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Fermions as a Gauge Model Ground State*

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Abstract. It is argued that the lepton sector in the Standard Model of electroweak and strong interactions can be represented as a non-trivial, anomalously quantized vacuum configuration of a pure $U(2)$ gauge model.

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1 Introduction

It is a common technique in Field Theory to consider models in the vicinity of a non-trivial solution of their equations of motion. Such solution plays the role of a vacuum and the deviation from it is quantized. The procedure is very straightforward when the vacuum is unique. However, an extra care is needed if the ground state depends on some parameters (collective coordinates). In this case a change of the path integral measure is required. In our present consideration the vacuum is so degenerate that it is described by a field. We argue for that in this case the functional integral is on both quantum and vacuum fields. In other words we quantize the vacuum as well. Our basic example is a pure $U(1)$ gauge model in four dimensional Euclidean space-time E^4 and the vacuum is (anti) self-dual. We show that it satisfies the Dirac equation and we use the results of Ref. [1] to quantize it anomalously[1] thus obtaining mass-less Quantum Electrodynamics. Our second example is a pure $U(2)$ gauge model again in E^4. The resulting theory looks very much like the lepton sector of the Standard Model.

2 Quantization in the Vicinity of Degenerate Vacuum

Suppose we have a model with Lagrangean L for the filed φ and a solution ϕ of the corresponding equation of motion. The field ϕ realizes an extremum of the action and can be used as a vacuum. In other words we decompose the initial field into a classical and quantum part $\varphi = \phi + \eta$. When the vacuum

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[1]The idea that instantons can change the statistics of some fields is not new. See, e.g. [2–4].

is unique this is a simple change of variables and the path integration measure $D\varphi$ and $D\eta$ coincide. In the case of a degenerate vacuum the path integral measure comprises of two terms $\tilde{D}\eta$ and $\tilde{D}\phi$ – a measure over quantum field and a measure over collective coordinates. Because a new gauge symmetry emerges if the vacuum is not unique [5–7] $\tilde{D}\eta$ is a path measure for a gauge model and incorporates a gauge fixing and Faddeev-Popov determinant. The measure $\tilde{D}\phi$ is the interesting one. When the vacuum is not known explicitly but is defined as a solution of some differential operator $\mathcal{L}$ which is *not* the equation of motion then it is

$$\tilde{D}\phi = \delta(\mathcal{L}\phi)D\phi. \tag{1}$$

Using Lagrange multipliers the delta function in Eq. (1) can be represented as a part of the Lagrangean. Thus we obtain the following Lagrangean:

$$L' = \bar{\phi}\mathcal{L}\phi + L(\phi, \eta) \tag{2}$$

Now, η, ϕ and $\bar{\phi}$ are independent fields.

We want to stress that it does not follow from the above considerations that when the Lagrangean L possesses *ad initium* gauge symmetry then L' has bigger symmetry. For example, the symmetry will be the same if the variation of the vacuum with respect to collective coordinates coincides with some of the zero modes due to the initial gauge freedom.

3 Quaternions and Quaternion Analyticity

Let us remind the definition of quaternion number $\mathcal{H}$: $\mathbb{H} \ni \mathcal{H} = h_\mu e_\mu$ where $h_\mu \in \mathbb{R}$, $\mu = 0, .., 3$ and e_μ are four non-commutative units such that:

$$e_0 e_0 = e_0, \ \ e_i e_0 = e_0 e_i = e_i, \ \ e_i e_j = -\delta_{ij} e_0 + \epsilon_{ijk} e_k. \tag{3}$$

Here ϵ is the totally anti symmetric third rank tensor and $i, j, k = 1, 2, 3$. Note that e_0 can be always set to 1. There is a natural operation of quaternion conjugation in $\mathbb{H}$ under which the conjugated to $\mathcal{H}$ is $\bar{\mathcal{H}} = h_0 - h_i e_i = h_\mu \bar{e}_\mu$.

The so called Fueter analyticity is the quaternion analog to the complex analyticity. To define it we need two first order differential operators $\mathcal{D}$ and $\bar{\mathcal{D}}$

$$\bar{\mathcal{D}} = e_\mu \partial_\mu, \ \ \mathcal{D} = \bar{e}_\mu \partial_\mu. \tag{4}$$

The function $\mathcal{F}$ is called Fueter analytic if it satisfies the equation $\bar{\mathcal{D}}\mathcal{F} = 0$ and Fueter anti analytic if it satisfies the equation $\mathcal{D}\mathcal{F} = 0$.

Hereafter we use two different representations of the quaternions. In the first one the quaternions are 2×2 matrices and the units are

$$e_0 = \mathit{1}, \ \ e_k = -i\sigma_k, \quad k = 1, 2, 3. \tag{5}$$

In the second one the quaternions are 4×4 real matrices and the units are

$$e^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e^1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, e^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad (6)$$

where $I = i\sigma_2$. Together eqs.(5,6) provide a representation of $\mathbb{H}^2$.

4 A Pure *U*(1) Gauge Model in E^4

This is our basic example which demonstrates the features of the anomalously quantized vacuum [8]. The model is defined by the following action

$$\mathbf{A} = -\frac{1}{4}\int F_{\mu\nu}F_{\mu\nu}. \qquad (7)$$

Here $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the field strength tensor for the potential A_μ. In E^4 we can define self dual $^+$ and anti self dual $^-$ field strengths $F^{\pm}_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\pm}_{\lambda\rho}$. The corresponding potentials are solutions of the equations of motion but are not taken into account by the perturbation theory. We, however, can use them as a vacuum and thus to estimate their contribution to the theory. The most general vacuum we can construct is $\phi = A^+ + A^-$ where A^+ and A^- are self-dual and anti self-dual potentials. Expanding the gauge potential around it we get

$$A_\mu = A^+_\mu + A^-_\mu + A'_\mu. \qquad (8)$$

The action (7) in the vicinity of the ground state $A^+ + A^-$ is $\mathbf{A} = -\frac{1}{4}\int F'_{\mu\nu}F'_{\mu\nu}$ where F' is the field strength for A'. There are no extra symmetry in this case. Therefore we can view A' as the standard $U(1)$ potential. The contribution of the vacuum modes to the transition amplitude is a multiplicative constant which gives the number of different (anti) self-dual configurations. In order to find this number we have to determine the gauge freedom of the vacuum.

Writing Eq. (8) we have assumed that A' is a connection as the field A. Therefore, the vacuum has to be gauge invariant. On the other hand the decomposition (8) introduces a Stuckelberg type symmetry $\delta'_\zeta(A^{\pm}_\mu) = \pm\partial_\mu\zeta$ in the vacuum which has to be fixed as well. So, we have to impose two conditions on A^+ and A^- and our choice is to use Lorentz gauge conditions $\partial_\mu A^{\pm}_\mu = 0$.

Let us translate the above discussion to the quaternion language. From the potential A_μ we construct a quaternion function $\mathcal{A} = A_\mu e_\mu$ (and $\bar{\mathcal{A}} = A_\mu \bar{e_\mu}$ as well). The action of the Fueter operators (4) on these functions gives:

$$\mathcal{D}\mathcal{A} = \partial_\mu A_\mu + (\partial_0 A_i - \partial_i A_0 - \epsilon_{ijk}\partial_j A_k)e_i, \qquad (9)$$
$$\bar{\mathcal{D}}\bar{\mathcal{A}} = \partial_\mu A_\mu + (-\partial_0 A_i + \partial_i A_0 - \epsilon_{ijk}\partial_j A_k)e_i. \qquad (10)$$

So, if the potential $\bar{\mathcal{A}}$ ($\mathcal{A}$) is Fueter (anti) analytic, then we have anti self-dual (self-dual) configuration in the Lorentz gauge.

Proposition: The requirements for Fueter anti-analyticity for $\mathcal{A}^+$ and Fueter analyticity for $\bar{\mathcal{A}}^-$ specify the possible $U(1)$ vacuums.

Now we want to rewrite the Lagrangean (7) into the form (2). According to Eq. (1) the integration measure over self-dual vacuum is something like

$$\delta(\bar{\mathcal{D}}\mathcal{A}^+) \;\; DA^+ \tag{11}$$

with an analogous expression for the anti self-dual vacuum potential. Here is the moment to fix the quaternion representation. We use the one given by Eqs. (5)[2].

Using the representation (5) we obtain in Eq. (11) a delta function of a matrix which has to be understood as a product of delta functions of each matrix entry. This gives four complex conditions for A^+ which exceeds the correct number. In order to get two complex conditions for A^+ we have to pick up a constant 2D vector (in fact — Weyl spinor) v and to use $\mathcal{D}\mathcal{A}^+ \cdot v$ as an argument of the delta function in Eq. (11). Note that the explicit form of any quaternion $\mathcal{H}$ in the representation (5) is $\mathcal{H} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ where $a = h_0 - ih_3$, $b = h_2 - ih_1$ and $\bar{a}$, $\bar{b}$ denoting the complex conjugates of a and b. So, if for some normalized spinor v we know $\mathcal{H} \cdot v$, then we know $\mathcal{H}$ itself.

Let us define two spinors $\psi^+ = \mathcal{A}^+ \cdot v$ and $\psi^- = \bar{\mathcal{A}}^- \cdot u$ and a 4D Dirac spinor $\psi = \begin{pmatrix} \psi^- \\ \psi^+ \end{pmatrix}$. Let us use the diagonal γ^5 representation of the 4D Euclidian gamma matrices in which the Dirac operator has the form

$$\partial_\mu \gamma_\mu = \begin{pmatrix} 0 & \partial_0 + i\partial_k \sigma_k \\ \partial_0 - i\partial_k \sigma_k & 0 \end{pmatrix} = \begin{pmatrix} 0 & \bar{\mathcal{D}} \\ \mathcal{D} & 0 \end{pmatrix}. \tag{12}$$

Then the transition amplitude for our $U(1)$ model plus vacuum contributions is

$$S = \int D\psi^+ D\psi^- \tilde{D}A' \; \delta(\bar{\mathcal{D}}\psi^+)\delta(\mathcal{D}\psi^-) \exp\left\{-\frac{1}{4}\int F'_{\mu\nu}F'_{\mu\nu}\right\} \tag{13}$$

which after some algebra can be rewritten in the following form

$$S = \int D\bar{\psi} D\psi \tilde{D}A' \exp\left\{-\int i\bar{\psi}\partial_\mu\gamma_\mu\psi + \frac{1}{4}F'_{\mu\nu}F'_{\mu\nu}\right\}. \tag{14}$$

Here is the moment to use some of the results of Ref. [1] and to quantize anomalously (as fermions) the field ψ. This is the most important step in our work. Note that the anomalous quantization effectively changes the trivial $U(1)$ bundle over E^4 to a one whose base B is a double cover of E^4 and therefore is with nontrivial fundamental group. This in turns justifies the existence of non-trivial (anti) self dual $U(1)$ field configurations.

[2] The use of representation (5) can be viewed as a change of variables such that the vector is described by bispinor.

There is a little flow in our recipe to pass from quaternions to spinors — the obtained Lagrangean is gauge non-invariant. However, we have started with a gauge invariant action, and we have to restore this invariance in Eq. (14). We can do this just prolonging the derivatives with A' and thus obtaining

$$S = \int D\bar{\psi} D\psi DA' \exp\left\{ -\int \bar{\psi}(i\partial_\mu + A'_\mu)\gamma_\mu \psi + \frac{1}{4} F'_{\mu\nu} F'_{\mu\nu} \right\}. \qquad (15)$$

It is possible to get Eq. (15) in a more elegant and consistent way without passing through Eq. (14) at all. For this we have to use in the definition of ψ^+ (ψ^-) a gauge invariant spinor $w = e^{-i\phi[\Gamma]} v$. Here $\phi\,[\Gamma]$ is some phase which compensates the gauge transformation of v. It is non-integrable [9], but possesses well defined derivatives $\partial_\mu \phi\,[\Gamma] = A'_\mu$. Formally $\phi = \int_\Gamma dl.A'$.

To summarize: we have obtained fully fledged mass-less Quantum Electrodynamics from a pure $U(1)$ gauge model in the most general non-trivial vacuum.

5 A Pure $U(2)$ Gauge Model in E^4

The action of our model is

$$\mathbf{A} = -\frac{1}{4} \int F^\alpha_{\mu\nu} F^\alpha_{\mu\nu}. \qquad (16)$$

In our notations the $U(1)$ charge is 1 and the $SU(2)$ charge is g, so that the corresponding field strengths are $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,[A_\mu, A_\nu]^a\,,\;\; a = 1,2,3$. Again, the (anti) self-dual fields are solutions of the equations of motion and we can use them as a vacuum. Generally, the strategy is the same as for the $U(1)$ model just considered. However, now we need two types of mutually commuting (e- and ξ-) quaternions. We associate to the potential A^α_μ four $e-$quaternion functions $\mathcal{A}^\alpha = e_\mu A^\alpha_\mu$, four $\xi-$quaternion functions $\mathfrak{A}_\mu = \xi^\alpha A^\alpha_\mu$ and one bi-quaternion function $\mathbb{A} = \xi^\alpha e_\mu A^\alpha_\mu$. We use the $e-$conjugated to $\mathcal{A}$ and $\mathbb{A}$ functions as well which we denote $\bar{\mathcal{A}}^\alpha$ and $\bar{\mathbb{A}}$ respectively. After some algebra, we get that the equations

$$(\mathcal{D} + \frac{g}{2}\bar{\mathbb{A}})\mathbb{A} = 0, \;\; (\bar{\mathcal{D}} + \frac{g}{2}\mathbb{A})\bar{\mathbb{A}} = 0. \qquad (17)$$

describes self-dual and anti self dual field respectively in a particular gauge.[3]

Here we present two solutions of Eqs. (17) for which the nonlinear term in them vanishes. This effectively reduces the non-Abelian (anti) self-dual conditions to Fueter (anti) analyticity conditions.

Solution (a): Let $\mathcal{A}^0$ is a Fueter anti analytic function and $\mathcal{A}^a = 0, \;\; a = 1,2,3$. This is the $U(1)$ solution we have discussed in the previous section.

[3] Any gauge condition can be obtained adding an arbitrary $\xi-$quaternion function to Eqs. (17).

Solution (b): Fix a $su(2)$ index, say a and let $\mathcal{A}^\alpha = 0 \ \ \forall\alpha \neq a$ while $\mathcal{A}^a$ is Fueter anti analytic. Again, this solution describes an $U(1)$ vacuum. There are two important differences between solutions (a) and (b): First, they are associated with two different $u(1)$ algebras and therefore have different charges. Second, there is a global $SO(3)$ covariance for solution (b) such that if $\mathcal{A}^a$ is a solution, so is $\mathcal{A}'^a = U^{ab}\mathcal{A}^b$ where U^{ab} is a $SO(3)$ matrix.

Solutions (a) and (b) can be freely combined with each other. Therefore, the largest $U(2)$ vacuum, we can construct out of them describes two mass-less Dirac spinors. We can speculate a little with the $SO(3)$ symmetry of solution (b). It indicates that this vacuum is not unique and the space of vacuums can be span by three basic vectors. We can numerate them simply by v^1, v^2, v^3 or, if you prefer, e, μ, τ. Note that only one of these vacuums can be used as an asymptotic state, or in other words there is only one stable particle.

6 Conclusions

It is known that the difference between two connections transforms as a matter field under the action of the gauge group. What we argue for here is that the matter *is* a difference of two gauge connections. We show that the anomalous quantization of the most general non-trivial vacuum of the pure U(1) model leads to fully fledged mass-less Electrodynamics. We apply the same approach to a pure $U(2)$ Yang–Mills theory. It is tempting to associate the constructed vacuums with the lepton sector of the Standard model.

There are two major problems to solve if we want to use the anomalous quantized vacuum as an origin of the leptons. First, this is the question of the vacuum charges which up to this point are arbitrary. Second is the question about the general solution of Eqs. (17). It is possible using path exponents to construct a rather general solution, but there are a lot of questions about its interpretation.

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