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Bianchi Type-III Cosmological Models in $f(T)$ Theory of Gravity

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Abstract. In this research paper, we have investigated on $f(T)$ theory of gravity,as an alternative theory of gravity with combination of Bianchi Type-III cosmological model. Our aim to focus on the field equations and analyze physical and kinematical parameters over red shift. With the help of these parameters, we obtain a model which reproduce the power-law expansion of the universe.

KEY WORDS: Bianchi type-III, $f(T)$ Gravity, Cosmology.

1 Introduction

In General Relativity gravity is described by geometry: the distribution of matter changes the space time curvature, which in turn dictates how matter moves. While General Relativity works great locally, mysteries like dark energy and dark matter imply we may need modify gravity on a universe scale [1–4]. Modified gravity [5, 6] theories propose an alternative: the acceleration might stem from changes in the fundamental nature of gravity itself, removing the need for that extra unexplained constant. Modified gravity has two main approaches: one describes the two accelerated phases of cosmic expansion at early and late times respectively, and the other approach introduces the concept of the dark energy in the General Relativity framework [4, 7]. There are many different types of modified theories of gravity, among them, one of the most popular is $f(R)$ gravity [8, 9], which is a modification of Einstein-Hilbert action, where R is a Ricci scalar and its equations are of order four. $f(R)$ gravity is extended to formulate $f(R,T)$ gravity [10], which is also a modified theory of gravity, where R is the Ricci scalar and T is the trace of energy momentum tensor. All these modified gravity theories are built upon the foundation of Einstein General Relativity, meaning they all begin with the concept of space time curvature [11].

At the present work, we discussed a new modified theory of gravity known as $f(T)$ theory of gravity [12, 13], where T is the Torsion. Torsion is the main cause that explains the universes late time accelerated expansion [14] and its equations are second order. $f(T)$ theory of gravity is the modification of teleparallel gravity. The simplest teleparallel gravity models works exactly the same as General Relativity [15]. The gravitational field can be described using either curvature or torsion, and both description involved the same fundamental aspects of gravity. Einstein had also constructed Teleparallel Equivalent of General Relativity [16–21], within the teleparallel framework all gravitational effects [22]. The General Relativity is considered as a theory of geometry, and its framework is based on the Levi Civita Connection which has no torsion and $f(T)$ theory is considered as a gauge theory [23, 24], and its framework is based on the Weitzenböck Connection which has no curvature. The theory of $f(T)$ gravity is a practical observation-driven expansion of standard teleparallel gravity conceptually borrowing from how $f(R)$ gravity expands upon General Relativity [25]. The Λ CDM model [26] is now the standard framework utilized by cosmologist to understand the universe large scale evaluation. Observation sources from various astronomical studies suggest that $f(T)$ gravity offer a competitive alternative to the Λ CDM model [27–36]. A significant body of research exists on $f(T)$ gravity, covering topics like, reconstruction methods [37, 38], observational constraints [34, 39], and perturbation [40], these are ultimately helps us for better understand the theory's cosmological dynamics, and viability.

In this paper, we are interested to study some cosmological solution in $f(T)$ gravity, specifically considered Bianchi Type-III metric. We are interested in determining the exact solutions of the field equations for $f(T)$ gravity and also interested in finding some physical and kinematics parameters in res shift to determine whether our universe is accelerating or not.

The section of the paper is as follows. In Section-2 our model with $f(T)$ gravity is defined, especially Bianchi Type-III space time and after modifying the action, the resulting gravitational field equations are presented. In Sections-3 and 4 we define the field equations and for analytic the kinematical and physical parameter over red shift. Reconstruction of $f(T)$ model for $f(T) = \alpha T + \beta$ and we examine the corresponding field equations and these solutions. Section-5 summarizes the discussion and final conclusion.

2 Metric and Equations of Motion

Here we define a brief introduction of $f(T)$ gravity, where the line element is describe by the geometry of space time. For a space time, metric is described by the components of $g_{\mu\nu}$, and the line element is defined as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

Here the metric tensor is defined by $g_{\mu\nu}$, which is symmetric and $dx^m u$ and $dx^n u$ are the differential space time coordinates.

Moreover, the line element can be changed into Minkowski description with the help of tetrad transformation

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{kl}\theta^k\theta^l\,, \tag{2}$$

$$dx^\mu = e_k^{\ \mu}\theta^k, \quad \theta^k = e^k_{\ \mu}dx^\mu\,, \tag{3}$$

where $\eta_{kl} = diag[1,-1,-1,-1]$ and $e_k^{\ \mu}e^k_{\ \nu} = \delta^\mu_\nu$ and the square root of the metric determinant is given by $\sqrt{-g} = det[e^k_{\ \mu}] = e$, where the tetrad denoted by $e_k^{\ \mu}$, which is represent the dynamic fields of the theory.

We define the Weitzenböck connection in terms of tetrad as

$$\Gamma^\alpha_{\ \mu\nu} = e_k^{\ \alpha}\partial_\nu e^k_{\ \mu} = -e^k_{\ \mu}\partial_\nu e_k^{\ \alpha}. \tag{4}$$

The antisymmetric part of the connection (4) is defined as

$$T^\alpha_{\ \mu\nu} = \Gamma^\alpha_{\ \mu\nu} - \Gamma^\alpha_{\ \nu\mu} = e_k^{\ \alpha}(\partial_\mu e^k_{\ \nu} - \partial_\nu e^k_{\ \mu})\,, \tag{5}$$

where, $T^\alpha_{\ \mu\nu}$ is the torsion tensor. The contorsion tensor is defined by

$$K^{\mu\nu}_{\ \ \ \alpha} = -\frac{1}{2}(T^{\mu\nu}_{\ \ \ \alpha} - T^{\mu\nu}_{\ \ \ \alpha} - T_\alpha^{\ \mu\nu}). \tag{6}$$

Now, we define an important tensor $S_\alpha^{\ \mu\nu}$ for making more clear the definition of the scalar equivalent to the scalar curvature of General Relativity.

$$S_\alpha^{\ \mu\nu} = \frac{1}{2}(K^{\mu\nu}_{\ \ \ \alpha} + \delta^\mu_\alpha T^{\lambda\nu}_{\ \ \ \lambda} - \delta^\nu_\alpha T^{\lambda\mu}_{\ \ \ \lambda}). \tag{7}$$

We define the torsion scalar as

$$T = T^\alpha_{\ \mu\nu}S_\alpha^{\ \mu\nu}. \tag{8}$$

By generalizing the teleparallel gravity, we define the functional action in $f(T)$ gravity

$$A = \int e[f(T) + L_{\text{matter}}]d^4x, \tag{9}$$

where $f(T)$ is define by algebraic function of the torsion scalar T, with respect to the tetrad, modifying the action (9) we obtain the following equation of motion [41]

$$S_\mu^{\ \nu\rho}\partial_\rho T f_{TT} + [e^{-1}e^k_{\ \mu}\partial_\rho(ee_k^{\ \alpha}S_\alpha^{\ \nu\rho}) + T^\alpha_{\ \lambda\mu}S_\alpha^{\ \nu\lambda}]f_T + \frac{1}{4}\delta^\nu_\mu f = 4\pi T^\nu_\mu, \tag{10}$$

where, $f_T = \dfrac{df(T)}{dT}$, $f_{TT} = \dfrac{d^2f(T)}{dT^2}$ and T^ν_μ is the energy momentum tensor,

these equation totally depend on the choice for the set of tetrad [42]. The energy momentum tensor is defined as

$$T_{\mu}^{\nu} = \text{diag}\,(\rho, -p_x, -p_y, -p_z), \tag{11}$$

here, ρ is the energy density and p_x, p_y, and p_z are the pressure along the x, y and z direction respectively.

3 Field Equations for Bianchi Type-III

A Bianchi Type-III universe metric is anisotropic cosmological model and provides a tool for understanding the universe evalution and expansion.

$$ds^2 = dt^2 - A^2(t)dx^2 - e^{-2\gamma x}B^2(t)dy^2 - C^2(t)dz^2\,, \tag{12}$$

where $A(t)$, $B(t)$ and $C(t)$ are the component of scale along the x, y and z direction respectively. Let us choose the following tetrad with the given related metric:

$$e_{\mu}^{\alpha} = \text{diag}\,[1, A, e^{-\gamma x}B, C]\,. \tag{13}$$

The determinant of the matrix (13) is

$$e = e^{-\gamma x}ABC. \tag{14}$$

The component of the torsion tensor (5) for the tetrad (13) are given by

$$T^{1}_{\ 41} = \frac{\dot{A}}{A}\,, \quad T^{2}_{\ 42} = \frac{\dot{B}}{B}\,, \quad T^{3}_{\ 43} = \frac{\dot{C}}{C}\,, \quad T^{2}_{\ 21}\,. \tag{15}$$

Now the components of the corresponding contorsion tensor (6) are

$$K^{41}{}_{1} = \frac{\dot{A}}{A}\,, \quad K^{42}{}_{2} = \frac{\dot{B}}{B}\,, \quad K^{12}{}_{2} = \frac{\gamma}{A^2}\,, \quad K^{43}{}_{3} = \frac{\dot{C}}{C}\,. \tag{16}$$

Now define the components of the new tensor $S_{\alpha}^{\mu\nu}$, in (7) are given by

$$\begin{aligned} S_{1}^{\ 14} &= \frac{1}{2}\Big(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big)\,, \qquad S_{2}^{\ 24} = \frac{1}{2}\Big(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\Big)\,, \\ S_{3}^{\ 34} &= \frac{1}{2}\Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\Big)\,, \qquad S_{4}^{\ 41} = S_{2}^{\ 21} = \frac{\gamma}{2A^2}\,. \end{aligned} \tag{17}$$

Therefore the corresponding torsion scalar T in (8) is given by

$$T = -2\Big(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\Big)\,. \tag{18}$$

The torsion scalar T is choosen such that it represents the dynamical expansion of the Bianchi Type-III spacetime. We adopt tetrad basis that eliminates the

boundary effects from the torsion scalar, while the spatial curvature γ is preserved in the field equations as a geometric curvature.

The field equation for the Bianchi Type-III space time, in (10) are obtained as

$$16\pi\rho = f(T) + 4[\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{\gamma^2}{A^2}]f_T(T)\,, \tag{19}$$

$$\begin{aligned} -16\pi p = f(T) &+ 2\Big[\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + 2\frac{\dot{B}\dot{C}}{BC}\Big]f_T(T) \\ &+ 2\Big[\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big]\dot{T}f_{TT}(T)\,, \end{aligned} \tag{20}$$

$$\begin{aligned} -16\pi p = f(T) &+ 2\Big[\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\Big]f_T(T) \\ &+ 2\Big[\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\Big]\dot{T}f_{TT}(T)\,, \end{aligned} \tag{21}$$

$$\begin{aligned} -16\pi p = f(T) &+ 2\Big[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{\gamma^2}{A^2}\Big]f_T(T) \\ &+ \Big[\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\Big]\dot{T}f_{TT}(T) \end{aligned} \tag{22}$$

$$\frac{\gamma}{2A^2}\Big[\Big(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\Big)f_T(T) - \dot{T}f_{TT}\Big] = 0\,, \tag{23}$$

$$\gamma\Big[\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\Big]f_T(T) = 0\,. \tag{24}$$

Where for the simplicity, we assumed $p_x = p_y = p_z = p$. We obtain the given relation from equation (24)

$$A = B\,. \tag{25}$$

By putting the value of equation (25) in equations (19)–(23), they become

$$16\pi\rho = f(T) + 4\Big[\Big(\frac{\dot{A}}{A}\Big)^2 + 2\frac{\dot{A}\dot{C}}{AC} - \frac{\gamma^2}{A^2}\Big]f_T(T)\,, \tag{26}$$

$$\begin{aligned} -16\pi p = f(T) &+ 2\Big[\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + (\frac{\dot{A}}{A})^2 + 3\frac{\dot{A}\dot{C}}{AC}\Big]f_T(T) \\ &+ 2\Big[\frac{\dot{A}}{C} + \frac{\dot{C}}{C}\Big]\dot{T}f_{TT}(T)\,, \end{aligned} \tag{27}$$

$$\begin{aligned} -16\pi p = f(T) &+ 2\Big[2\frac{\ddot{A}}{A} + 2\Big(\frac{\dot{A}}{A}\Big)^2 + 2\frac{\dot{A}\dot{C}}{AC} - \frac{\gamma^2}{A^2}\Big]f_T(T) \\ &+ 4\frac{\dot{A}}{A}\dot{T}f_{TT}(T)\,, \end{aligned} \tag{28}$$

$$\dot{T}f_{TT} = 0\,. \tag{29}$$

On equating (26)-(27), and put the value of (28), we get

$$\rho = \frac{1}{16\pi}[f(T) + 4((\frac{\dot{A}}{A})^2 + 2\frac{\dot{A}\dot{C}}{AC} - \frac{\gamma^2}{A^2})f_T(T)]\,, \tag{30}$$

$$p = -\frac{1}{16\pi}\Big[f(T) + 2\Big(3\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + 3\Big(\frac{\dot{A}}{A}\Big)^2 + 5\frac{\dot{A}\dot{C}}{AC} - \frac{\gamma^2}{A^2}\Big)f_T(T)\Big]\,. \tag{31}$$

The overall expansion or contraction of the universe is represented by the average scale factor, which is mathematically as the geometric mean of the scale factor along different spatial direction. Average scale factor (a) and volume (V) defined as

$$a = \sqrt[3]{ABC}\,, \qquad V = a^3\,. \tag{32}$$

To describe the expansion rate of the universe, we generalize the Hubble parameter

$$H = \frac{1}{3}[H_1 + H_2 + H_3]\,, \tag{33}$$

where H_1, H_2 and H_3 are the directional Hubble parameters along x, y, and z directions respectively. To analyze whether the model approaches isotropic or not, we define the mean anisotropic parameter as

$$A_m = \frac{1}{3}\sum_{i=1}^{n}\Big(\frac{H_i - H}{H}\Big)^2, \quad i = 1, 2, 3\,. \tag{34}$$

The expansion scalar (θ) represents the rate at which is the volume of universe expand

$$\theta = 3H = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\,. \tag{35}$$

The shear scalar (σ^2), is characterizes distortion the rate in the evolution of the spatial metric

$$\sigma^2 = \frac{3}{2}A_m H^2 = \frac{1}{2}(H_1^2 + H_2^2 + H_3^2 - \frac{\theta^2}{3})\,. \tag{36}$$

The deceleration parameter q is characterizes that the universe expansion is accelerating or decelerating over time

$$q = \frac{\ddot{a}a}{\dot{a}^2} = -1 + \frac{d}{dt}\frac{1}{H}\,, \tag{37}$$

where $q < 0$ signifies expansion, $q > 0$ slowing down expansion and $q = 0$ indicate constant rate of expansion of universe.

4 Solution of the Field Equation; Physical and Kinematical Parameters

From (26)-(28) we have three differential equation five unknown variables (A, C, f, ρ, p), now we assume the relation between the metric coefficient as

$$A = C^n \,. \tag{38}$$

To the determination of $a(t)$, we choose the relation between Hubble Parameter and average scale factor by [43]

$$H = ma^{\frac{-1}{m}} \,, \tag{39}$$

$m > 0$, solving the equations (33) and (39), we get

$$a = (t + c)^m \,, \tag{40}$$

where c is constant.

From (32), (38) and (40), we have the following values:

$$A = (t + c)^{\frac{3mn}{2n+1}} \,, \tag{41}$$

$$C = (t + c)^{\frac{3m}{2n+1}} \,. \tag{42}$$

To compare the theoretical and practical studies we will define redshift

$$\frac{a_0}{a} = 1 + z \,, \tag{43}$$

where $a_0 =$ present value of scale factor and we will fix it using the usual rules.

4.1 Kinematic parameters

To understand the expansion of the universe, we have described the value of the kinematic parameter using the equations. From equation (33)-(37), (40) and (43), we obtains

$$H = m(1 + z)^{\frac{1}{m}} \,, \tag{44}$$

$$\theta = 3m(1 + z)^{\frac{1}{m}} \,, \tag{45}$$

$$A_m = \frac{2(n-1)^2}{(2n+1)^2} \,, \tag{46}$$

$$\sigma^2 = \frac{(n-1)^2}{(2n+1)^2} 3m^2(1 + z)^{\frac{2}{m}} \,. \tag{47}$$

Here equation (40) provides the constant value of deceleration parameter

$$q = -1 + \frac{1}{m} \,. \tag{48}$$

$q > 0$ indicates universe is accelerating, and q lies in the range $+1 < q < 0$, so for the accelerating universe we have $m > 1$.

4.2 Physical parameters

From equations (18), (41) and (42) torsion T is as follows:

$$T = -18m^2 n\Big[\frac{(n+2)}{(2n+1)^2(t+c)^2}\Big]. \tag{49}$$

4.2.1 Action $f(T) = \alpha T + \beta$

$$f(T) = -18m^2 n\alpha\frac{(n+2)}{(2n+1)^2(t+c)^2} + \beta\,. \tag{50}$$

From equations (49) and (43), we get

$$T = 18m^2 n[\frac{(n+2)}{(2n+1)^2}](1+z)^{\frac{2}{m}}\,. \tag{51}$$

From equations (30)-(31) and (48), we obtain

$$\rho = \frac{1}{16\pi}\Big[\frac{18m^2 n\alpha(n+2)}{(2n+1)^2(t+c)^2} - \frac{2\gamma^2\alpha}{(t+c)^{\frac{6mn}{2n+1}}} + \beta\Big], \tag{52}$$

$$p = \frac{1}{16\pi}\Big[\frac{\alpha(3m(3m(4n^2+n+1)) - (6n^2+5n+1))}{(t+c)^2} - \frac{\gamma^2(2n+1)^2}{(t+c)^{\frac{6mn}{2n+1}}} + \beta(2n+1)^2\Big]. \tag{53}$$

5 Result and Discussion

In this section we investigated the dynamical behavior of $f(T)$ gravity and its associated cosmological parameters. At late times the exponential models converges to de sitter expansion. By assuming an action of the form $f(T) = \alpha T + \beta$, we analyzed our results as follows.

1. Figure 1 illustrates the Hubble parameter and shows that if $x = 1$ at $z = 0$ (the dimensionless parameter) then it representing the current expansion rate, setting $m < 1$, the model exhibits a rapid growth of $H(z)$ as redshift and for $m > 1$ $H(z)$ is slow increase if we move to higher redshift, this type of behavior is key signature of modified gravity,suggesting the late time acceleration of the universe, driven primarily by dark energy domination.
2. Figure 2 illustrate the expansion scalar (θ) and shear scalar (σ^2). The expansion scalar relates to the Hubble parameter,as z increases, θ also increases, it signifies a faster expansion during during past epoch and gradually increasing in shear scalar indicate that the early universe was intrinsically very homogeneous (smooth).

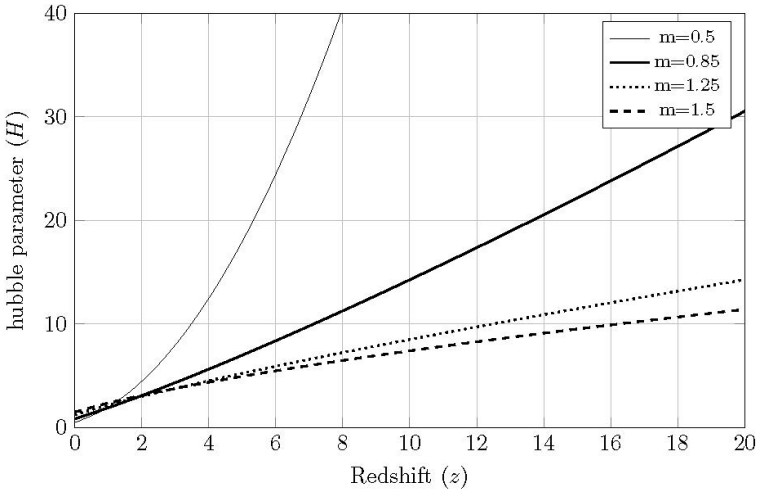

Figure 1. Pictorial illustration for Hubble parameter (H) with red shift (z).

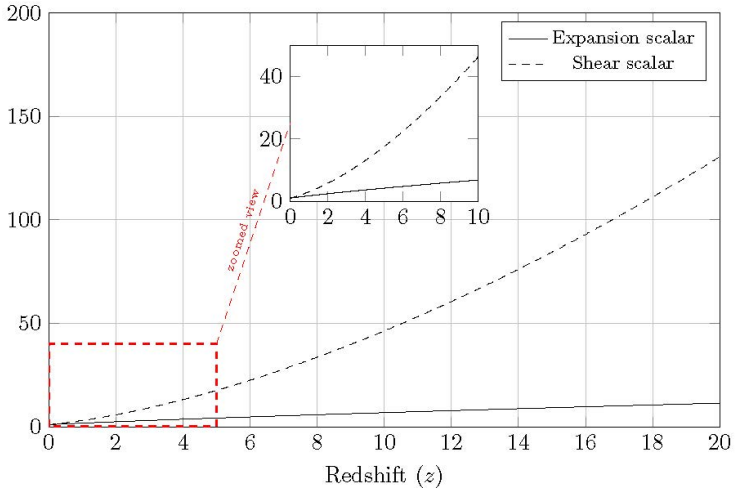

Figure 2. Pictorial illustration for expansion scalar θ, shear scalar σ^2 with redshift for $m = 1.25$, and $n = 1.5$.

3. Figure 3 illustrates that the torsion scalar (T) shows a negative behavior with redshift and if z increase, T increases negative, this align with teleparallel equivalent of gravity. It indicates that the torsion effect was at its peak in the high density early universe.
4. Figure 4 displays the behavior of energy density (ρ) with redshift (z). Energy density increase with redshift is consistent with standard cosmology highlighting the high-density nature of the early universe.

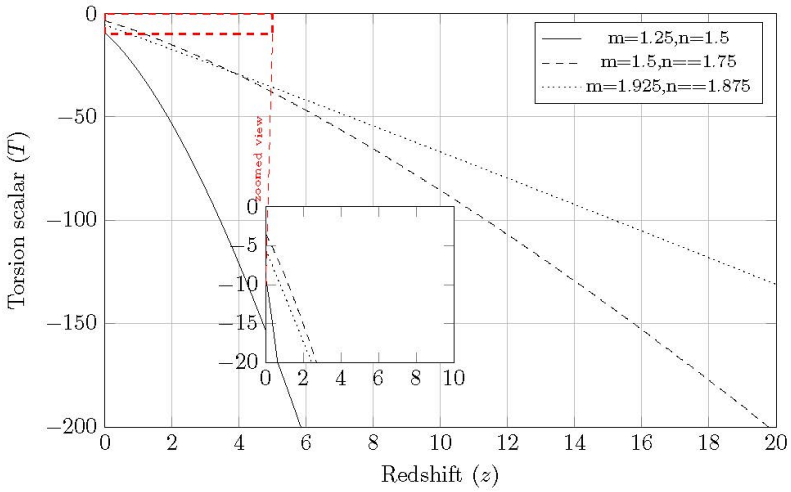

Figure 3. Pictorial illustration of torsion scalar T with red shift z.

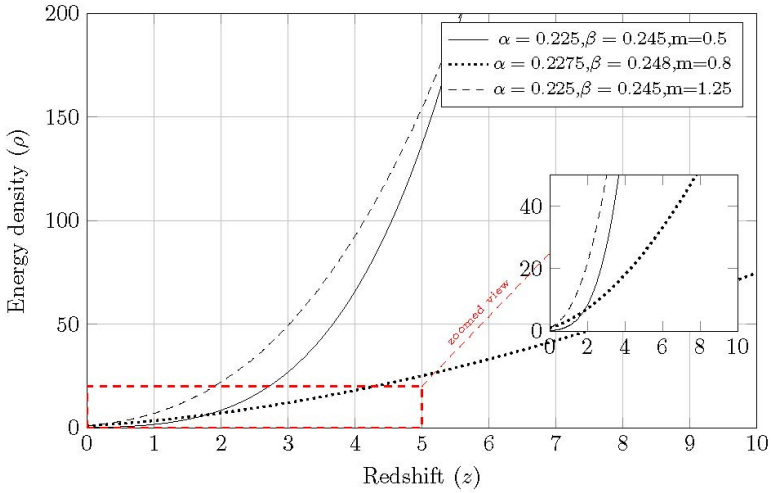

Figure 4. Pictorial illustration of energy density (ρ) with red shift (z), where $\gamma = 0.01$, $n = 1.5$.

5. Figure 5 illustrates the pressure (p) with redshift (z). The persistence of negative pressure at higher redshift suggests that the repulsive gravitational effect of dark energy is dominance's, leading to a continuously accelerating expansion of the universe.
6. Figure 6 illustrates the stability of model under the condition $m < 1.25$, whereas for $m \geq 1.25$ the model becomes unstable.

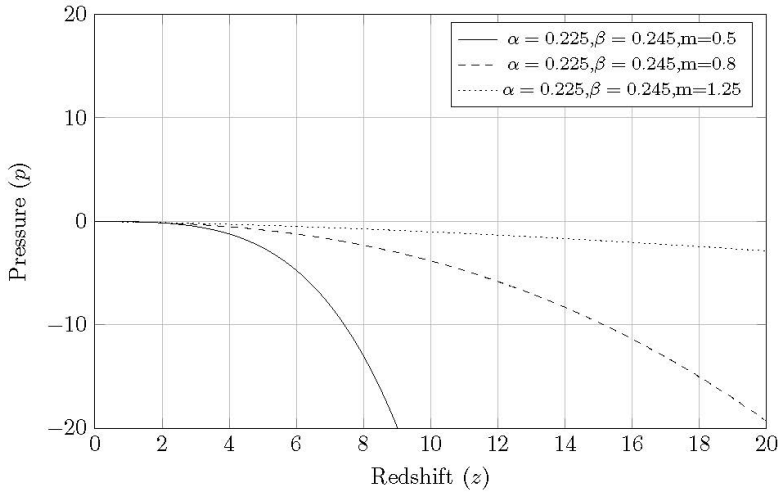

Figure 5. Pictorial illustration for pressure (p) with red shift, $\gamma = 0.01$, $n = 1.5$, $\alpha \in (0, 0.2256)$, and $0 < m < 1.265$.

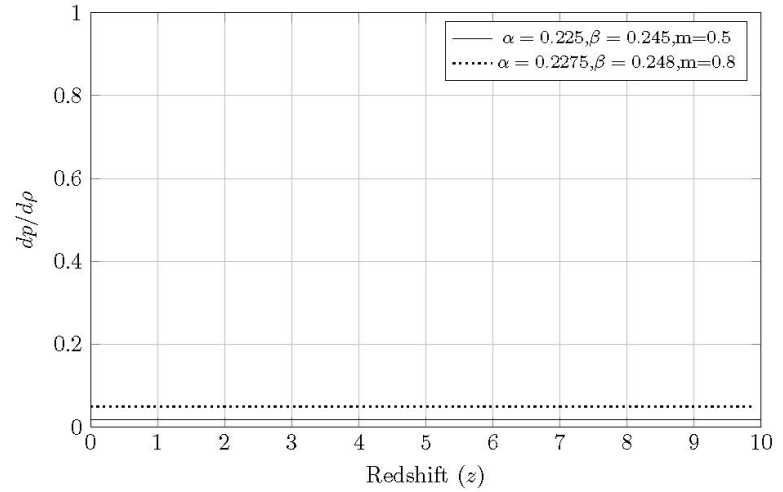

Figure 6. Stability factor $\gamma = 0.01$, $n = 1.5$.

6 Conclusion

We investigate the $f(T)$ gravity dynamics in the context of the Bianchi Type-III metric along with exponential models featuring different parameters. We consider the model $f(T) = \alpha T + \beta$, and examine the physical and kinematical behavior of $f(T)$ gravity . Based on the findings, the study indicates that the rate

of expansion of the universe is accelerating. The results shows that the rate of expansion depends on the cosmological parameters like energy density (ρ) and pressure (p), which characterizes the energy-momentum distribution within the universe. The analysis shows that energy density increases while the pressure decreases across redshift, confirming the accelerated expansion of the universe. This is further supported by the deceleration parameter, which converges to a constant value. Additionally the shear scalar shows a minimal increase at a very slow rate, indicating a universe that is a highly dynamics yet maintains a high degree of uniformity. this approach yields significant properties for modified theories of gravity.

References

[1] A.G. Riess, et al. (1998) Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.* **116**(3) 1009.

[2] S. Perlmutter (1999)Measurements of Ω and Λ from 42 high-redshift supernovae. *Astrophys. Journal* **517**(2)565.

[3] P.J.E. Peebles, Bharat Ratra (2003) The cosmological constant and dark energy. *Rev. Mod. Phys.* **75**(2) 559.

[4] E.J. Copeland, M. Sami, S. Tsujikawa (2006) Dynamics of dark energy. *Int. J. Mod. Phys. D* **15**(11) 1753-1935.

[5] S. Nojiri, S.D. Odintsov (2007) Introduction to modified gravity and gravitational alternative for dark energy. *Int. J. Geom. Methods Mod. Phys.* **4** 115-145.

[6] S. Capozziello, M. De Laurentis (2011) Extended theories of gravity. *Phys. Rep.* **509**(4-5) 167-321.

[7] Y.-F. Cai, E.N. Saridakis, M.R. Setare, J.-Q. Xia (2010) Quintom cosmology: theoretical implications and observations. *Phys. Rep.* **493**(1) 1-60.

[8] S. Nojiri, S.D. Odintsov (2011) Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models. *Phys. Rep.* **505** 59-144.

[9] K. Bamba, S. Capozziello, S. Nojiri, S.D. Odintsov (2012) Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests. *Astrophys. Space Sci.* **342**(1) 155-228.

[10] T. Harko, F.S.N. Lobo, S. Nojiri, S.D. Odintsov (2011) $f(R,T)$ gravity. *Phys. Rev. D* **84**(2) 024020.

[11] E.N. Saridakis (2018) Introduction to teleparallel and $f(T)$ gravity and cosmology. In: *The Fourteenth Marcel Grossmann Meeting On Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories: Proceedings of the MG14 Meeting on General Relativity, University of Rome "La Sapienza", Italy, 12-18 July 2015*, pp 1135-1140.

[12] G.R. Bengochea, R. Ferraro (2009) Dark torsion as the cosmic speed-up. *Phys. Rev. D* **79**(12) 124019.

[13] E.V. Linder (2010) Einstein's other gravity and the acceleration of the universe. *Phys. Rev. D* **81**(12) 127301.

[14] S.S. Nerkar, M.V. Dawande (2022) Dark Energy Bianchi Type-III Cosmological Models in $f(T)$ Theory of Gravity. *Bulg. J. Phys.* **49**(3) 209-218.

[15] J.W. Maluf (2013) The teleparallel equivalent of general relativity. *Ann. Phys. (Berlin)* **525**(5) 339–357.

[16] A. Einstein (1928) Riemannian geometry with maintaining the notion of distant parallelism. *Sitz. Preuss. Akad. Wiss* **217** 224.

[17] A. Unzicker, T. Case (2005) Translation of Einstein's attempt of a unified field theory with teleparallelism. arXiv preprint physics/0503046.

[18] C. Møller (1961) Conservation laws and absolute parallelism in general relativity. *Nordita Publ.* **64** 50.

[19] C. Pellegrini, J. Plebanski (1963) Tetrad fields and gravitational fields. *Kgl. Danske Videnskab. Selskab, Mat. Fys. Skrifter* **2**(4).

[20] K. Hayashi, T. Shirafuji (1979) New general relativity. *Phys. Rev. D* **19**(12) 3524.

[21] R. Aldrovandi, J.G. Pereira (2012) *Teleparallel gravity: an introduction*, Springer Science & Business Media.

[22] H.I. Arcos, J.G. Pereira (2004) Torsion gravity: a reappraisal. *Int. J. Mod. Phys. D* **13**(10) 2193-2240.

[23] S. Bahamonde, K.F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J.L. Said, J. Mifsud, E. Di Valentino (2023) Teleparallel gravity: from theory to cosmology. *Rep. Prog. Phys.* **86**(2) 026901.

[24] T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis (2012) Modified gravity and cosmology. *Phys. Rep.* **513** 1-189.

[25] B. Li, T.P. Sotiriou, J.D. Barrow (2011) Large-scale structure in $f(T)$ gravity. *Phys. Rev. D* **83**(10) 104017.

[26] V. Sahni, A.A. Sen (2017) A new recipe for $Lambda$CDM.*Eur. Phys. J. C* **77**(4) 1-8.

[27] P. Wu, H. Yu (2010) Observational constraints on $f(T)$ theory. *Phys. Lett. B* **693**(4) 415-420.

[28] S. Nesseris, S. Basilakos, E.N. Saridakis, L. Perivolaropoulos (2013) Viable $f(T)$ models are practically indistinguishable from Λ CDM. *Phys. Rev. D* **88**(10) 103010.

[29] S. Camera, V.F. Cardone, N. Radicella (2014) Detectability of torsion gravity via galaxy clustering and cosmic shear measurements. *Phys. Rev. D* **89**(8) 083520.

[30] S. Capozziello, O. Luongo, E.N. Saridakis (2015) Transition redshift in $f(T)$ cosmology and observational constraints. *Phys. Rev. D* **91**(12) 124037.

[31] S. Basilakos (2016) Linear growth in power law $f(T)$ gravity. *Phys. Rev. D* **93**(8) 083007.

[32] R.C. Nunes, S. Pan, E.N. Saridakis (2016) New observational constraints on $f(T)$ gravity from cosmic chronometers. *J. Cosmol. Astropart. Phys.* **2016**(08) 011.

[33] J.-Z. Qi, S. Cao, M. Biesiada, X. Zheng, Z.-H. Zhu (2017) New observational constraints on $f(T)$ cosmology from radio quasars. *Eur. Phys. J. C* **77**(8) 502.

[34] R.C. Nunes, A. Bonilla, S. Pan, E.N. Saridakis (2017) Observational constraints on $f(T)$ gravity from varying fundamental constants. *Eur. Phys. J. C* **77**(4) 230.

[35] B. Xu, H. Yu, P. Wu (2018) Testing viable $f(T)$ models with current observations. *Astrophys. J.* **855**(2) 89.

[36] S. Basilakos, S. Nesseris, F.K. Anagnostopoulos, E.N. Saridakis (2018) Updated constraints on $f(T)$ models using direct and indirect measurements of the Hubble parameter. J. Cosmol. Astropart. Phys. **2018**(08) 008.

[37] M.E. Rodrigues, M.J.S. Houndjo, D. Saez-Gomez, F. Rahaman (2012) Anisotropic universe models in $f(T)$ gravity. *Phys. Rev. D* **86**(10) 104059.

[38] R. El Ouardi, A. Bouali, I. El Bojaddaini, A. Errahmani, T. Ouali (2025) Model-Independent Reconstruction of $f(T)$ Gravity Using Genetic Algorithms. *Chin. Phys. C* **49** 115106.

[39] F.B.M. dos Santos, J.E. Gonzalez, R. Silva (2022) Observational constraints on $f(T)$ gravity from model-independent data. *Eur. Phys. J. C* **82**(9) 823.

[40] K. Izumi, Y.C. Ong (2013) Cosmological perturbation in $f(T)$ gravity revisited. *J. Cosmol. Astropart. Phys.* **2013**(06) 029.

[41] M.E. Rodrigues, M.H. Daouda, M.J.S. Houndjo, R. Myrzakulov, M. Sharif (2014) Inhomogeneous universe in $f(T)$ theory. *Gravit. Cosmol.* **20**(2) 80-89.

[42] C. Deliduman, B. Yapiskan (2011) Absence of relativistic stars in $f(T)$ gravity. arXiv:1103.2225.

[43] M.S. Berman (1983) A special law of variation for Hubble's parameter. *Nuovo Cim. B* **74**(2) 182-186.