angular variables in the group element factorizes and it can be written as

$$g(r, \tilde{\phi}_i) = M(\tilde{\phi}_i)\hat{g}(r_i)M(\tilde{\phi}_i), \qquad (69)$$

where

$$M = e^{\tilde{\Phi}},$$

$$\tilde{\Phi} = \begin{pmatrix} \tilde{\phi}_1+\tilde{\phi}_2+\tilde{\phi}_3 & 0 & 0 & 0 \\ 0 & -\tilde{\phi}_1-\tilde{\phi}_2+\tilde{\phi}_3 & 0 & 0 \\ 0 & 0 & \tilde{\phi}_1-\tilde{\phi}_2-\tilde{\phi}_3 & 0 \\ 0 & 0 & 0 & -\tilde{\phi}_1+\tilde{\phi}_2-\tilde{\phi}_3 \end{pmatrix} \qquad (70)$$

$$\hat{g}(r_i) = \begin{pmatrix} 0 & r_3 & r_2 & r_2 \\ -r_3 & 0 r_2 & -r_1 & \\ -r_1 & -r_2 & 0 & r_3 \\ -r_2 & r_1 & -r_3 & 0 \end{pmatrix}, \quad \hat{g}^{-1} = -\hat{g}.$$

The dependence on $\tilde{\Phi}$ in the currents then takes the form

$$J_\alpha(r_i, \tilde{\phi}_i) = M^{-1} \hat{J}_\alpha(r_i, \partial\tilde{\phi}_i) M, \qquad (71)$$

where

$$\hat{J}_\alpha(r_i, \partial\tilde{\phi}_i) = -\hat{g}\partial_\alpha\hat{g} - \hat{g}\partial\tilde{\Phi}\hat{g} + \partial_\alpha\tilde{\Phi}. \qquad (72)$$

Written in this form, it is clear that the dependence on $M$ can be gauged away

$$D_\alpha \;\rightarrow\; M D_\alpha M^{-1} = \partial_\alpha - \hat{\mathcal{A}}_\alpha - \partial_\alpha\tilde{\Phi}. \qquad (73)$$

It is also obvious that the gauge transformed Lax connection is flat since it depends on $\partial\tilde{\phi}_i$ but not on $\tilde{\phi}_i$.

The conclusion is that all we have to do to obtain the Lax representation is to express $\tilde{\phi}_i$ (before TsT transformations) in terms of $\partial\phi_i$ (after TsT transformation). The explicit relation between these quantities is given in (...).

One can repeat the same logic to obtain the Lax representation in the case of TsT transformations on all isometry directions. The resulting expressions are rather complicated, but still demonstrating the existence of integrable structures in the deformed theory.

### 2.4 A Generalization

The deformations of the string background certain numbers of isometries can be obtained by transforming with the T-duality matrix (7) which is also equivalent to performing a coordinate transformation, a factorized duality along one isometry direction, a geometric shift, then dualizing back along the same isometry direction and finally the inverse coordinate transformation. Frolov applied the latter on all the 2-tori embedded in the 3-torus and found a 3-parameter family of deformations of the $AdS_5 \times S^5$ background [5]. As one might guess from above, it is equivalent to generating a new solution by the action of the T-duality matrix

$$\mathcal{T} = \begin{pmatrix} 1_3 & 0_3 \\ \Gamma & 1_3 \end{pmatrix}, \qquad (74)$$

where $\Gamma$ is now[2]

$$\begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}. \tag{75}$$

If we look at the solution generating matrix (74) we can easily see that it has a unique factorization in terms of the generators of $O(d, d, \mathbb{R})$

$$\begin{pmatrix} 1_3 & 0_3 \\ \Gamma & 1_3 \end{pmatrix} = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1_3 & \Gamma \\ 0 & 1_3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix}. \tag{76}$$

So, the action of (74) is equivalent to a factorized duality along all the isometry directions, then doing a $\Theta$ shift[3] and finally an inverse factorized duality along all the isometry directions.

We should also discuss the transformation of the vector of momentum and winding charges since they play a crucial role. The momenta are conserved and if $w_i$ are the winding numbers of the undeformed background then the deformed background has

$$\phi_1(2\pi) - \phi_1(0) \;=\; 2\pi(w_1 + \gamma_2 J_3 - \gamma_3 J_2) \tag{77}$$

$$\phi_2(2\pi) - \phi_2(0) \;=\; 2\pi(w_2 + \gamma_3 J_1 - \gamma_1 J_3) \tag{78}$$

$$\phi_3(2\pi) - \phi_3(0) \;=\; 2\pi(w_3 + \gamma_1 J_2 - \gamma_2 J_1). \tag{79}$$

Here $(J_1, J_2, J_3)$ are the conserved angular momenta[4].

## 3 Deformations of the Background

This transformation can be seen as a solution-generating technique to obtain the gravity duals of the deformed gauge theories. In particular, on a supergravity solution of a type II theory, it reduces to a simple TsT transformation. The latter is described as follows. Let us parametrize the two torus by $(\phi_1, \phi_2)$, the transformation then consists of a T-duality along $\phi_1$, followed by a shift $\phi_2 \to \phi_2 + \gamma\phi_1$ in the T-dual background, and finally by another T-duality along $\phi_1$. The results can be immediately applied to a number of background having these properties, in particular in the contexts string/gauge theory duality. We distinguish three cases of deformations:

1. In the first case, we take the two isometries necessary for the TsT-transformation to be along the brane. In this case, the product of the fields

[2] We note that $d(d-1)/2$ parameter deformations of a background with $d$ commuting $U(1)$ isometries were studied in [20]. We would expect that such deformed solutions should be generated by the action of an $O(d, d, \mathbb{R})$ matrix of the same form, where $\Gamma$ is now a $d$-dimensional antisymmetric matrix with $d(d-1)/2$ independent parameters.

[3] It is standard to call the generators of $O(d, d, \mathbb{R})$ of this type $\Theta$ shifts [10].

[4] Here we use the letter $J$, not $p$, following [5], where $p$ is used to name the world-sheet current.

in the dual gauge theory becomes:

$$(f*g)(x) = e^{-i\pi\gamma\left(\frac{\partial}{\partial x^1}\frac{\partial}{\partial y^2}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial y^1}\right)} f(x)g(y)_{|x=y}$$
$$= f(x)g(x)i\pi\gamma\left(\partial_1 f(x)\partial_2 g(x) - \partial_2 f(x)\partial_1 g(x)\right) + \cdots \quad (80)$$

This deformed product can be recognized as the appropriate Moyal product for a non-commutative two-torus. One should note that this deformation yields a non-commutative theory, which is non-local and breaks Lorentz invariance and causality. Nevertheless, this picture, and its generalizations, opens the road to interesting string realizations of non-commutative theories.

2. In the second case, let us suppose that there is a global $U(1)$ isometry along the D-brane, but the other to be used for TsT-transformations is transversal to the brane. If we associate with the isometries charges, say $Q^i$, one can map the string picture to the gauge theory dual. In this case the deformed product of the fields can be read off

$$(f*g)(x) = e^{\pi\gamma\left(Q^g\frac{\partial}{\partial x}-Q^f\frac{\partial}{\partial y^1}\right)} = f(x+\pi\gamma Q^g)g(x\pi\gamma Q^f). \quad (81)$$

This is called dipole deformation, and is clearly non-local although living on a commutative space-time. Here we applied TsT-transformation shifting a single direction $x$, but we can obtain more general deformations by introducing "dipole vectors" $L^{M\mu} = 2\pi\gamma Q^M L^\mu$ for the various fields (here $L^\mu$ is a constant vector). In this case the product of the fields is

$$(f*g)(x) = f(x-\frac{1}{2}L^g)g(x+\frac{1}{2}L^f). \quad (82)$$

3. In the last case, let us suppose there are two isometries transverse to the D-brane.The dual gauge theory picture changes so that the product of the fields reduces to:

$$(f*g)(x) = e^{i\pi\gamma(Q_1^f Q_2^g - Q_2^f Q_1^g}fg. \quad (83)$$

In this case the product deformation yields an ordinary commutative and local theory since the only effect of the deformed product (83) is to introduce some phases in the interactions. The superconformal gauge theories arising from the product (83) are classified by Leigh and Strassler and are called the $\beta$-deformed. One can note that the deformation reduces the amount of supersymmetry and TsT-transformations can serve as supersymmetry breaking procedure.

Let us describe these cases separately.

### 3.1 Non-Commutative Theories from Deformations

As we pointed out above, these type theories appear when we use the solution generating technique to isometries on the brane. The simplest example of a nontriviality of the duality transformations and generating new solutions in the presence of background fields is the torus.

Consider compactification on $T^2$ with a constant Neveu-Schwarz two-form field B. The resulting eight dimensional theory has an exact $SL(2,Z)\times SL(2,Z)$ symmetry which acts on the complex structure of the torus and on the Kähler parameter $\tau = B + i\sqrt{g}$. There are two coordinates $\varphi_1, \varphi_2$ and the two $U(1)$ symmetries act on these two coordinates as shifts of $\varphi_i$. Then we will have a two torus parametrized by $\varphi_i$, which, in general, will be fibered over an eight dimensional manifold. Under simultaneous T-duality of all coordinates, the combination $(G+B)_{ij}$ is inverted, where $G$ is the metric expressed in string units. Consider a square torus and take the limit

$$R_1 = R_2 \to 0; \quad B\alpha' = 0 \text{ fixed.} \tag{84}$$

The T-dual torus has

$$G + B = \frac{1}{R_1^2 R_2^2 + B^2}\begin{pmatrix} R_2^2 & -B \\ B & R_1^2 \end{pmatrix} \tag{85}$$

and the T-dual radii go to zero as $\tilde{R}_1 = \tilde{R}_2 \sim R_1/B$. If we T-dualize a single dimension, the procedure will exchange the complexified Kähler form $B + i\sqrt{g}$ and the complex structure of the torus $\tau = R2/R1$, to produce a theory with fixed volume and $B = 0$, while the torus becomes highly anisotropic: $\tau' = B + iR_1R_2$. Let us think of this torus as $\mathbb{R}^2$ quotiented by the two translations $x \to x+1$ and $x \to x+\tau'$. In the limit $R_1R_2 \to 0$, the original $T^2$ appears to degenerate to $\mathbb{R}\times S^1$ with coordinate $0 \le x^1 < 1$ quotiented by an additional translation $x^1 \to x^1 + B$. [5] Next point is to see what happens when two open strings interact. The interaction of two open strings means that the end point of the first string coincides with the beginning point of the second one. This can be express as follows. Using the string coordinates $\phi_1, \phi_2, \phi_3$ to describe the interaction process, one can write the interaction as

$$\begin{aligned} S_{\text{int}} &= \sum_{\o_2,\o_2'} \int dx^1 \phi_1(x, -\o_2 - \o_2')\phi_2(x^1,\o_2)\phi_3(x^1 - B\o_2/R_1, \o_2') \\ &= \sum_{\o_2,\o_2'} \int dx^1 \phi_1(x, -\o_2 - \o_2')\phi_2(x^1,\o_2) \exp\Big(\frac{-B\o_2}{R_1}\frac{\partial}{\partial x^1}\Big)\phi_3(x^1,\o_2'), \end{aligned} \tag{86}$$

[5] The nature of such a quotient depends radically on whether B is rational or irrational; if we interpret it in the latter case naively as a pointwise identification, it will not lead to a conventional Hausdorff manifold.

which after the redefinition $\sigma^i \equiv R_i x^i$ and Fourier transformation becomes

$$S_{\text{int}} = \int d\sigma^1 d\sigma^2 \phi_1(\sigma^1, \sigma^2) \exp\Big(\frac{B}{2\pi i}\frac{\partial^2}{\partial\sigma^{1\prime}\partial\sigma^{2\prime}}\Big)\phi_2(\sigma^1, \sigma^{2\prime})\phi_3(\sigma^{1\prime}, \sigma^2)\big|_{\sigma^{i\prime}=\sigma^i}. \tag{87}$$

Going back to [Connes, Douglas, Schwartz, 9211162] one can conclude that the general connection on the non-commutative torus is a sum $\nabla_i + A_i$, where $\nabla_i$ is a constant curvature connection and $A_i$ are elements of the $C^*$ algebra defined by

$$U^2 U^1 = e^{2\pi i B} U^1 U^2. \tag{88}$$

Therefore, the classical torus is described in terms of the set of single valued functions. These are of course generated by the Fourier modes $U_i = e^{2\pi i x_i}$ . To define the noncommutative torus in d dimensions, we start from the functions, which again are taken to be generated by Fourier modes. We will also denote these modes by $U_i$. Obviously, the algebra of functions is generated by these d operators. Although in the classical case these modes obviously commute, for the noncommutative torus we assume that they do not commute, but have commutation relations of the form

$$U_j U_k = e^{2\pi i\theta^{jk}} U_k U_j, \quad \text{or } [x^j, x^k] = \frac{\theta^{jk}}{2i}, \tag{89}$$

where $\theta^{ij}$ is a constant anti-symmetric tensor. It is called the noncommutative parameter, as it characterizes the noncommutativeness of the torus. The noncommutative torus defined by this relation is denoted $\mathbb{T}^{\theta}_{d}$ . Note that the above commutation relation can also be expressed in terms of a commutation relation between the local coordinates $x^i$ as $[x^j, x^k] = 2\pi i\theta^{jk}$, which is essentially the commutation relation on a quantum phase space. Therefore the noncommutative torus is often referred to as the quantum torus. We will not use this name here, as we want to see the noncommutative geometry really as a classical geometry. Except for the functions on the torus, we will also need a set of derivations. These will effectively replace the ordinary derivative operators on the torus, and can be considered as such. They are defined through the commutation relation with the algebra of functions. It is obviously enough to give the commutation relations with the coordinates $x^k$

$$[\partial_j, x^k] = \delta_j^k. \tag{90}$$

Note that this is taken over without change from the classical situation. Also, remark that these derivations commute among themselves. This property will be very convenient, and allows one to consistently define differential forms on the noncommutative torus [19, 7].

Going back to the problem, at the supergravity level we recognize an $SL(2,R)\times SL(2,R)$ symmetry, which is *not* symmetry of the full string theory. One of these is acting on the Kähler parameter while the other is acting on the complex structure. It is a well known trick to use the $SL(2,R)$ symmetries of supergravity as solution generating technique. Actually the T-duality transformations can generate a rich structures in string theory. The particular transformation which is playing important role is the one acting as

$$\tau \to \tau' = \frac{\tau}{1+\gamma\tau} \in SL(2,R), \tag{91}$$

where $\tau = B + i\sqrt{g}$. If we start with non-singular ten dimensional geometry, one can show that the above transformation is the only $SL(2,R)$ transformation preserving this property of the geometry.

Let us consider a D-brane on the original background that is invariant under both $U(1)$ symmetries. Such a brane will be left invariant under the action of (91). In other words, there is a corresponding brane on the new background. The most natural question to ask is what is the theory on this brane in the new background. The conjectured answer suggested in [0502086 lunin-malda] is as follows. Suppose that the original brane, on the original background, gave rise to a certain open string field theory. Then the open string field theory on the brane living on the new background is given by changing the start product

$$f *_\gamma g \equiv e^{i\pi\gamma(Q^1_f Q^2_g - Q^2_f Q^1_g)} f *_0 g\,, \tag{92}$$

where $*_0$ is the original star product and $Q^i_{f,g}$ are the $U(1)$ charges of the fields $f$ and $g$. This conjecture is supported from the above considerations and the Seiberg-Witten paper on non-commutative theories. Namely, in the presence of a $B$ field the open string field theory is defined in terms of an open string metric and non-commutativity parameter

$$G^{ij}_{open} + \Theta^{ij} = \left(\frac{1}{g+B}\right)^{ij} \sim \frac{1}{\tau}\,, \tag{93}$$

where the last expression is schematic. Note that under the transformation $1/\tau \;\to\; 1/\tau' \;=\; 1/\tau + \gamma$. All that happens is that we introduce a non-commutativity parameter $\Theta^{12} = \gamma$. The open string metric remains the same. The reason we call this a “conjecture” rather than a derivation is that in [S-W] they considered a constant metric and $B$ field while here we are applying their formulae in a case where these fields vary in space-time.

### 3.2 Dipole Deformations

The gravity duals of dipole theories appear when we apply the TsT transfomations to one isometry along the brane and one in the transverse space.

Dipole-theories can be thought of as a generalization of field theories on commutative or noncommutative spaces. They are constructed by modifying the ordinary (or noncommutative) product of functions to the $\tilde{\star}$-product defined as follows. To each field, $\Phi(x)$, we assign a dipole-vector, $L^i$. The complex conjugate field, $\Phi^\dagger(x)$ is assigned the dipole-vector $-L^i$. If $\Phi_1(x)$ and $\Phi_2(x)$ have dipole-vectors $L_1$ and $L_2$ respectively, we define their dipole-product to be [16]

$$(\Phi_1 \tilde{\star} \Phi_2)_{(x)} \equiv \Phi_1(x - \frac{L_2}{2})\Phi_2(x + \frac{L_1}{2}). \tag{94}$$

For associativity, we have to make sure that the dipole-vector is additive, *i.e.* $\Phi_1 \tilde{\star} \Phi_2$ is defined to have dipole-vector $L_1 + L_2$. Intuitively, the dipole field $\Phi(x)$ represents a dipole of length $L$ starting at the point $x - \frac{L}{2}$ and ending at $x + \frac{L}{2}$. To see this, let us add a $U(1)$ gauge field, $A(x)$, and define it to have dipole-vector zero. The covariant derivative is then

$$\begin{aligned} D_i\Phi(x) &= \partial_x i\Phi(x) - iA_i(x)\tilde{\star}\Phi(x) + i\Phi(x)\tilde{\star}A_i(x) \\ &= \partial_x i\Phi(x) - iA_i(x - \frac{L}{2})\Phi(x) + i\Phi(x)A_i(x + \frac{L}{2}). \end{aligned}$$

So $\Phi(x)$ transforms nontrivially under the subgroup $U(1)_{(x-\frac{L}{2})} \times U(1)_{(x+\frac{L}{2})}$ of the gauge group, where $U(1)_{(x)}$ is the local transformation group at $x$.

To ensure associativity we can start with an ordinary theory that has a global $U(1)$ symmetry and assign to the field $\Phi_I$ ($I$ is an arbitrary index that labels the field) a dipole-vector in the form $L_I = Q_I L$ where $L$ is a fixed vector, common to all the fields, and $Q_I$ is the $U(1)$ charge of $\Phi_I$. More generally, working on $\mathbb{R}^d$, we can have global charges, $Q_{Ia}$, $a = 1\ldots l$ and a fixed $d \times l$ matrix, $\Theta^{ia}$ ($i = 1\ldots d$), such that the field $\Phi_I$ is assigned a dipole-vector $\sum_a \Theta^{ia}Q_{Ia}$.

The claim that field-theories on a noncommutative space are a special case of the dipole-theories can be interpreted in two ways. First, the $\tilde{\star}$-product can be defined to modify a noncommutative $\star$-product. We just have to interpret the products on the RHS of (94) as $\star$-products. Moreover, starting with a commutative space, we can take the charges $Q_{Ia}$ above to be the components of the momentum of the field $\Phi_I$ and then $a = 1\ldots d$. If $\Theta^{ia}$ is chosen to be anti-symmetric we recover the familiar field-theory on a noncommutative $\mathbb{R}^d$. Let us also note that gauge theories on a noncommutative space can be recast in terms of bi-local fields [17] which might be reminiscent of the dipole-fields.

### 3.3 $\beta$-Deformations

The last, and probably most studied case is the when the isometries to which we apply the solution generating technique are transverse to the brane. Then the dual gauge theories are ordinary ones, but since the symmetries of the transverse space serve as R-symmetries the dual gauge theory will be less supersymmetric. Actually this is what is usually understood under Lunin-Maldacena deformations, or solution generating techniques. This case is of special interest to AdS/CFT correspondence since it describes gravity duals of gauge theories with less supersymmetry.

The procedure of obtaining the deformed backgrounds is the very same as above and we will not repeat it here. Instead we will write down the resulting theory for deformed $AdS_5 \times S^5$ background with three parameters (NS sector). Applying the T-duality matrix (74) to the background matrix of (99) we obtain exactly the solution of Frolov, whose NS sector is

$$\begin{aligned} ds^2 &= R^2[ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + G\mu_1^2\mu_2^2\mu_3^2(\sum_i \hat{\gamma}_i d\phi_i)^2] \\ B &= R^2 G(\hat{\gamma}_3\mu_1^2\mu_2^2 d\phi_1 \wedge d\phi_2 + \hat{\gamma}_1\mu_2^2\mu_3^2 d\phi_2 \wedge d\phi_3 + \hat{\gamma}_2\mu_3^2\mu_1^2 d\phi_3 \wedge d\phi_1) \\ e^{2\Phi'} &= Ge^{2\Phi} \\ G &= (1 + (\hat{\gamma}_3^2\mu_1^2\mu_2^2 + \hat{\gamma}_1\mu_2^2\mu_3^3 + \hat{\gamma}_2\mu_1^2\mu_3^2))^{-1}, \quad \hat{\gamma}_i = R^2\gamma_i. \end{aligned} \tag{95}$$

As discussed above, the deformed background has

$$\phi_1(2\pi) - \phi_1(0) = 2\pi(w_1 + \gamma_2 J_3 - \gamma_3 J_2), \tag{96}$$
$$\phi_2(2\pi) - \phi_2(0) = 2\pi(w_2 + \gamma_3 J_1 - \gamma_1 J_3), \tag{97}$$
$$\phi_3(2\pi) - \phi_3(0) = 2\pi(w_3 + \gamma_1 J_2 - \gamma_2 J_1), \tag{98}$$

where $(J_1, J_2, J_3)$ are the conserved angular momenta and $w_i$ are the winding numbers of the undeformed background.

We note that one can easily make a lift to M-theory and apply the techniques described here (see for instance [lunin-maldacena].

## 4 AdS/CFT in Deformed Backgrounds: Examples

Here we will present mainly results from $\beta$-deformed backgrounds. Other examples one can find in [21]- [24].

### Case of $AdS_5 \times S^5$

We consider the background $AdS_5 \times S^5$ whose metric is

$$\frac{ds^2}{R^2} = ds^2_{AdS_5} + \sum_{i=1}^{3} d\mu_i^2 + \mu_i^2 d\phi_i^2, \quad \text{with} \quad \sum_i \mu_i^2 = 1. \tag{99}$$

Applying TsT transformations we end up with [2]

$$ds^2 = R^2[ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \hat{\gamma}^2 G\mu_1^2\mu_2^2\mu_3^2(\sum_i d\phi_i)^2] \tag{100}$$
$$B = \hat{\gamma} R^2 G(\mu_1^2\mu_2^2 d\phi_1 \wedge d\phi_2 + \mu_2^2\mu_3^2 d\phi_2 \wedge d\phi_3 + \mu_3^2\mu_1^2 d\phi_3 \wedge d\phi_1) \tag{101}$$
$$e^{2\Phi'} = Ge^{2\Phi} \tag{102}$$
$$G = (1 + \hat{\gamma}^2(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^3 + \mu_1^2\mu_3^2))^{-1}, \qquad \hat{\gamma} = R^2\gamma. \tag{103}$$

As an application to AdS/CFT correspondence let us consider the motion of a rigid string on $S^3_\gamma$. This space can be thought of as a subspace of the $\gamma$-deformation of $AdS_5 \times S^5$ presented above

$$\mu_3 = 0, \quad \phi_3 = 0 \quad \text{i.e.} \quad \psi = 0, \quad \phi_3 = 0. \tag{104}$$

The relevant part of the $\gamma$-deformed $AdS_5 \times S^5$ is

$$ds^2 = -dt^2 + d\theta^2 + G\sin^2\theta d\phi_1^2 + G\cos^2\theta d\phi_2^2, \tag{105}$$

where $G = \dfrac{1}{1+\tilde{\gamma}^2\sin^2\theta\cos^2\theta}$ and due to the series of T-dualities there is a non-zero component of the B-field

$$B_{\phi_1\phi_2} = \tilde{\gamma}G\sin^2\theta\cos^2\theta. \tag{106}$$

The ansatz

$$t = \kappa\tau, \quad \theta = \theta(y), \quad \phi_1 = \omega_1\tau + \tilde{\phi}_1(y), \quad \phi_2 = \omega_2\tau + \tilde{\phi}_2(y) \tag{107}$$

describes the motion of rigid strings on the deformed 3-sphere, here we have defined a new variable $y = \alpha\sigma + \beta\tau$. One can substitute the above ansatz in the equations of motion and use one of the Virasoro constraints to find three first order differential equations for the unknown functions:

$$\begin{aligned}
\tilde{\phi}_1' =& \frac{1}{\alpha^2-\beta^2}\Big(\frac{A}{G\sin^2\theta} + \beta\omega_1 - \tilde{\gamma}\alpha\omega_2\cos^2\theta\Big)m \\
\tilde{\phi}_2' =& \frac{1}{\alpha^2-\beta^2}\Big(\frac{B}{G\cos^2\theta} + \beta\omega_2 + \tilde{\gamma}\alpha\omega_1\sin^2\theta\Big), \\
(\theta')^2 =& \frac{1}{(\alpha^2-\beta^2)^2}\Big[(\alpha^2+\beta^2)\kappa^2 - \frac{A^2}{G\sin^2\theta} - \frac{B^2}{G\cos^2\theta} \\
& - \alpha^2\omega_1^2\sin^2\theta - \alpha^2\omega_2^2\cos^2\theta + 2\tilde{\gamma}\alpha(\omega_2 A\cos^2\theta - \omega_1 B\sin^2\theta)\Big].
\end{aligned} \tag{108}$$

$A$ and $B$ are integration constants and the prime denotes derivative with respect to $y$. The other Virasoro constraints provides the following relation between the parameters

$$A\omega_1 + B\omega_2 + \beta\kappa^2 = 0. \tag{109}$$

This system has three conserved quantities - the energy and two angular momenta:

$$\begin{aligned}
E &= 2T\frac{\kappa}{\alpha}\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}\,, \\
J_1 &= 2\frac{T}{\alpha}\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}G\sin^2\theta[\omega_1 + \beta\tilde{\phi}_1' + \tilde{\gamma}\alpha\cos^2\theta\tilde{\phi}_2']\,, \\
J_2 &= 2\frac{T}{\alpha}\int_{\theta_0}^{\theta_1}\frac{d\theta}{\theta'}G\cos^2\theta[\omega_2 + \beta\tilde{\phi}_2' + \tilde{\gamma}\alpha\sin^2\theta\tilde{\phi}_1']\,,
\end{aligned} \tag{110}$$

where the integration is performed over the range of the coordinate $\theta$. For the magnon-type string solutions the dispersion relations are obtained to be

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2} - \pi\beta\right)}, \quad \kappa^2 = \omega_1^2, \tag{111}$$

where $p$ is the momentum of the magnon excitation. This relation is invariant under $p \to p+2\pi$ and $\beta \to \beta+1$ as is required by the spin chain analysis [19,25]. In [23] a detailed analysis of the infinite limit of the charges, as well as finite size corrections, is presented. It was argued that in the limit $\mathcal{J} = J/g \to \infty$ the dispersion relations do not feel the deformation since it shows up just as a shift by $\pi\gamma$. It is important however that the deformation produces a non-trivial twist in the boundary conditions for the isometry directions which is proportional to $\sim \gamma J$. The latter has non-trivial consequences, the analysis of which can be seen in [23].

The string profile with one single spike and a large winding number is realized when $\beta^2\kappa^2 = \alpha^2\omega_1^2$ and hence $A = -\omega_1\alpha^2/\beta$ [27]. It is natural to expect the existence of a rigid string solution solution on $S^3_\gamma$ which is the analogue of the single spike solution on $S^3$ found in [27]. The relation between the conserved charges becomes

$$J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\cos^2\theta_1}\,. \tag{112}$$

This looks identical to the corresponding expression in the undeformed case, the dependence on the deformation parameter $\tilde{\gamma}$ is buried in the definition of $\cos\theta_1$. In analogy with the giant magnon solution we can identify $\cos\theta_1 = \sin\left(p/2 - \pi\beta\right)$.

For the relation between $E$ and $\Delta\phi_1$ we find

$$E - T\Delta\phi_1 = \frac{\sqrt{\lambda}}{\pi}\left(\frac{\pi}{2} - \theta_1\right) - \tilde{\gamma}\frac{\sqrt{\lambda}}{\pi}\frac{\sqrt{\omega_1^2 - \Omega_1^2}}{\Omega_1}\cos\theta_1. \tag{113}$$

As should be expected that in the limit $\tilde{\gamma} \to 0$ this expression reduces to the one for the single spike solution on undeformed $S^3$.

### Case of $AdS_5 \times T^{1,1}$

We proceed with $AdS_5 \times T^{1,1}$ background, which arises from a stack of N D-branes at the tip of the conifold, the conic Calabi-Yau 3-fold whose base space is $T^{1,1}$ [26]. The manifold $T^{1,1}$ is the coset space $(SU(2) \times SU(2))/U(1)$. This supergraviry solution is the dual to the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. The metric of $AdS_5 \times T^{1,1}$ is defined as

$$\frac{ds^2}{R^2} = ds^2_{AdS} + \frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2. \quad (114)$$

This is clearly an $S^1$ Hopf fibration over $S^2 \times S^2$ with $\psi$ parameterizing the fiber circle which winds over the two base spheres once[6]. The isometry group of $T^{1,1}$ is $SU(2)\times SU(2)\times U(1)$ and in particular, it has three commuting Killing vectors: $\partial/\partial\phi_1, \partial/\partial\phi_2, \partial/\partial\psi$. [2] used the first two of them in order to generate new solutions corresponding to exactly marginal real $\gamma$-deformations of the field theory.

After the deformation the background has the form given by

$$\frac{ds^2}{R^2} = ds^2_{AdS} + G\left[\frac{1}{6}\sum_{i=1}^{2}(G^{-1}d\theta_i^2 + \sin^2\theta_i\, d\phi_i^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \tilde{\gamma}^2\frac{\sin^2\theta_1\sin^2\theta_2}{324}d\psi^2\right] \quad (115)$$

Due to the T-dualities a non-trivial B-field is generated

$$\frac{B}{R^2} = \tilde{\gamma}G\left[\left(\frac{\sin^2\theta_1\sin^2\theta_2}{36} + \frac{\cos^2\theta_1\sin^2\theta_2 + \cos^2\theta_2\sin^2\theta_1}{54}\right)d\phi_1\wedge d\phi_2 + \frac{\sin^2\theta_1\cos\theta_2}{54}d\phi_1\wedge d\psi - \frac{\cos\theta_1\sin^2\theta_2}{54}d\phi_2\wedge d\psi\right]. \quad (116)$$

The conformal factor in the metric and the B-field has the form[7]

$$G^{-1} = 1 + \tilde{\gamma}^2\left(\frac{\cos^2\theta_1\sin^2\theta_2 + \cos^2\theta_2\sin^2\theta_1}{54} + \frac{\sin^2\theta_1\sin^2\theta_2}{36}\right). \quad (117)$$

Now we turn to the AdS/CFT correspondence applications. This means that we have to find the dispersion relations that are supposed to give the anomalous dimensions of the gauge theory operators at strong coupling. One can check by direct inspection that the following ansatz is a consistent truncation of the complete background

$$\theta_2 = \text{const}, \quad \phi_2 = \text{const}. \quad (118)$$

To further simplify considerations we choose $\theta_2 = 0$. Next we choose the following *ansatz* for solitonic string configurations

$$t = \kappa\tau, \quad \theta_2 = 0, \quad \phi_2 = \text{const}.$$
$$\Psi = \varnothing_\psi\tau + \psi(y); \quad \Phi = \varnothing_\phi\tau + \phi(y), \quad \theta = \theta(y), \quad (119)$$

[6] When the Hopf numbers of the $S^1$ bundle over the base 2-spheres are $p$ and $q$ the above generalizes to the manifolds $T^{p,q}$.

[7] We skip here the rest of the field content since it will not be used in what follows.

where $y = c\sigma - d\tau$, $\Psi$ is the $U(1)$ fiber coordinate while $\Phi \equiv \phi_1$. With this choice the metric becomes (we set $R^2 = 1$)

$$ds^2 = -dt^2 + \frac{1}{6}d\theta^2 + \frac{G}{6}\sin^2\theta d\phi^2 + \frac{G}{9}(d\psi + \cos\theta d\phi)^2 \tag{120}$$

and the B-field takes the form

$$B = G\tilde{\gamma}\frac{\sin^2\theta}{54}. \tag{121}$$

In (120) and (121) the factor $G$ can be read off from (117)

$$G^{-1} = 1 + \tilde{\gamma}^2\frac{\sin^2\theta}{54}. \tag{122}$$

We are looking for solutions with the profile of arc or spike moving along the isometry directions and described by (119).

The corresponding conserved string charges are defined by

$$J_\psi = \int_{-\infty}^{\infty}\frac{dy}{c}P_\psi = \frac{T}{9}\int_{-\infty}^{\infty}\frac{dy}{c}G\Big[\o_\psi - d\psi' + \cos\theta(\o_\phi - d\phi') + \tilde{\gamma}c\frac{\sin^2\theta}{6}\phi'\Big], \tag{123}$$

$$J_\phi = \int_{-\infty}^{\infty}\frac{dy}{c}P_\phi = \frac{T}{9}\int_{-\infty}^{\infty}\frac{dy}{c}\Big[\big(1 + \frac{\sin^2\theta}{2}\big)\big(\o_\phi - d\phi'\big)$$
$$+ \cos\theta(\o_\psi - d\psi') - \tilde{\gamma}c\frac{\sin^2\theta}{6}\psi'\Big], \tag{124}$$

$$E = -\int_{-\infty}^{\infty}\frac{dy}{c}P_t = T\int_{-\infty}^{\infty}\frac{dy}{c}\kappa. \tag{125}$$

The giant magnons are characterized with certain conditions, namely for giant magnon string solutions we have

$$A_\phi = \frac{2}{9}d(\o_\phi - \o_\psi), \quad A_\psi = -A_\phi$$

which combined with the second Virasoro constraint gives

$$\kappa = \frac{\o_\phi - \o_\psi}{3}, \tag{126}$$

Using the equations of motion and the Virasoro constraints one can find the following form of the dispersion relations

$$\frac{\sqrt{3}}{2}(\mathcal{E} - 3\mathcal{J}_\phi) = \frac{\cos\left[\frac{\sqrt{3}}{2}(\mathcal{E} + 3\mathcal{J}_\psi)\right] - \cos\big(\Delta\varphi - \tilde{\gamma}/2(\mathcal{J}_\phi + \mathcal{J}_\psi)\big)}{\sin\left[\frac{\sqrt{3}}{2}(\mathcal{E} + 3\mathcal{J}_\psi)\right]}. \tag{127}$$

We note that the transcendental behavior of the dispersion relation persists in the deformed background. The deformation parameter enters the expression by shifting the angle amplitude by term proportional to $\gamma$ times the total spin [8]. The BMN and basic giant magnon analysis considered in [28] can be easily repeated with the same conclusions (up to the $gamma$ shift). Note that each conserved charge depends on the $\gamma$ parameter, but this dependence is hidden in the dispersion relations.

Let us proceed with the condition for single spike string solutions

$$A_\phi = \frac{2c^2}{9d}(\varnothing_\phi - \varnothing_\psi),$$

which combined with the second Virasoro constraint gives

$$\kappa = \frac{c(\varnothing_\phi - \varnothing_\psi)}{3d}. \tag{128}$$

One can use again Virasoro constraints and the equations of motion supplied with the appropriate boundary conditions to obtain the string solutions. For the dispersion relations one obtains

$$-\frac{3\sqrt{3}}{2}\mathcal{J}_\phi = \frac{\cos\left(\frac{3\sqrt{3}}{2}\mathcal{J}_\psi\right) - \cos\Delta\delta}{\sin\left(\frac{3\sqrt{3}}{2}\mathcal{J}_\psi\right)}. \tag{129}$$

Here

$$\Delta\delta \equiv \Delta\varphi - \frac{3}{2}\left(\frac{E}{T}\right) - \frac{\tilde{\gamma}}{2}\left(\frac{J_\psi}{T} + \frac{J_\phi}{T}\right). \tag{130}$$

The transcendental behavior of the dispersion relations persists as expected. The non-trivial shift of the angle amplitude is of the same form as in the case of the giant magnons and therefore has an universal form.

The above considerations are to serve for starting point of thorough investigations of the string/gauge theory dualities for less supersymmetric cases which on other hand are more realistic playground for unified picture leading to observable physics.

## 5 Conclusions

In this paper we considered string inspired solution generating method and discussed some general properties on specific examples related to string/gauge theory duality. The method has potentially wide range of applications. We hope that, although not pretending to be complete, the review part as well as the new

[8] In [23] another regime where the $\gamma$ deformation scales to zero is realized

results will be helpful to get oriented in the vast literature and find reformulation of this method in the context of many physical problems. Let us make a short recap.

We presented in some details a simple explicit introduction to the solution generating technique suggested in [2]. The general strategy was applied to several simple (but important) examples of exactly marginal deformations of (super)conformal gauge theories, such as the deformations of $\mathcal{N} = 4$ Super Yang-Mills theory considered in [3], whose $\beta$-deformed gravity duals (the so-called $\beta$-deformations) were derived in [2]. Another case of such interesting deformations is provided by gravity duals of non-commutative gauge theories thus, proving some universality of the method. We briefly mentioned another often studied case, namely the one of "dipole" theories [16–18], which are non-local theories living on an ordinary commutative space. Although technically not very complicated, the examples studied here have strikingly rich physical consequences and deserve further study.

The idea of generating scales in the physical theories is not new and is broadly discussed from various points of view in more than half of a century. Its realizations, in a broader sense, have many faces and one of them led to the constructions of dipole deformed, thus non-local, and theories on non-commutative spacetimes. In any case, the good news is that such theories doesn't suffer from divergencies. The bad news is that it is still very difficult to construct, more over to study them. The examples of the $\beta$-deformation of gauge theories and of their type II gravity duals are important in several ways. First of all, they provide an elegant method of constructing such theories. Secondly, the method making use of string theory provides an unified way of treating dipole and non-commutative theories. The approach offers broad list of applications. The most attractive one seems to be the string/gauge theory duality, in particular AdS/CFT correspondence. The remarkable progress made in studying strong coupling phenomenon via AdS/CFT at zero and finite temperature makes the results and the future research on the issues even more attractive.

## Acknowledgments

I would like to thank my colleagues and collaborators for spending many hours of helpful discussions and communications on the subject over the last years.

This work was supported in part by the Austrian Research Fund FWF # P22000, NSFB VU-F-201/06 and DO 02-257.

## References


[1] J.M. Maldacena (1998) *Adv. Theor. Math. Phys*. **2**) 231-252;
— (1999) *Int. J. Theor. Phys*. **38** 1113-1133, hep-th/ 9711200;
E. Witten (1998) *Adv. Theor. Math. Phys*. **2** 253, hep-th/9802150;
S.S. Gubser, I.R. Klebanov and A.M. Polyakov (1998) *Phys. Lett*. **B428** 105, hep-th/9802109.

[2] O. Lunin and J.M. Maldacena (2005) *JHEP* **0505** 033, [arXiv:hep-th/0502086].

[3] R.G. Leigh and M.J. Strassler (1995) *Nucl. Phys. B* **447** 95, [arXiv:hep-th/9503121].

[4] N. Seiberg and E. Witten (1999) *JHEP* **9909** 032, [arXiv:hep-th/9908142].

[5] S. Frolov (2005) *JHEP* **0505** 069, [arXiv:hep-th/0503201].

[6] A. Catal-Ozer (2006), *JHEP* **0602** 026, [arXiv:hep-th/0512290].

[7] A. Hashimoto and N. Itzhaki (1999) *Phys. Lett. B* **465** ) 142, [arXiv:hep-th/9907166].

[8] J.M. Maldacena and J.G. Russo (1999) *JHEP* **9909** 025, [arXiv:hep-th/9908134].

[9] C.M. Hull and P.K. Townsend (1995) "Unity of superstring dualities," *Nucl. Phys. B* **438** 109, [arXiv:hep-th/9410167].

[10] A. Giveon, M. Porrati and E. Rabinovici (1994) *Phys. Rep.* **244** 77, [arXiv:hep-th/9401139].

[11] E. Bergshoeff, C.M. Hull, and T. Ortin (1995) *Nul. Phys. B* **451** 547–578, [`http://xxx.lanl.gov/abs/hep-th/9504081hep-th/9504081`].

[12] M. Cvetic, H. Lu, C.N. Pope, and K.S. Stelle (2000) *Nucl. Phys. B* **573** 149-176, [`http://xxx.lanl.gov/abs/hep-th/9907202hep-th/9907202`].

[13] M. Fukuma, T. Oota, and H. Tanaka (2000) *Prog. Theor. Phys.* **103** 425–446, [`http://xxx.lanl.gov/abs/hep-th/9907132hep-th/9907132`].

[14] S.F. Hassan (2000) *Nucl. Phys. B* **568** 145-161, [`http://xxx.lanl.gov/abs/hep-th/9907152hep-th/9907152`].

[15] P. Sundell (2001) *Int. J. Mod. Phys. A* **16** 3025-3040, [`http://xxx.lanl.gov/abs/hep-th/0011283hep-th/0011283`].

[16] A. Bergman and O.J. Ganor: "Dipoles, Twists and Noncommutative Gauge Theory", [arXiv:hep-th/0008030].

[17] S. Iso, H. Kawai and Y. Kitazawa (2000) *Nucl. Phys. B* **576** 375, [hep-th/0001027].

[18] A. Bergman, K. Dasgupta, O.J. Ganor, J.L. Karczmarek and G. Rajesh (2002) *Phys. Rev. D* **65** 066005, hep-th/0103090.

[19] S. A. Frolov, R. Roiban and A. A. Tseytlin, JHEP **0507** (2005) 045 [arXiv:hep-th/0503192].

[20] L.F. Alday, G. Arutyunov and S. Frolov: "Green-Schwarz Strings in TsT-transformed backgrounds", [arXiv:hep-th/0512253].

[21] U. Gursoy and C. Nunez (2005) *Nucl. Phys. B* **725** 45, [arXiv:hep-th/0505100].

[22] N.P. Bobev, H. Dimov and R.C. Rashkov: "Semiclassical strings in Lunin-Maldacena background", arXiv:hep-th/0506063.

[23] D.V. Bykov and S. Frolov (2008) *JHEP* **0807** 071, [arXiv:0805.1070 [hep-th]].

[24] H. Dimov, M. Michalcik and R.C. Rashkov (2009) *JHEP* **0910** 019, [arXiv:0908.3065 [hep-th]].

[25] N. Beisert and R. Roiban (2005) *JHEP* **0508** 039, [arXiv:hep-th/0505187].

[26] I.R. Klebanov and E. Witten (1998) *Nucl. Phys. B* **536** 199, [arXiv:hep-th/9807080].

[27] R. Ishizeki and M. Kruczenski (2007) *Phys. Rev. D* **76** 126006, [arXiv:0705.2429 [hep-th]].

[28] S. Benvenuti and E. Tonni (2009) *JHEP* **0902** 041, [arXiv:0811.0145 [hep-th]].

[29] D. Berenstein and V. Jejjala and R.G. Leigh (2000) *Nucl. Phys. B* **589** 196, hep-th/0005087;

S.S. Razmat: "Marginal deformations of N=4 SYM and of its supersymmetric orbifold descendants", hep-th/0204043;

O/ Aharony and S.S. Razmat (2002) *JHEP* **0205**, 029, hep-th/0204045;

O. Aharony, B. Kol and S. Yankielowicz (2002) *JHEP* **0206** 039, hep-th/0205090;

B. Kol (2002) *JHEP* **0209** 046, hep-th/0205141;

N. Dorey, T.J. Hollowood and S. Prem Kumar (2002) *JHEP* **0212** 003, hep-th/0210239;

V. Niarchos, N. Prezas (2003) *JHEP* **0306** 015, hep-th/0212111;

N. Dorey (2004) *JHEP* **0408** 043, hep-th/0310117;

F. Benini (2004) *JHEP* **0412** 068, hep-th/0411057;

N. Dorey and T.J. Hollowood: "On the Coulomb branch of a marginal deformation of N=4 SUSY Yang-Mills", hep-th/0411163;

S. Benvenuti, A. Hanany: "Conformal manifolds for the conifold and other toric field theories", hep-th/0502043.