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SU(n) WZNW fusion and a Q-algebra*

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Abstract. The quantum group covariant quantization of the chiral parts of the Wess–Zumino–Novikov–Witten (WZNW) model on a compact Lie group G gives rise to an extension of the $2D$ unitary model involving matrix algebras (with non-commutative entries) generated by chiral "zero modes" $a^i_\alpha, \bar{a}^\beta_j$. The quantum group invariant combinations of the latter $Q^i_j = a^i_\alpha \otimes \bar{a}^\alpha_j$ represent the internal symmetry of the model in a setting that generalizes the axiomatic approach to gauge theories. Here we sketch the concept of the Q-operators for $G = SU(n)$ starting with $n = 2$ outline the steps to follow for arbitrary n and discuss the relation of the Q-algebra with the WZNW fusion.

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1 Introduction

The main peculiarities of the $2D$ space(-time) are the presence of two disconnected space-like regions, the infinite dimensional conformal symmetry, the presence of generalized (braid group, in place of permutational) statistics, the appearance of "internal" (gauge) symmetries not corresponding to (compact) groups. These features are closely related and they also influence the notions of charges and superselection sectors. In the framework of the algebraic (local, relativistic) QFT [1] charges are defined as quantities conserved independently of the concrete dynamics, i.e., represented by operators commuting not only with the Hamiltonian but with all observables, and superselection sectors are inequivalent representation spaces $\mathcal{H}_p$ of the algebra of observables, generated from the vacuum by charged fields. In space-time dimensions $D \geq 3+1$ internal (gauge) symmetries (whose action leaves observables invariant) generated by (localizable) charges are classified by the Doplicher-Roberts theorem [2]: according to it any such symmetry is given by a *compact* gauge group G. The full Hilbert

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space $\mathcal{H}$ of the theory decomposes in superselection sectors as

$$\mathcal{H} = \bigoplus_p \mathcal{H}_p \otimes \mathcal{V}_p \ , \qquad d_p := \dim \mathcal{V}_p < \infty \ . \tag{1}$$

Here $\mathcal{V}_p$ are finite dimensional representations of G and statistics dimensions d_p obey *fusion rules* $d_{p_1} d_{p_2} = \sum_p N_{p_1 p_2}^p \, d_p$ following from their tensor product (Clebsch-Gordan) decomposition: $\mathcal{V}_{p_1} \otimes \mathcal{V}_{p_2} = \oplus_p N_{p_1 p_2}^p \, \mathcal{V}_p \ , \ N_{p_1 p_2}^p \in \mathbb{Z}_+$. The lower dimensional substitutes of the gauge symmetry are much more involved, and our study reflects a step to the clarification of this problem.

The WZNW models [3] describe the conformal invariant dynamics of strings on a Lie group G [4] characterized by a single natural number k, the level. It is customary to extend it writing the $2D$ field as a product of two independent chiral fields, $G_B^A(z,\bar{z}) = g_\alpha^A(z) \otimes \bar{g}_B^\alpha(\bar{z})$ (all indices belong to the set $\{1, \dots, n\}$) which are *quasiperiodic*, e.g. the left one satisfies $g_\alpha^A(e^{2\pi i} z) = g_\beta^A \, M_\alpha^\beta$. In the particular case when the monodromy is diagonal, we will denote the chiral field and its monodromy matrix by $u_i^A(z)$ and M_p , respectively [5]. The *zero modes* (a and $\bar{a}$) are then defined as the corresponding intertwiners, e.g. $M_p \, a = a \, M$ so that, in effect $g_\alpha^A(z) = u_i^A(z) \otimes a_\alpha^i$, $\bar{g}_B^\alpha(\bar{z}) = \bar{a}_j^\alpha \otimes \bar{u}_B^j(\bar{z})$ and

$$G_B^A(z,\bar{z}) = u_i^A(z) \otimes Q_j^i \otimes \bar{u}_B^j(\bar{z}) \ , \quad Q_j^i = a_\alpha^i \otimes \bar{a}_j^\alpha \ , \quad (Q_j^i) =: Q \ . \tag{2}$$

For compact G (the case of rational CFT [6]) with Lie algebra $\mathcal{G}$ the model is unitary, the label p in (1) belongs to a finite set $\mathcal{S}$ and d_p coincide with the *quantum dimensions* of the irreducible "physical representations" $\{\mathcal{V}_p\}_{p \in \mathcal{S}}$ of the quantum group $U_q(\mathcal{G})$ which, as a rule, are *non-integer*. (Here we will only consider $G = SU(n)$ in which case the deformation parameter is $q = e^{-i\frac{\pi}{h}}$, $h := k + n$.) In addition, the tensor ring generated by the "physical representations" is infinite dimensional and contains indecomposable representations. Our idea is to show that although $U_q(\mathcal{G})$ is not a proper candidate for the gauge symmetry of the unitary model, it plays a similar role in an extended construction reminiscent to the axiomatic approach to gauge theories [7].

2 Chiral zero modes and the quantum group

In our case the chiral zero modes' matrix $a = (a_\alpha^i)$ forms, together with a set of commuting operators $\{p_i\}_{i=1}^n$, a quantum matrix algebra of $SL(n)$ type [5,8,9]:

$$[q^{p_i}, q^{p_j}] = 0, \quad \prod_{j=1}^n q^{p_j} = 1, \quad q^{p_{j\ell}} a_\alpha^i = a_\alpha^i q^{p_{j\ell} + \delta_j^i - \delta_\ell^i} \quad (p_{j\ell} = p_j - p_\ell),$$

$$R_{12}(p) \, a_1 \, a_2 = a_2 \, a_1 \, R_{12} \quad \Leftrightarrow \quad \hat{R}_{12}(p) \, a_1 \, a_2 = a_1 \, a_2 \, \hat{R}_{12} \ . \tag{3}$$

Here R_{12} is a Drinfeld-Jimbo quantum R-matrix satisfying the quantum Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ and the quantum *dynamical* R-matrix $R_{12}(p)$ satisfies a dynamical analog of it. It follows that the operators

$\hat{R}_{ii+1} = P_{ii+1} R_{ii+1}$ (where P is the permutation operator) satisfy the braid group relations:

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{4}$$

The exchange relations (3) can be rewritten as

$$\begin{gathered} a^j_\beta a^i_\alpha \,[p_{ij}-1] = a^i_\alpha a^j_\beta \,[p_{ij}] - \,a^i_\beta\, a^j_\alpha\, q^{\epsilon_{\alpha\beta}p_{ij}} \quad (i\neq j \ \text{ and } \alpha\neq\beta), \\ [a^j_\alpha, a^i_\alpha] = 0\ , \qquad a^i_\alpha a^i_\beta = q^{\epsilon_{\alpha\beta}}\, a^i_\beta a^i_\alpha\ , \quad i,j = 1,\dots,n\ . \end{gathered} \tag{5}$$

The quantum group emerges from the factorization of the monodromy matrix $M = q^{\frac{1}{n}-n} M_+ M_-^{-1}$ such that $diag(M_+) = diag(M_-^{-1}) =: D\,,\ \det(D) = 1$. The exchange relations of $M_\pm$

$$R_{12} M_{\pm 2} M_{\pm 1} = M_{\pm 1} M_{\pm 2} R_{12}\ , \quad R_{12} M_{+2} M_{-1} = M_{-1} M_{+2} R_{12} \tag{6}$$

define [10] an n-fold cover of $U_q(s\ell(n))$ while those with the zero mode matrix

$$M_{\pm 2}\, a_1 = a_1\, R^{\mp}_{12}\, M_{\pm 2} \quad (\,R^-_{12} = R_{12}\ ,\ R^+_{12} = R^{-1}_{21}\,) \tag{7}$$

imply that the rows of the latter transform as "q-tensor operators".

The Fock representation of the chiral matrix algebra is built on a quantum group invariant vacuum vector characterized by $a^i_\alpha \mid 0\rangle = 0$ for $i \geq 2$. For q generic the chiral Fock space $\mathcal{F}$ is a *model space* for $U_q(s\ell(n))$, i.e. splits into a direct sum of all its irreducible representations with multiplicity one [5]. The action of a^i_α on a vector associated with a *semistandard* $su(n)$ Young tableau Y_Λ is described by adding a box with label α to the i-th row, while the eigenvalues of $p = (p_1,\dots,p_n)$ coincide with the corresponding shifted weights $p = \Lambda + \rho$ in a "barycentric" basis ($\sum_{i=1}^n p_i = 0$).

3 The Q-algebra and its vacuum representation for n=2

The Fock representation of the matrix algebra for $n = 2$ and $q = e^{\pm i\frac{\pi}{h}}$ has been studied in [11,12]. The Fock space then carries a representation of the $2h^3$-dimensional *restricted* quantum group $\overline{U}_q = \overline{U}_q(s\ell(2))$ generated by E, F, K such that $E^h = 0 = F^h\,,\ K^{2h} = 1$. Further, the matrix algebra has a non-trivial finite-dimensional quotient characterized by $(a^i_\alpha)^h = 0$; the corresponding quotient $\mathcal{F}^{(h)}$ of the Fock space $\mathcal{F}$ is h^2-dimensional.

The matrix of $2D$ zero modes' (quantum group invariant) operators

$$Q = (Q^i_j) = \begin{pmatrix} Q^1_1 & Q^1_2 \\ Q^2_1 & Q^2_2 \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{8}$$

has the following properties. If $(a^i_\alpha)^h = 0 = (\bar{a}^\alpha_i)^h \ \ \forall\ i,\alpha \in \{1,2\}$, then $(Q^i_j)^h = 0$. Further, diagonal and off-diagonal elements of Q commute:

$$AB = BA\,,\ CA = AC\,,\ BD = DB\,,\ CD = DC\ . \tag{9}$$

Moreover, they generate two (commuting) $\overline{U}_q$ algebras:

$$\begin{array}{llll} [B,C]=[N], & NB=q^2BN, & NC=q^{-2}CN, & N^{\pm 1}:=-q^{\pm p}\otimes q^{\mp\bar{p}} \\ [A,D]=[L], & LA=q^2AL, & LD=q^{-2}DL, & L^{\pm 1}:=-q^{\pm p}\otimes q^{\pm\bar{p}} \\ A^h=D^h=0=B^h=C^h, & & L^{2h}=1=N^{2h} & (p=p_{12},\bar{p}=\bar{p}_{12}). \end{array} \tag{10}$$

So the quantum group comes in again from the back door! Then, while the vacuum representation of the off-diagonal Q-algebra is one-dimensional,

$$B\,|0\rangle=0=C\,|0\rangle\,,\quad N\,|0\rangle=-\,|0\rangle\quad(\Rightarrow\ [N]\,|0\rangle=0\,) \tag{11}$$

the diagonal one generates a certain h-dimensional representation of $\overline{U}_q$,

$$\begin{aligned} &A\mid m\rangle=[m+1]\mid m+1\rangle\,,\\ &D\mid m\rangle=[m+1]\mid m-1\rangle\,,\\ &(L+\,q^{2(m+1)})\mid m\rangle=0 \end{aligned} \tag{12}$$

for

$$\mid m\rangle:=\frac{A^m}{[m]!}\,|0\rangle\,,\ m=0,\ldots,h-1\quad(D\,|0\rangle=0)\,.$$

There is a scalar product on it which is Hermitean and semidefinite: $(m'\mid m)=[m+1]\,\delta_{mm'}$.

This picture contains typical ingredients of the axiomatic approach to gauge theories [7] – an extended state space $\mathcal{F}^{(h)}\otimes\bar{\mathcal{F}}^{(h)}$, a pre-physical subspace $\mathcal{F}'$ with a positive semidefinite scalar product, a subspace of zero-norm vectors $\mathcal{F}''$, and a physical subquotient

$$\mathcal{F}^{phys}=\mathcal{F}'/\mathcal{F}''\simeq\oplus_{p=1}^{h-1}\mathcal{F}_p^{phys}\,,\qquad\mathcal{F}_p^{phys}:=\mathbb{C}\,A^{p-1}\,|0\rangle\,.$$

The entries of Q generate the algebra of observables and the (inverse) coproducts $\Delta'((M_\pm)^\alpha_{\ \beta})=(M_\pm)^\sigma_{\ \beta}\otimes(\bar{M}_\pm)^\alpha_{\ \sigma}$, the gauge symmetry leaving it invariant.

The $2D$ unitary model is recovered with the “physical” $2D$ field (2) acting on

$$\mathcal{H}^{phys}=\oplus_{p=1}^{h-1}\,\mathcal{H}_p\otimes\,\mathcal{F}_p^{phys}\otimes\bar{\mathcal{H}}_p \tag{13}$$

($\mathcal{H}_p$ and $\bar{\mathcal{H}}_p$ carrying integrable representations of the left and right current algebras, respectively) which is monodromy invariant and periodic,

$$\big(G(e^{2\pi i}\,z,e^{-2\pi i}\,\bar{z})-G(z,\bar{z})\big)\ \mathcal{H}^{phys}\ =0\,. \tag{14}$$

4 The arbitrary n case

We proceed by displaying some encouraging results for arbitrary $n\geq 3$ which make us believe that repeating the steps of the program described above for $n=2$ is feasible in general. First of all, it is easy to prove again that

$$(a^i_\alpha)^h=0=(\bar{a}^\alpha_i)^h\quad\forall\,i,\alpha\in\{1,\ldots,n\}\quad\Rightarrow\quad(Q^i_j)^h=0\,. \tag{15}$$

Further, it seems reasonable to make the following

Conjecture *For any* n *a* Q*-monomial containing off-diagonal entries of* Q *annihilates the vacuum vector.*

Definition Let $\mathcal{F}'$ be the subspace of the space of "diagonal Q-vectors" $\mathcal{F}^{diag}$ which is annihilated by any off-diagonal element of Q:

$$\mathcal{F}' \subset \mathcal{F}^{diag} \,, \quad Q^r_s \, \mathcal{F}' = 0 \quad \text{for} \quad r \neq s \,. \tag{16}$$

(The above conjecture is thus equivalent to $\mathcal{F}' = \mathcal{F}^{diag}$.) We will use the weak relation $A \approx B$ for $(A-B)\, v = 0 \;\; \forall \, v \in \mathcal{F}'$. Note that the eigenvalues of p_{ij} and $\bar{p}_{ij}$ on $\mathcal{F}^{diag}$ are equal.

We have been able to prove so far the following important results.

Lemma *The diagonal entries of* Q *commute with the off-diagonal ones from the same row or column:*

$$[Q^j_i, Q^i_i] = 0 = [Q^i_j, Q^i_i] \quad \text{for} \quad j \neq i \,. \tag{17}$$

Bilinear in Q exchange relations can be derived from (5) and its bar counterpart:

$$[p_{ij}-1] \otimes [\bar{p}_{\ell m}] \, Q^j_\ell \, Q^i_m - [p_{ij}] \otimes [\bar{p}_{\ell m}-1] \, Q^i_m \, Q^j_\ell = [p_{ij} - \bar{p}_{\ell m}] \, Q^i_\ell \, Q^j_m \tag{18}$$

for $i \neq j$, $\ell \neq m$. They imply

$$\begin{aligned} &[p_{ij}-1] \otimes [\bar{p}_{i\ell}] Q^j_\ell Q^i_i = [p_{ij}] \otimes [\bar{p}_{i\ell}+1] Q^i_i \, Q^j_\ell - [p_{ij} + \bar{p}_{i\ell}] Q^i_\ell Q^j_i \,, \\ &[p_{ij}] \otimes [\bar{p}_{i\ell}-1] Q^j_\ell Q^i_i = [p_{ij}+1] \otimes [\bar{p}_{i\ell}] Q^i_i \, Q^j_\ell - [p_{ij} + \bar{p}_{i\ell}] Q^j_i Q^i_\ell \end{aligned} \tag{19}$$

for $i \neq j \neq \ell \neq i$. Starting with $Q^j_\ell \, |0\rangle = 0$ and then using the statement (17) of the Lemma and Eqs.(19) allows us to carry out an induction in the number of the diagonal Q-operators applied to the vacuum to generate a vector $v \in \mathcal{F}'$ whenever $Q^j_\ell \, Q^i_i$ is applied to such v and $[p_{ij}-1]\, v \neq 0$ or $[p_{i\ell}-1]\, v \neq 0$.

On the other hand, the weak relation (that also follows from (19))

$$[\hat{p}_{ij}+1] \, Q^i_i \, Q^j_j \;\approx\; [\hat{p}_{ij}-1] \, Q^j_j \, Q^i_i \tag{20}$$

is a powerful tool to explore the structure of $\mathcal{F}^{diag}$. One should also have in mind a *determinant condition* on $Q^n_n \, Q^{n-1}_{n-1} \dots Q^1_1$ as well as certain *trilinear* relations for diagonal Q-operators implied by the braid relations (4). It would be reasonable to foresee that $\mathcal{F}^{diag}$ is again finite dimensional, with basis vectors labelled by $su(n)$ Young diagrams that fit into the rectangle of size $(h-1) \times (n-1)$. (The unitary case involves *all* diagrams in a rectangular $k \times (n-1)$.)

5 Q-algebra and WZNW fusion

For $n=2$ the (binary) *fusion* matrices $F^{(\lambda)}_h$ encoding the action of the operator $(A+D)^\lambda$ (that corresponds to a primary field of weight $\lambda = 0, 1, \dots, k$) in the basis $|m\rangle$ (12) have Perron-Frobenius eigenvalue $[\lambda + 1]$ and provide a representation of the fusion ring of the unitary model.

It would be intriguing to look for a possible connection, for n arbitrary, of the emerging diagonal Q-algebra with the algebra of the (phase model) "*hopping operators*" $\{Q_1, \dots, Q_n\}$ on a circle (also called "*affine local plactic algebra*"). The latter is characterized by the relations

$$\begin{array}{rl} [Q_i, Q_j] = 0 & \text{if } \ i \neq j \pm 1 \bmod n \\ Q_i Q_j^2 = Q_j Q_i Q_j , \quad Q_i^2 Q_j = Q_i Q_j Q_i , & \text{if } \ i = j + 1 \bmod n \end{array} \tag{21}$$

and provides a description of the (unitary) $\widehat{su}(n)_k$ affine fusion ring [13, 14].

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