

# Entanglement Entropy for the Charged BTZ Black Hole

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**Abstract.** Using the AdS/CFT correspondence we calculate the explicit form of the entanglement entropy for the charged BTZ (Banados-Teitelboim-Zanelli) black hole. The leading term in the large temperature expansion of the entropy function for this black hole reproduces its Bekenstein-Hawking entropy and the subleading term, representing the first corrections due to quantum entanglement, behaves as a logarithm of the BH entropy. It has also been obtained an inverse of area term in subleading order similar to the reported when considering Hawking radiation as quantum tunneling of particles through the event horizon.

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## 1 Introduction

Recently, the idea of holography has gained a great attention. It claims that the degrees of freedom in  $(d + 2)$ -dimensional quantum gravity will be comparable to those of quantum many body systems in  $(d + 1)$  dimensions. This was essentially found by remembering that the entropy of a black hole is not proportional to its volume, but to the area of its event horizon (Bekenstein-Hawking entropy),

$$S^{BH} = \frac{A}{4G}, \quad (1)$$

where  $G$  is the Newton constant.

The discovery of the AdS/CFT correspondence show explicit examples where the holography is manifestly realized. This correspondence states that the quantum gravity on  $(d+2)$ -dimensional anti-de Sitter space-time ( $\text{AdS}_{d+2}$ ) is equivalent to a certain conformal field theory in  $d + 1$  dimensions ( $\text{CFT}_{d+1}$ ). However, the fundamental mechanism of the AdS/CFT correspondence is not known today.

In this paper we investigate quantum entanglement in the context of three-dimensional (3D) AdS gravity, in particular of the charged Banados-Teitelboim-Zanelli (BTZ) black hole [1], using the AdS3/CFT2 correspondence. In order to obtain the entropy function we will use the standard method for studying correlations in QFT described in [2] in which two external length-scales are introduced in the boundary 2D CFT. One is a thermal wavelength  $\beta = 1/T$  (with  $T$  the CFT's temperature) and a spatial length  $\gamma$  which is the measure of the observable spatial region of the 2D universe.

## 2 Entanglement Entropy

Consider a  $2D$  space-time with a compact space-like dimension with  $S^1$  topology and length  $\ell$ . If the only part of the universe accessible for measurement is a space-like slice  $\Sigma$  with length  $\gamma$ , we loose information about the degrees of freedom localized in the complementary region  $\Sigma'$ . The entanglement entropy originated by tracing over the unobservable degrees of freedom is given by the von Neumann entropy

$$S^{\text{ent}} = -\text{Tr}_{\Sigma} [\rho_{\Sigma} \ln \rho_{\Sigma}], \quad (2)$$

where  $\rho_{\Sigma}$  is the reduced density matrix, obtained by tracing the density matrix  $\rho$  over the states of the  $\Sigma'$  region

$$\rho_{\Sigma} = \text{Tr}_{\Sigma'} \rho. \quad (3)$$

The entanglement entropy for the ground state of the  $2D$  CFT at zero temperature, with a space-like dimension with  $S^1$  topology and infinite time-like direction (cylinder), is given in [2, 3]

$$S_{\text{cyl}}^{\text{ent}} = \frac{c + \bar{c}}{6} \ln \left( \frac{\ell}{\epsilon\pi} \sin \frac{\pi\gamma}{\ell} \right), \quad (4)$$

where  $c$  and  $\bar{c}$  are the central charges of the  $2D$  CFT and  $\epsilon$  is an ultra-violet cutoff used to regularize the divergence that appears because of the sharp boundary between the regions  $\Sigma$  and  $\Sigma'$ .

When  $\ell \gg \gamma$ , the space-like dimension becomes infinite and the entanglement entropy is independent of  $\ell$ . Therefore, the entanglement entropy for a  $2D$  CFT at zero temperature on a plane is given by

$$S_{\text{cyl}}^{\text{ent}} = \frac{c + \bar{c}}{6} \ln \left( \frac{\gamma}{\epsilon} \right). \quad (5)$$

However, if we consider a  $2D$  CFT at finite temperature  $T = 1/\beta$  on a plane, the entanglement entropy is

$$S_{\text{cyl}}^{\text{ent}} = \frac{c + \bar{c}}{6} \ln \left( \frac{\beta}{\epsilon\pi} \sinh \frac{\pi\gamma}{\beta} \right). \quad (6)$$

As is well-known, from the correspondence between 3D AdS gravity and 2D CFT (AdS<sub>3</sub>/CFT<sub>2</sub>) there follows that the entanglement entropy of a 2D CFT contains information about quantum gravity correlations of its 3D gravity holographical dual. In this particular case, the CFT has a central charge given in [3,4]

$$c = \bar{c} = \frac{3L}{2G}, \quad (7)$$

where  $L$  is the AdS length and  $G$  is the 3D Newton constant.

### 3 The Charged BTZ Black Hole

The (2+1)-dimensional BTZ (Banados-Teitelboim-Zanelli) black holes have obtained a great importance in recent years because they provide a simplified model for exploring some conceptual issues, not only about black hole thermodynamics [5, 6] but also about quantum gravity and string theory. The charged BTZ black hole is a solution of the (2+1)-dimensional gravity theory with a negative cosmological constant  $\Lambda = -1/L^2$ . The metric is given in [1]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \quad (8)$$

where

$$f(r) = -M + \frac{r^2}{L^2} - \frac{Q^2}{2} \ln \left[ \frac{r}{L} \right] \quad (9)$$

is known as the lapse function and  $M$  and  $Q$  are the mass and electric charge of the BTZ black hole, respectively. The lapse function vanishes at the radii  $r = r_{\pm}$ , where  $r_+$  gives the position of the event horizon. The electric potential of this black hole is

$$\Phi = \frac{\partial M}{\partial Q} = -Q \ln \left[ \frac{r}{L} \right] \quad (10)$$

while the Hawking temperature is given as usual

$$T_H = \frac{1}{4\pi} \left| \frac{df(r)}{dr} \right|_{r=r_+} = \frac{r_+}{2\pi L^2} - \frac{Q^2}{8\pi r_+}, \quad (11)$$

and its entropy by

$$S_{\text{BTZ}}^{\text{BH}} = \frac{A}{4G} = \frac{\pi r_+}{2G}. \quad (12)$$

The spinless charged BTZ black hole can be considered as the thermalization at temperature  $T = T_H = \frac{1}{\beta_H}$  of the charged AdS space-time. On the 2D boundary of the AdS space-time, and in the large temperature limit ( $r_+ \gg L$ ), this thermalization corresponds to a plane/cylinder transformation that maps the

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CFT on the plane in the CFT on the cylinder. The conformal map plane/cylinder has the form

$$z = \exp \left[ \frac{2\pi}{\beta_H} (\varphi + it) \right]. \quad (13)$$

The above transformation is the asymptotic form of the map between the BTZ black hole and AdS<sub>3</sub> in Poincare coordinates, and corresponds to the conformal transformation that maps the entanglement entropy of a CFT at zero temperature (on the plane) in that of a CFT at finite temperature (in the cylinder). Correspondingly, the holographic entanglement entropy of the charged AdS space-time becomes the holographic entanglement entropy of the BTZ black hole

$$\tilde{S}_{\text{BTZ}}^{\text{ent}} = S_{\text{CFT}}^{\text{ent}}(\gamma = 2\pi L, \beta = \beta_H) \quad (14)$$

$$= \frac{c}{3} \ln \left[ \frac{8r_+ L^2}{\epsilon (4r_+^2 - L^2 Q^2)} \sinh \left( \frac{\pi r_+}{L} - \frac{\pi L Q^2}{4r_+} \right) \right]. \quad (15)$$

This entanglement entropy depends on the UV cutoff given by the parameter  $\epsilon$ . To obtain a renormalized entropy we will subtract the contribution of the vacuum (i.e. the zero mass, zero temperature BTZ black hole solution). This is given by equation (5),

$$S_{\text{vac}}^{\text{ent}} = S_{\text{cyl}}^{\text{ent}}(\gamma = 2\pi L) = \frac{c}{3} \ln \left( \frac{2\pi L}{\epsilon} \right). \quad (16)$$

Therefore, the renormalized entanglement entropy is given by

$$S_{\text{BTZ}}^{\text{ent}} = \tilde{S}_{\text{BTZ}}^{\text{ent}} - S_{\text{vac}}^{\text{ent}} \quad (17)$$

$$S_{\text{BTZ}}^{\text{ent}} = \frac{c}{3} \ln \left[ \frac{4r_+ L}{\pi (4r_+^2 - L^2 Q^2)} \sinh \left( \frac{\pi r_+}{L} - \frac{\pi L Q^2}{4r_+} \right) \right], \quad (18)$$

or using (7),

$$S_{\text{BTZ}}^{\text{ent}} = \frac{L}{2G} \ln \left[ \frac{4r_+ L}{\pi (4r_+^2 - L^2 Q^2)} \sinh \left( \frac{\pi (4r_+^2 - L^2 Q^2)}{4r_+ L} \right) \right]. \quad (19)$$

Note that, as expected, the renormalized entanglement entropy goes to zero as  $4r_+^2 - L^2 Q^2 \rightarrow 0$  or  $r_+ \rightarrow LQ/2$  (this is the condition for the charged BTZ black hole ground state). The entanglement entropy for the charged BTZ black hole reduces to the entropy obtained for the spinless BTZ solution reported in [2] and using the relation between 2D and 3D Newton constant

$$G_2 = \frac{L}{4G}, \quad (20)$$

it reproduces exactly the result in [7].

In the large temperature limit ( $r_+ \gg L$ ), equation (19) can be expanded as

$$S_{\text{BTZ}}^{\text{ent}} \approx \frac{\pi r_+}{2G} - \frac{\pi}{2G} \frac{L^2 Q^2}{4r_+} - \frac{L}{2G} \ln \left[ \frac{\pi (4r_+^2 - L^2 Q^2)}{4r_+ L} \right] + O\left(\frac{L}{r_+}\right)^2 \quad (21)$$

$$S_{\text{BTZ}}^{\text{ent}} \approx \frac{\pi r_+}{2G} - \frac{L}{2G} \ln \left[ \frac{\pi r_+}{L} \right] + O\left(\frac{L}{r_+}\right) \quad (22)$$

$$S_{\text{BTZ}}^{\text{ent}} \approx S_{\text{BTZ}}^{\text{BH}} - \frac{L}{2G} \ln \left[ \frac{\pi r_+}{L} \right] + O\left(\frac{L}{r_+}\right). \quad (23)$$

The first term in the expansion is the Bekenstein-Hawking entropy for the BTZ black hole,  $S_{\text{BTZ}}^{\text{BH}} = \frac{\pi r_+}{2G}$ , and describes the situation in which thermal fluctuations dominate, *i.e.* the entanglement entropy becomes the thermodynamical entropy in this limit. The second term describes the first corrections to the entropy due to quantum entanglement. This behavior reproduces results of Cadoni and Melis [2], as well as the logarithmic term obtained in [8, 9].

It is also interesting that the second term in (21) is an inverse of area term in subleading order similar to the one obtained by S.K. Modak in the equation (63) in [8] for the rotating BTZ black hole and also in the general case studied by Akbar and Saifullah [10]. In this paper, the authors calculate the entropy of the black hole considering the quantum tunneling of particles through the event horizon, using two unknown parameters,  $\alpha_1$  and  $\alpha_2$ . The present analysis let us fix those parameters for the charged BTZ black hole as

$$\alpha_1 = \frac{L}{8\pi G} \quad (24)$$

$$\alpha_2 = \frac{L^3 Q^2}{64G}. \quad (25)$$

## 4 Conclusion

We have derived the entanglement entropy function for the charged BTZ black hole, obtaining an expansion with a leading term that corresponds to the thermodynamical entropy and subleading terms that describe the corrections due to quantum entanglement. This result sheds light on the meaning of the holographic entanglement entropy for the charged BTZ black hole. The resulting function for the entropy seems to have a meaning only in the regime  $r_+ \gg L$ , *i.e.* for macroscopic black holes and the quantum corrections are similar to those obtained when considering Hawking radiation as quantum tunnelling of particles through the event horizon.

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