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Advance of Planetary Perihelion in Post-Newtonian Gravity

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Abstract. We present an elementary derivation of the planetary advance of the perihelion for a general spherically symmetric line element in the post-newtonian approximation.

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1 Introduction

Einstein field equations are nonlinear, and therefore cannot in general be solved exactly. By imposing the symmetry requirements of time independence and spatial isotropy, the Schwarzschild solution can be obtained, but this solution can not actually describe in the solar system because it is not static and isotropic. The post-newtonian approximation is a systematic method in which we can describe a system of slowly moving particles bound together by gravitational forces. In this method we will use an expansion of geometric objects as the metric tensor, in inverse powers of the speed of light using the parameter $\beta = v/c$.

Several works had calculated the perihelion precession of planetary orbits based on Einstein's equations with the line element of Schwarzschild and the use of Euler-Lagrange equations. In this paper we present a simplified derivation using a post-newtonian approximation for a general spherically symmetric line element. In order to obtain the precession [1], we compare the Keplerian orbit with the curved space trajectory and making use of the invariance of Kepler's second law.

2 Post-Newtonian Metric

The Minkowski metric in polar coordinates is given by

$$ds^2 = dt^2 - dr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta \, d\varphi^2, \tag{1}$$

whereas a general curved metric with spherical symmetry has a line element that can be written as

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta d\varphi^2. \tag{2}$$

In the post-newtonian approximation the metric components g_{tt} and g_{rr} are expanded up to the fourth order in $\beta = v/c$ in the form

$$g_{tt} = 1 + \overset{(2)}{g}_{tt} + \overset{(4)}{g}_{tt} + O\left(\beta^6\right) \tag{3}$$

$$g_{rr} = -1 + \overset{(2)}{g}_{rr} + \overset{(4)}{g}_{rr} + O\left(\beta^6\right). \tag{4}$$

Thus, the relation between flat and curved spaces for the radial and time coordinates can be considered as the transformation

$$dt' = \left[1 + \frac{1}{2}\overset{(2)}{g}_{tt} + \frac{1}{2}\overset{(4)}{g}_{tt}\right] dt \tag{5}$$

and

$$dr' = \left[1 - \frac{1}{2}\overset{(2)}{g}_{rr} - \frac{1}{2}\overset{(4)}{g}_{rr}\right] dr, \tag{6}$$

where the binomial approximation has been applied.

3 The Advance of the Perihelion

In order to obtain the advance of the perihelion we will consider two elliptical orbits. First, the Keplerian orbit in flat space with coordinates (t, r) for which the element of area is given by

$$dA = \int_0^R r dr d\vartheta = \frac{R^2}{2} d\vartheta. \tag{7}$$

This equation gives the second law of Kepler,

$$\frac{dA}{dt} = \frac{R^2}{2}\frac{d\vartheta}{dt}. \tag{8}$$

The second elliptical orbit that we will consider is obtained in the coordinates (t', r') of the curved space. Here, the element of area is

$$dA' = \int_0^R r dr' d\vartheta \tag{9}$$

$$dA' = \int_0^R r \left[1 - \frac{1}{2}\overset{(2)}{g}_{rr} - \frac{1}{2}\overset{(4)}{g}_{rr}\right] dr d\vartheta \tag{10}$$

$$dA' = \frac{R^2}{2}\left(1 - \frac{1}{R^2}\int_0^R r \left[\overset{(2)}{g}_{rr} + \overset{(4)}{g}_{rr}\right] dr\right) d\vartheta, \tag{11}$$

where we have used (6). Kepler's second law is now

$$\frac{dA'}{dt'} = \frac{R^2}{2}\left(1 - \frac{1}{R^2}\int_0^R r\left[\overset{(2)}{g}_{rr} + \overset{(4)}{g}_{rr}\right]dr\right)\frac{d\vartheta}{dt'}, \tag{12}$$

which, using equation (5), gives

$$\frac{dA'}{dt'} = \frac{R^2}{2}\left(1 - \frac{1}{R^2}\int_0^R r\left[\overset{(2)}{g}_{rr}(r) + \overset{(4)}{g}_{rr}(r)\right]dr\right) \times\left[1 + \frac{1}{2}\overset{(2)}{g}_{tt}(R) + \frac{1}{2}\overset{(4)}{g}_{tt}(R)\right]^{-1}\frac{d\vartheta}{dt}. \tag{13}$$

Using again the binomial approximation it gives

$$\frac{dA'}{dt'} = \frac{R^2}{2}\Phi(R)\frac{d\vartheta}{dt}, \tag{14}$$

where we have defined

$$\Phi(R) = \left(1 - \frac{1}{R^2}\int_0^R r\left[\overset{(2)}{g}_{rr}(r) + \overset{(4)}{g}_{rr}(r)\right]dr\right) \times\left[1 - \frac{1}{2}\overset{(2)}{g}_{tt}(R) - \frac{1}{2}\overset{(4)}{g}_{tt}(R)\right]. \tag{15}$$

Applying this increase to change $d\vartheta$ to $d\vartheta'$ in a single orbit gives the expression

$$\int_0^{\Delta\vartheta'} d\vartheta' = \Delta\vartheta' = \int_0^{\Delta\vartheta = 2\pi}\Phi(R)d\vartheta. \tag{16}$$

In order to perform the integration in the right hand side we will consider the Keplerian ellipse using $R(\vartheta) = \dfrac{l}{1+\epsilon\cos\vartheta}$, where l is the *latus rectum* and ϵ is the eccentricity.

4 Particular Cases

4.1 Schwarzschild Metric

Up to second order in β^2, Schwarzschild's metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 + \frac{2M}{r}\right)dr^2 - r^2 d\vartheta^2 - r^2\sin^2\vartheta d\varphi^2, \tag{17}$$

and therefore

$$\overset{(2)}{g}_{rr}(r) = \overset{(2)}{g}_{tt}(r) = -\frac{2M}{r} \tag{18}$$

$$\overset{(4)}{g}_{rr}(r) = \overset{(4)}{g}_{tt}(r) = 0. \tag{19}$$

Function $\Phi(R)$ is given by

$$\Phi(R) = \left(1 - \frac{1}{R^2}\int_0^R r\overset{(2)}{g}_{rr}(r)\,dr\right)\left[1 - \frac{1}{2}\overset{(2)}{g}_{tt}(R)\right] \tag{20}$$

$$\Phi(R) = \left(1 + \frac{1}{R^2}\int_0^R 2M dr\right)\left(1 + \frac{M}{R}\right) \tag{21}$$

$$\Phi(R) = \left(1 + \frac{2M}{R}\right)\left(1 + \frac{M}{R}\right) \tag{22}$$

$$\Phi(R) = 1 + \frac{3M}{R} + \frac{2M^2}{R^2} \tag{23}$$

and therefore the increasing $\Delta\vartheta'$ is

$$\Delta\vartheta' = \int_0^{\Delta\vartheta=2\pi}\left[1 + \frac{3M}{R} + \frac{2M^2}{R^2}\right]d\vartheta. \tag{24}$$

In order to perform the integration, we will the Keplerian ellipse using $R(\vartheta) = \frac{l}{1+\epsilon\cos\vartheta}$, so

$$\Delta\vartheta' = \int_0^{\Delta\vartheta=2\pi}\left[1 + \frac{3M}{l}(1+\epsilon\cos\vartheta) + \frac{2M^2}{l^2}(1+\epsilon\cos\vartheta)^2\right]d\vartheta \tag{25}$$

$$\Delta\vartheta' = 2\pi + \frac{6\pi M}{l} + \frac{2\pi M^2}{l^2}\left(2+\epsilon^2\right). \tag{26}$$

Hence, the obtained perihelion advance for Schwarzschild's metric has the standard value $6\pi M/l$ plus an additional term of order β^4.

4.2 Reissner-Nordstrm Metric

Up to fourth order in β, the metric for an electrically charged and spherically symmetric object is

$$ds^2 = \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2-\left(1+\frac{2M}{r}-\frac{Q^2}{r^2}\right)dr^2-r^2d\vartheta^2-r^2\sin^2\vartheta d\varphi^2, \tag{27}$$

and therefore we will take the approximation terms

$$\overset{(2)}{g}_{rr}(r) = \overset{(2)}{g}_{tt}(r) = -\frac{2M}{r} \tag{28}$$

$$\overset{(4)}{g}_{tt}(r) = \frac{Q^2}{r^2}. \tag{29}$$

This time function $\Phi(R)$ is

$$\Phi(R) = \left(1 - \frac{1}{R^2}\int_0^R r\,\overset{(2)}{g}_{rr}(r)\,dr\right)\left[1 - \frac{1}{2}\overset{(2)}{g}_{tt}(R) - \frac{1}{2}\overset{(4)}{g}_{tt}(R)\right] \tag{30}$$

$$\Phi(R) = \left(1 + \frac{1}{R^2}\int_0^R 2M dr\right)\left(1 + \frac{M}{R} - \frac{Q^2}{2R^2}\right) \tag{31}$$

$$\Phi(R) = \left(1 + \frac{2M}{R}\right)\left(1 + \frac{M}{R} - \frac{Q^2}{2R^2}\right) \tag{32}$$

$$\Phi(R) = 1 + \frac{3M}{R} + \frac{2M^2}{R^2} - \frac{Q^2}{2R^2} - \frac{MQ^2}{R^3} \tag{33}$$

and the increasing $\Delta\vartheta'$ is now

$$\Delta\vartheta' = \int_0^{\Delta\vartheta=2\pi}\left[1 + \frac{3M}{R} + \frac{2M^2}{R^2} - \frac{Q^2}{2R^2} - \frac{MQ^2}{R^3}\right]d\vartheta. \tag{34}$$

In order to perform the integration we consider again $R(\vartheta) = \dfrac{l}{1+\epsilon\cos\vartheta}$, and therefore

$$\Delta\vartheta' = \int_0^{\Delta\vartheta=2\pi}\Big[1 + \frac{3M}{l}(1+\epsilon\cos\vartheta) + \Big(\frac{2M^2}{l^2} - \frac{Q^2}{2l^2}\Big)(1+\epsilon\cos\vartheta)^2 - \frac{MQ^2}{l^3}(1+\epsilon\cos\vartheta)^3\Big]d\vartheta \tag{35}$$

$$\Delta\vartheta' = 2\pi + \frac{6\pi M}{l} + \frac{2\pi M^2}{l^2}(2+\epsilon^2) - \frac{\pi Q^2}{2l^2}(2+\epsilon^2) - \frac{\pi M Q^2}{l^3}(2+3\epsilon^2) \tag{36}$$

We have obtained again the standard value $6\pi M/l$ and the β^2 order term for Schwarzschild's solution given above, but now we also obtain an additional term of order β^4 solely due to the electric charge of the central body which agree with the perihelion precession reported in [2] and [3]. Finally we obtain a term that is proportional to the product MQ^2 that is not reported in these papers.

Finally, it is also important to note that the terms that contain the electric charge have a negative contribution to the perihelion precession.

4.3 Schwarzschild-de Sitter and Schwarzschild-Anti de Sitter Metric

The Schwarzschild-de Sitter metric is

$$ds^2 = \left(1 - \frac{2M}{r} + \Lambda r^2\right)dt^2 - \left(1 + \frac{2M}{r} + \Lambda r^2\right)dr^2 - r^2 d\vartheta^2 - r^2\sin^2\vartheta d\varphi^2 \tag{37}$$

where Λ is the cosmological constant and therefore we will take

$$\overset{(2)}{g}_{rr}(r) = \overset{(2)}{g}_{tt}(r) = -\frac{2M}{r} \tag{38}$$

$$\overset{(4)}{g}_{rr}(r) = \overset{(4)}{g}_{tt}(r) = \Lambda r^2. \tag{39}$$

This time function $\Phi(R)$ is

$$\Phi(R) = \left(1 - \frac{1}{R^2}\int_0^R r[\overset{(2)}{g}_{rr}(r) + \overset{(4)}{g}_{rr}(r)]dr\right)\times \\ \times\left[1 - \frac{1}{2}\overset{(2)}{g}_{tt}(R) - \frac{1}{2}\overset{(4)}{g}_{tt}(R)\right] \tag{40}$$

$$\Phi(R) = \left(1 + \frac{1}{R^2}\int_0^R [2M - \Lambda r^3]dr\right)\left(1 + \frac{M}{R} - \frac{\Lambda R^2}{2}\right) \tag{41}$$

$$\Phi(R) = \left(1 + \frac{2M}{R} - \frac{1}{4}\Lambda R^2\right)\left(1 + \frac{M}{R} - \frac{\Lambda R^2}{2}\right) \tag{42}$$

$$\Phi(R) = 1 + \frac{3M}{R} + \frac{2M^2}{R^2} - \frac{5\Lambda MR}{4} - \frac{3\Lambda R^2}{4} + \frac{\Lambda^2 R^4}{8} \tag{43}$$

and the increasing $\Delta\vartheta'$ is now

$$\Delta\vartheta' = \int_0^{\Delta\vartheta=2\pi}\left[1 + \frac{3M}{R} + \frac{2M^2}{R^2} - \frac{5\Lambda MR}{4} - \frac{3\Lambda R^2}{4} + \frac{\Lambda^2 R^4}{8}\right]d\vartheta. \tag{44}$$

In order to perform the integration we consider again $R(\vartheta) = \frac{l}{1+\epsilon\cos\vartheta}$, and therefore

$$\Delta\vartheta' = \int_0^{\Delta\vartheta=2\pi}\left[1 + \frac{3M}{l}(1+\epsilon\cos\vartheta) + \frac{2M^2}{l^2}(1+\epsilon\cos\vartheta)^2 \\ - \frac{5\Lambda Ml}{4(1+\epsilon\cos\vartheta)} - \frac{3\Lambda l^2}{4(1+\epsilon\cos\vartheta)^2} + \frac{\Lambda^2 l^4}{8(1+\epsilon\cos\vartheta)^4}\right]d\vartheta \tag{45}$$

which for small eccentricities can be integrated as

$$\Delta\vartheta' = 2\pi + \frac{6\pi M}{l} + \frac{2\pi M^2}{l^2}\left(2+\epsilon^2\right) - \frac{5\pi\Lambda Ml}{2} - \frac{3\pi\Lambda l^2}{2} + \frac{\pi\Lambda^2 l^4}{4} \tag{46}$$

where the three last terms are the corrections due to the cosmological constant. The classical advance in the perihelion is recuperated for zero cosmological constant ($\Lambda \to 0$) and the complete expression agrees with the one reported in [4]. This derivation is much simpler than the obtained in the work of G. Kraniotis and S. Whitehouse [5], where the evaluation has been done by means of the genus 2 siegelsche modular forms and including the Mercury's data.

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