

A Pseudo-Unitary Version of Schwinger's Symbolism of Atomic Measurements and a Prospect for a New Relativistic Quantum Information Theory

J.G. Cardoso 

Department of Mathematics, Centre for Technological Sciences-UDESC,
89223-100 Joinville Santa Catarina, Brazil

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Abstract. We transcribe into a relativistic dynamical framework proposed earlier the measurement processes that are traditionally described within the realm of non-relativistic quantum mechanics. The correlations between observations registered in different spacetime frames, concerning intermediate steps and outcomes of microscopic measurements, are attained through the implementation of the orthochronous proper Poincaré subgroup of an appropriate realization of $SU(2, 2)$. Such correlations turn out to support the view whereby the physical inner structure of the Schrödinger quantum mechanical picture may be formally described within special relativity. It appears that the overall work affords a fundamental background to the construction of covariant quantum computational gates whilst making it feasible to elaborate upon the observed formation of entangled states in processes of creation-annihilation pairs. The important question as to whether quantum computation should bear an invariant character is then raised.

KEY WORDS: covariant gates, relativistic quantum information theory, relativistic quantum mechanics

1 Introduction

The leading assumption upon which the construction of the classical theory of measurement rests, lies in the fact that the interaction between the measuring devices and the systems being observed can always be made arbitrarily small. It follows that every classical measurement process may be carried out without disturbing significantly any physical attribute of the system involved in the process. On the other hand, the interaction between measurement instruments and atomic systems produces disturbances on the systems that can not be fully

controlled and predicted [1–7], whence any such disturbance can not be made arbitrarily small or even precisely compensated by any means. Any process aimed at measuring an observable attribute of some atomic system thus holds up indivisibility in the sense that any attempt to trace the history of the system during the measurement would entail modifying the intrinsic characteristics of the process, while likewise producing unavoidable changes in previously known values for other attributes of the system [2]. It is this property of microscopic measurements which typically precludes dynamical variables of atomic systems from being represented by classical numbers¹.

In the present paper, the symbolism of atomic measurements that had been developed by Schwinger a long ago [2, 3] towards describing the traditional class of non-relativistic quantum measurement processes, is transcribed into the dynamical framework built up in Ref. [13]. The key structure upon which the construction of this framework stands amounts to Cartan's space, the four-valued representation space of the restricted conformal group for special relativity [14]. A fundamental physical character was ascribed to Cartan's space thereabout, and a supposed relativistic quantum measurement process was thus described within a finite-dimensional framework wherein the dynamics of free particles and antiparticles could seemingly be formulated in a purely quantum mechanical way.

What has supplied the underlying basis for the ascription of an elementary character to Cartan's space, stems primarily from the existence within the theoretical structure of special relativity of a complex indefinite inner product space endowed with a representative group of pseudo-unitary linear transformations, which could thereby bring forward in a non-trivial way the essential role that is usually played by complex numbers in any quantum mechanical formulation. A fairly complete algebraic description of Cartan's space had been given earlier in Refs. [15, 16] where a detailed covariant account is provided of $SU(2, 2)$, the group of unimodular pseudo-unitary (4×4) -matrices which effectively acts on any copy of Cartan's space as a fifteen-parameter group of linear transformations that leave invariant quadratic forms of signature $(+ + --)$. It then became apparent that the overall framework could supply a relativistic quantum mechanical description of free twofold systems.

Remarkably enough, the pseudo-unitary framework of Ref. [13] was thought of as greatly unifying the description of free particles and antiparticles of any spin, whence it has appeared to differ profoundly from the famous van der Waerden two-component formulation of Dirac's equation which at first describes the helicity degrees of freedom of one and the same spin-1/2 particle. The crude absence of unitarity from Dirac's theory thus seemed to have been bypassed inside a manifestly mechanical framework. Just based upon Naimark's theorems [17, 18], which ensure the existence in Cartan's space of common eigen-

¹The most remarkable application of the theory of quantum measurements has thus far been addressed to a non-relativistic approach often designated in the literature as measurement-based quantum computation (see, for instance, Refs. [8–12]).

vectors for commuting linear operators, every particle was physically characterized through a complete set of compatible observables. Spaces of state vectors for any free particles and antiparticles were accordingly taken as two-complex-dimensional subspaces that occur in orthogonal decompositions of isomorphic copies of Cartan's space, whereas all dynamical variables were taken to carry a symbolic coordinate-free character having twofold discrete spectra, and thence to operate linearly on spaces of states independently of the action of any generator of the background restricted conformal group. Each of these variables shows up as an operator restriction that has one of the two pieces borne by the orthogonal decomposition of a copy of Cartan's space as its principal invariant eigenspace, with the other piece being considered as the zero subspace of the variable in question. Orbital angular momenta were assumed to be absent from dynamical sets such that only spin-helicity and polarization degrees of freedom should be accounted for as regards the arrangements of angular-momentum contributions to eventual preparations of states. Thus, electric and all the other flavour-colour charges, spins, helicities, polarizations as well as total energies, were the only quantities that should be taken to constitute any complete sets of observables.

We will see that a set of symbolic correlations, which directly arises out of the implementation of the operationality of an adequate realization of $SU(2, 2)$, turns out to support the contextual view whereby the physical inner structure of the Schrödinger quantum mechanical picture can be formally described within the observational framework of special relativity, while maintaining the concept of Schrödinger's quantum mechanical states along with the old postulated interpretation of Born's probabilities. The work to be exhibited herein may presumptively enable an implementation in spacetime frames of several measurement processes on a particular class of physical systems, whose outcomes are subjected to clearly defined observational correlations. We believe that the work bears an inherent physical significance in that it may pave the way for the construction of a fundamental background which could allow systematic settlements of covariant quantum computational gates to be obtained.

Indeed, we will consider only the mechanical situation of composite ensembles constituted by pairs of companion particles and antiparticles that can be measured separately. All dynamical spectra shall have neither geometric nor algebraic degeneracy. As emphasized in Ref. [2], the usefulness of measurement processes generally requires the presence of incompatible observables during the occurrences of the processes, but we will avoid this limitation by implementing measurement operators that suitably absorb amplitudes of eigenstates for compatible observables whilst supplying covariantly non-trivial statistical relations between intermediate-stage and outgoing states. It appears that the creation and destruction of states eventually caused by local measurements in the absence of spectral degeneracies, can be made into invariant properties of relativistic observations while Born probabilities can not.

It shall be necessary for us to call upon some of the configurations displayed in Refs. [13, 15, 16] before going into the dynamics of measurement processes. Some review material concerning Cartan's space as well as the algebraic structure of $SU(2, 2)$ and traces of operator restrictions in Cartan's space, is thus brought in Sections 2 through 4. In Section 5, we define prototypes for states, observables and local spectra prior to our transcription. All the corresponding observational correlations are shown in Section 6. In Section 7, we will draw up an outlook related to some important quantum mechanical issues and to our belief in the physical significance of the work. The terminology borne by the theory of linear operators in vector spaces equipped with indefinite inner products [19–23], will be partially used throughout the work. For the sake of convenience, we will make use of the “reflected” bra-ket and index notations explained in Section 1 of Ref. [13]. Particularly, bra-kets carrying double or single angular brackets shall denote Hilbert or indefinite inner products, respectively. In case an operator occurs in a bra-ket product, the vector on which it acts will bear a single bar and the specification of its action will be made up by attaching a double bar to the other vector. When there are operators acting on both sides of a bra-ket product, their action will be stipulated by inserting a single bar between the apposite operator blocks. Basis vectors, measurement operators and created states will be labelled by lower-case Greek letters surrounded by round brackets, while the labels for any vector components, matrix entries and eigenvalues will be taken as non-bracketed lower-case Greek letters. Some further conventions will be explained occasionally. The works of Refs. [13, 15, 16] shall be used many times in Sections 2 and 3, but they will henceforth be referred to explicitly just a few times.

2 Cartan's Space

We initially consider the Hilbert space $\mathcal{H} = (\mathbb{C}^4, \mathcal{D}_I)$, with $\mathcal{D}_I$ being the usual positive-definite inner product on $\mathbb{C}^4$. In the canonical basis $\{ \langle e_{(\mu)} | \}$ for $\mathbb{C}^4$, we write the dual linear combinations for some $\langle x | \in \mathbb{C}^4$

$$\langle x | = x^\mu \langle e_{(\mu)} |, \quad | x \rangle = | e_{(\nu)} \rangle \overline{x^\nu}, \quad (1)$$

such that²

$$\mathcal{D}_I(\langle x |, \langle y |) = \ll x | y \gg = x^\mu \Delta_{\mu\nu} \overline{y^\nu}, \quad (2)$$

with $(\Delta_{\mu\nu})$ denoting the identity (4×4) -matrix whose entries are expressed as

$$\Delta_{\mu\nu} = \ll e_{(\mu)} | e_{(\nu)} \gg. \quad (3)$$

Cartan's space is defined as the pair $\mathfrak{C} = (\mathbb{C}^4, \mathcal{J}_g)$. In this definition, $\mathcal{J}_g$ stands

²Here as elsewhere we have adopted the summation convention. A horizontal bar lying over a symbol should denote the ordinary operation of complex conjugation.

for an indefinite inner product on $\mathbb{C}^4$ which is specified by a completely invertible linear operator g according to the prescription

$$g : (x^0, x^1, x^2, x^3) \mapsto (x^0, x^1, -x^2, -x^3). \quad (4)$$

Thus, $g = g^{-1}$ throughout $\mathbb{C}^4$ whereas

$$\mathcal{J}_g(< x |, < y |) = < x | y >_g = x^\mu g_{\mu\nu} \overline{y^\nu}, \quad (5)$$

with the representation³

$$g_{\mu\nu} = < e_{(\mu)} | e_{(\nu)} >_g, \quad (g_{\mu\nu}) = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix} \quad (6)$$

and the inner-product relationship

$$< x | y >_g = \ll x | g || y \gg. \quad (7)$$

The operator g bears Hermiticity and unitarity with respect to $\mathcal{D}_I$ as well as pseudo-Hermiticity and pseudo-unitarity with respect to $\mathcal{J}_g$. An invertible linear operator A in $\mathfrak{C}$ is said to be pseudo-Hermitian iff $A = A^\star$, in conjunction with the definition

$$< x | A || y >_g = < x || A^\star | y >_g. \quad (8)$$

If A is also pseudo-unitary, we can write

$$A^{-1} = A^\star = A, \quad (9)$$

whence

$$< x | A | A | y >_g = < x || A^\star A | y >_g = < x | A A^\star || y >_g = < x | y >_g. \quad (10)$$

One of the main geometric properties of $\mathfrak{C}$ concerns the existence of pairs of g -orthogonal pseudo-Hermitian projectors in it [19, 21]. To any pair (P^+, P^-) of such projectors, there corresponds a direct-sum splitting like

$$\mathfrak{C} = \mathfrak{C}^+ \oplus \mathfrak{C}^-, \quad (11)$$

where⁴

$$\mathfrak{C}^\pm = \{ < x | \in \mathfrak{C} \text{ such that } < x | x >_g \gtrless 0 \text{ or } < x | = < 0 | \}. \quad (12)$$

We thus have the vector splitting

$$< x | = < x^+ | + < x^- |, \quad (13)$$

³The symbols 0_2 and I_2 denote the zero and identity (2×2) -matrices.

⁴We have denoted the zero vector of $\mathfrak{C}$ as $< 0 |$. The splitting (11) is designated as a *fundamental decomposition* of $\mathfrak{C}$.

together with the property

$$\langle x | P^\pm || x \rangle_g = \langle x || P^\pm | x \rangle_g \quad (14)$$

and the projections

$$\langle e_{(\mu)} | P^\pm = \langle e_{(\mu)}^\pm |, \quad \langle x | P^\pm = \langle x^\pm |, \quad (15)$$

such that

$$\langle x^+ | = (x^0, x^1, 0, 0), \quad \langle x^- | = (0, 0, x^2, x^3). \quad (16)$$

Hence, we can reexpress $\mathcal{J}_g$ as either of the prescriptions

$$\langle x | y \rangle_g = \ll x^+ | y^+ \gg - \ll x^- | y^- \gg \quad (17)$$

and

$$g_{\mu\nu} = \ll e_{(\mu)}^+ | e_{(\nu)}^+ \gg - \ll e_{(\mu)}^- | e_{(\nu)}^- \gg, \quad (18)$$

while decomposing A as

$$\langle x | AP^+ = \langle x^+ | A^{++} + \langle x^- | A^{-+} \quad (19)$$

and

$$\langle x | AP^- = \langle x^+ | A^{+-} + \langle x^- | A^{--}, \quad (20)$$

with the defining scheme

$$\begin{aligned} A^{++} : \mathfrak{E}^+ &\rightarrow R(A) \cap \mathfrak{E}^+, & A^{+-} : \mathfrak{E}^+ &\rightarrow R(A) \cap \mathfrak{E}^- \\ A^{-+} : \mathfrak{E}^- &\rightarrow R(A) \cap \mathfrak{E}^+, & A^{--} : \mathfrak{E}^- &\rightarrow R(A) \cap \mathfrak{E}^-, \end{aligned} \quad (21)$$

and $R(A)$ denoting the range of A . A formal decomposition of the matrix entries

$$A_{\mu\nu} = \langle e_{(\mu)} | A || e_{(\nu)} \rangle_g = A_\mu^\lambda g_{\lambda\nu} \quad (22)$$

thus emerges out of implementing (21). This procedure leads one, in effect, to the following (2×2) -block representation for the operator A

$$(A_{\mu\nu}) = \begin{pmatrix} \langle e_{(\mu)}^+ | A^{++} || e_{(\nu)}^+ \rangle_{g^+} & \langle e_{(\mu)}^+ | A^{+-} || e_{(\nu)}^- \rangle_{g^-} \\ \langle e_{(\mu)}^- | A^{-+} || e_{(\nu)}^+ \rangle_{g^+} & \langle e_{(\mu)}^- | A^{--} || e_{(\nu)}^- \rangle_{g^-} \end{pmatrix}. \quad (23)$$

A *restriction* $\text{Res}A_{\mathfrak{E}^\pm}$ of A on $\mathfrak{E}^\pm$ is an invertible linear operator whose domain and range are prescribed as [19]

$$D(\text{Res}A_{\mathfrak{E}^\pm}) = \mathfrak{E} \cap \mathfrak{E}^\pm, \quad R(\text{Res}A_{\mathfrak{E}^\pm}) \subset \mathfrak{E}^\pm \subset R(A). \quad (24)$$

When A is restricted to $\mathfrak{E}^\pm$, we obtain the operator array⁵

$$A = \begin{pmatrix} \text{Res}A_{\mathfrak{E}^+} & 0 \\ 0 & \text{Res}A_{\mathfrak{E}^-} \end{pmatrix}, \quad (25)$$

⁵In Sections 5 and 6, the ranges of restrictions will be identified with $\mathfrak{E}^\pm$.

together with the expansion

$$\langle x | A = \langle x^+ | \text{Res} A_{\mathfrak{C}^+} + \langle x^- | \text{Res} A_{\mathfrak{C}^-}. \quad (26)$$

For operators A, B and C in $\mathfrak{C}$, we can therefore write

$$\text{Res}(ABC)_{\mathfrak{C}^\pm} = \text{Res} A_{\mathfrak{C}^\pm} \text{Res} B_{\mathfrak{C}^\pm} \text{Res} C_{\mathfrak{C}^\pm}. \quad (27)$$

The restrictions on $\mathfrak{C}^\pm$ of the identity operator I of $\mathfrak{C}$ lead to the representative entries

$$\langle e_{(\mu)}^\pm | \text{Res} I_{\mathfrak{C}^\pm} | e_{(\nu)}^\pm \rangle_{g^\pm} = g_{\mu\nu}^\pm = \pm \langle \langle e_{(\mu)}^\pm | e_{(\nu)}^\pm \rangle \rangle, \quad (28)$$

which produce the reduced (2×2) -block matrices⁶

$$(g_{\mu\nu}^+) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (g_{\mu\nu}^-) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (29)$$

with $g^\pm = \text{Res} g_{\mathfrak{C}^\pm}$. It follows that, by implementing (16) together with the reduced identifications

$$x^0 = x_+^0, \quad x^1 = x_+^1, \quad x^2 = x_-^0, \quad x^3 = x_-^1, \quad (30)$$

and setting the second of (15) as

$$\langle x^\pm | = x_\pm^\mu \langle e_{(\mu)}^\pm |, \quad (31)$$

we recast the inner products on $\mathfrak{C}^\pm$ into the definite configurations

$$\mathcal{J}_{g^\pm} (\langle x^\pm |, \langle y^\pm |) = \langle x^\pm | y^\pm \rangle_{g^\pm} = x_\pm^\mu g_{\mu\nu}^\pm \overline{y_\pm^\nu}, \quad (32)$$

which agree with the pseudo-Hermiticity property

$$\langle x^\pm | g^\pm | y^\pm \rangle_{g^\pm} = \langle x^\pm | g^\pm | y^\pm \rangle_{g^\pm}. \quad (33)$$

Therefore, in agreement with (28), one can also spell out the $\mathcal{J}_g$ -expressions

$$\langle e_{(\mu)} | P^\pm | e_{(\nu)} \rangle_g = \langle e_{(\mu)}^\pm | \text{Res} I_{\mathfrak{C}^\pm} | e_{(\nu)}^\pm \rangle_{g^\pm}, \quad (34)$$

whereas the representation of $\text{Res} I_{\mathfrak{C}^\pm}$ with respect to $\mathcal{D}_I$ is constituted by

$$\langle \langle e_{(\mu)}^\pm | \text{Res} I_{\mathfrak{C}^\pm} | e_{(\nu)}^\pm \rangle \rangle = \langle \langle e_{(\mu)} | P^\pm | e_{(\nu)} \rangle \rangle = \Delta_{\mu\nu}^\pm, \quad (35)$$

which right away yields the reduced matrices

$$(\Delta_{\mu\nu}^+) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (\Delta_{\mu\nu}^-). \quad (36)$$

⁶We should stress that any g -restriction is allowed to be defined only on $\mathfrak{C}^\pm$.

The first adjoint $\mathfrak{C}^* = (\mathbb{C}^4, \mathcal{J}_{g^*})$ of $\mathfrak{C}$ is defined in such a way that each element of $\mathfrak{C}$ enters a one-to-one linear mapping which produces the basis relationships

$$\langle e_{(\mu)}^\pm | \mapsto g_{\mu\lambda}^\pm \langle e_\pm^{*(\lambda)} |, \quad \langle e_\pm^{*(\mu)} | \mapsto g_\pm^{*\mu\lambda} \langle e_{(\lambda)}^\pm |. \quad (37)$$

It should be evident that the operator rule for g^* is formally the same as the one for g such that it is reasonable to bring forth the splitting

$$\mathfrak{C}^* = \mathfrak{C}_+^* \oplus \mathfrak{C}_-^*, \quad (38)$$

with $\mathfrak{C}_\pm^*$ thus holding formally the same definition as that given by (12). Whence, for some element $\langle x^* |$ of $\mathfrak{C}^*$, we can take account of the component identifications

$$x_0 = x_0^+, \quad x_1 = x_1^+, \quad x_2 = x_0^-, \quad x_3 = x_1^- \quad (39)$$

for rewriting the adjoint linear combination

$$\langle x^* | = x_\mu \langle e^{*(\mu)} | \quad (40)$$

in terms of the configurations

$$\langle x_\pm^* | = x_\mu^\pm \langle e_\pm^{*(\mu)} |, \quad x_\mu^\pm = x_\pm^\lambda g_{\lambda\mu}^\pm, \quad x_\pm^\mu = x_\lambda^\pm g_\pm^{*\lambda\mu}. \quad (41)$$

This procedure promptly yields

$$\langle x^\pm | e_{(\mu)}^\pm \rangle_{g^\pm} = x_\mu^\pm, \quad \langle x_\pm^* | e_\pm^{*(\mu)} \rangle_{g_\pm^*} = x_\pm^\mu, \quad (42)$$

and likewise brings out the reflexiveness property

$$\langle x^\pm | x^\pm \rangle_{g^\pm} = \langle x_\pm^* | x_\pm^* \rangle_{g_\pm^*}. \quad (43)$$

Equations (37) carry forward the canonical character of $\langle e_{(\mu)} |$ to $\langle e^{*(\mu)} |$ such that the entries⁷

$$g_\pm^{*\mu\nu} = \langle e_\pm^{*(\mu)} | e_\pm^{*(\nu)} \rangle_{g_\pm^*}, \quad \Delta_\pm^{*\mu\nu} = \ll e_\pm^{*(\mu)} | e_\pm^{*(\nu)} \gg, \quad (44)$$

coincide with those for $\text{Res} I_{\mathbb{C}^\pm}$. Additionally, the entries (44) obey the relations⁸

$$g_{\mu\nu}^\pm = \Delta_{\mu\lambda}^\pm g_\pm^{*\lambda\sigma} \Delta_{\sigma\nu}^\pm, \quad g_\pm^{*\mu\nu} = \Delta_\pm^{*\mu\lambda} g_{\lambda\sigma}^\pm \Delta_\pm^{*\sigma\nu}, \quad (45)$$

along with the ones that are obtained from (45) by just interchanging the kernel letters Δ and g . For the restrictions carried by (28), we thus have the decomposition

$$\text{Res} I_{\mathfrak{C}^\pm} = | e_{(\mu)}^\pm \rangle_{g^\pm} g_\pm^{*\mu\nu} \langle e_{(\nu)}^\pm |, \quad (46)$$

⁷It is clear that $\mathfrak{C}^\pm \simeq \mathbb{C}^2 \simeq \mathfrak{C}_\pm^*$ such that the entry labels carried by any reduced or restricted structures must assume the values 0 and 1.

⁸It is of some interest to remark that the $*$ -adjointness obeys the same rearrangement of factors as the one that occurs in pseudo-Hermitian and Hermitian conjugations of matrix products. This will be used sometimes in Section 6.

which shows us that $\langle e_{(\mu)}^\pm |$ possesses a completeness property.

The unrestricted adjoint entries

$$A^{*\mu\nu} = \langle e_{(\mu)}^{*\mu} | A^* || e^{*(\nu)} \rangle_{g^*}, \quad (47)$$

satisfy

$$A_{\mu}^{\nu} = g_{\mu\lambda} A^{*\lambda\nu} \implies A_{\mu\nu} = g_{\mu\lambda} A^{*\lambda\sigma} g_{\sigma\nu} \quad (48)$$

and

$$A^{*\mu}_{\nu} = g^{*\mu\lambda} A_{\lambda\nu} \implies A^{*\mu\nu} = g^{*\mu\lambda} A_{\lambda\sigma} g^{*\sigma\nu}. \quad (49)$$

Hence, by adapting to (22) the notation of (29), likewise taking up (47), one obtains

$$A_{\mu\nu}^\pm = \langle e_{(\mu)}^\pm | A^\pm || e_{(\nu)}^\pm \rangle_{g^\pm}, \quad A_{\pm}^{*\mu\nu} = \langle e_{\pm}^{*(\mu)} | A_{\pm}^* || e_{\pm}^{*(\nu)} \rangle_{g_{\pm}^*} \quad (50)$$

and

$$A^\pm = | e_{(\mu)}^\pm \rangle A_{\pm}^{*\mu\nu} \langle e_{(\nu)}^\pm |, \quad A_{\pm}^* = | e_{\pm}^{*(\mu)} \rangle A_{\mu\nu}^\pm \langle e_{\pm}^{*(\nu)} |, \quad (51)$$

along with

$$A_{\mu\nu}^\pm = g_{\mu\lambda}^\pm A_{\pm}^{*\lambda\sigma} g_{\sigma\nu}^\pm, \quad A_{\pm}^{*\mu\nu} = g_{\pm}^{*\mu\lambda} A_{\lambda\sigma}^\pm g_{\pm}^{*\sigma\nu} \quad (52)$$

and

$$\text{Res} I_{\mathfrak{C}_{\pm}^*}^* = | e_{\pm}^{*(\mu)} \rangle g_{\mu\nu}^\pm \langle e_{\pm}^{*(\nu)} |. \quad (53)$$

The entire set of relationships between Hermitian and pseudo-Hermitian conjugations in $\mathfrak{C}$, is supplied in Ref. [16]. We next recall just the ones which may play a major role in Sections 5 and 6. A simpler procedure for starting with involves using the restricted form of Eq. (7), that is to say,

$$\langle x^\pm | y^\pm \rangle_{g^\pm} = \ll x^\pm | g^\pm || y^\pm \gg, \quad (54)$$

for pointing up that the definition

$$\langle e_{(\mu)}^\pm | A^{\pm\star} || e_{(\nu)}^\pm \rangle_{g^\pm} = \langle e_{(\mu)}^\pm || A^\pm | e_{(\nu)}^\pm \rangle_{g^\pm} \quad (55)$$

amounts to the same thing as⁹

$$\begin{aligned} \ll e_{(\mu)}^\pm | A^{\pm\star} g^\pm || e_{(\nu)}^\pm \gg &= \ll e_{(\mu)}^\pm | g^\pm | A^\pm | e_{(\nu)}^\pm \gg \\ &= \ll e_{(\mu)}^\pm | g^\pm A^{\pm\dagger} || e_{(\nu)}^\pm \gg. \end{aligned} \quad (56)$$

We are thus led to the operator relationship

$$A^{\pm\star} g^\pm = g^\pm A^{\pm\dagger}. \quad (57)$$

⁹From now onwards the "dagger" will be used to denote the Hermitian conjugation for the Hilbert inner product.

It follows that, setting¹⁰

$$A_{\mu\nu}^{\pm\star} = \langle e_{(\mu)}^{\pm} | A^{\pm\star} | e_{(\nu)}^{\pm} \rangle_{g^{\pm}} = \overline{A_{\mu\nu}^{\pm T}} \quad (58)$$

and

$$a_{\mu\nu}^{\pm\dagger} = \ll e_{(\mu)}^{\pm} | A^{\pm\dagger} | e_{(\nu)}^{\pm} \gg = \overline{a_{\mu\nu}^{\pm T}}, \quad (59)$$

as well as making use of the formulae

$$g_{\mu}^{\pm\nu} = \Delta_{\mu\lambda}^{\pm} g_{\pm}^{*\lambda\nu}, \quad g_{\pm}^{*\mu\nu} = \Delta_{\pm}^{*\mu\lambda} g_{\lambda\nu}^{\pm} \quad (60)$$

and

$$A_{\mu\nu}^{\pm\dagger} = \langle e_{(\mu)}^{\pm} | A^{\pm\dagger} | e_{(\nu)}^{\pm} \rangle_{g^{\pm}} = a_{\mu\lambda}^{\pm\dagger} g_{\pm}^{*\lambda\nu}, \quad (61)$$

yields the associations

$$A_{\mu\nu}^{\pm\star} = g_{\mu}^{\pm\lambda} A_{\lambda\sigma}^{\pm\dagger} g_{\pm}^{*\sigma\nu} \quad (62)$$

and

$$A_{\pm}^{*\star\mu\nu} = g_{\pm}^{*\mu\lambda} A_{\lambda\sigma}^{*\dagger\lambda\sigma} g_{\sigma}^{\pm\nu}, \quad A_{\pm}^{*\dagger\mu\nu} = a_{\pm}^{*\dagger\mu\lambda} g_{\lambda}^{\pm\nu}. \quad (63)$$

A pair of interesting relationships arises when we bring together (52), (60) and (62). We have, in effect,

$$A_{\mu\nu}^{\pm\star} = \Delta_{\mu\lambda}^{\pm} A_{\pm}^{*\dagger\lambda\sigma} \Delta_{\sigma\nu}^{\pm}, \quad A_{\pm}^{*\star\mu\nu} = \Delta_{\pm}^{*\mu\lambda} A_{\lambda\sigma}^{\pm\dagger} \Delta_{\pm}^{*\sigma\nu}, \quad (64)$$

whence the entry arrays that constitute the matrices for each of the pairs $(A_{\mu\nu}^{\pm\star}, A_{\pm}^{*\dagger\lambda\sigma})$ and $(A_{\pm}^{*\star\mu\nu}, A_{\lambda\sigma}^{\pm\dagger})$, coincide with one another.

3 The Group $SU(2,2)$

From the algebraic point of view, $SU(2,2)$ is the group constituted by the usual operation of matrix multiplication and the set of unimodular matrices that represent either of the classes $\{\mathcal{U}, \mathcal{U}^*\}$ of pseudo-unitary operators in $\mathfrak{C}$ and $\mathfrak{C}^*$. The group $SU(2,2)$ appears as [23–29] as the *special* subgroup of $U(2,2)$ whose action leaves in a broader manner the definitions (4)-(6) invariant. We have

$$SU(2,2) = \{\check{u} \in U(2,2) \text{ with } \det(\check{u}) = 1\}, \quad (65)$$

whence we can state that

$$SU(2,2) = U(2,2) \cap SL(4, \mathbb{C}) \subset SL(4, \mathbb{C}). \quad (66)$$

The maximal compact subgroup of $SU(2,2)$ amounts to [24]

$$K = SU(2) \times U(1) \times SU(2) \subset SU(4), \quad (67)$$

¹⁰Henceforth the superscript T will denote the usual operation of matrix transposition.

where¹¹

$$SU(n) = U(n) \cap SL(n, \mathbb{C}). \quad (68)$$

An important property of $SU(2, 2)$ comes from Cartan's decomposition theorem [14, 24, 27], which asserts that there exists a one-to-one homeomorphism of $SU(2, 2)$ onto the topological product

$$T_C = (SU(2, 2) \cap U(4)) \times (SU(2, 2) \cap P(4)), \quad (69)$$

where $P(4)$ is the group of positive-definite Hermitian (4×4) -matrices. This statement may be schematically displayed as $SU(2, 2) \ni \check{u} \mapsto (U, H) \in T_C$, $U \in (SU(2, 2) \cap U(4))$, $H \in (SU(2, 2) \cap P(4))$. Accordingly, we have the unique coupling prescription

$$\check{u} = UH, \quad (70)$$

such that we can likewise write

$$\check{u}^\dagger \check{u} = H^2, \quad \check{u} \check{u}^\dagger = UH^2U^{-1}. \quad (71)$$

It is shown in Ref. [15] that any adjoint realizations of $SU(2, 2)$ are just the same and possess closedness with respect to both pseudo-Hermitian and Hermitian conjugations. In any case, the only admissible bases for representing pseudo-unitary operators in $\mathfrak{C}$ and $\mathfrak{C}^*$ are $SU(2, 2)$ -related to each other.¹² In what follows, we will reproduce to some extent the presentation of the unstarred g -realization of $SU(2, 2)$ as exhibited in Refs. [13, 15]. If $u \in \mathcal{U}$, we may thus write the configuration¹³

$$uu^\star = I \Leftrightarrow ugu^\dagger = g, \quad (72)$$

where I denotes the identity operator of $\mathfrak{C}$ such as in Eq. (28). Hence, utilizing a decomposition for each of u and $u^\star$ of the same type as that given by (21), yields the $\star$ -invariant operator relations

$$u^{++}u^{++\star} + u^{+-}u^{+-\star} = I^+, \quad u^{--}u^{--\star} + u^{-+}u^{-+\star} = I^- \quad (73)$$

and

$$u^{++}u^{-+\star} + u^{+-}u^{--\star} = 0 = u^{-+}u^{++\star} + u^{--}u^{+-\star}, \quad (74)$$

where $I^\pm = \text{Res}I_{\mathfrak{C}^\pm}$.

In any admissible basis, the entries of the matrices which represent the relations (73) and (74) are expressed in much the same way as those of (22) and (23). We have the defining constraints

$$u_{\mu\lambda}^{++} g_+^{*\lambda\sigma} u_{\sigma\nu}^{++\star} + u_{\mu\lambda}^{+-} g_-^{*\lambda\sigma} u_{\sigma\nu}^{+-\star} = g_{\mu\nu}^+ \quad (75)$$

¹¹In (68), $U(n)$ denotes the group of unitary $(n \times n)$ -matrices for the Hilbert inner product, with $U(1)$ being particularly the multiplicative group of unit-modulus complex numbers.

¹²The basis $\{< e_{(\mu)} | \}$ is the so-called [13] *computational basis* of $\mathfrak{C}$.

¹³The g -invariance exhibited by (72) will be made explicit in Section 6.

and

$$u_{\mu\lambda}^{--} g_{-}^{*\lambda\sigma} u_{\sigma\nu}^{--\star} + u_{\mu\lambda}^{-+} g_{+}^{*\lambda\sigma} u_{\sigma\nu}^{-+\star} = g_{\mu\nu}^{-}, \quad (76)$$

along with

$$u_{\mu\lambda}^{++} g_{+}^{*\lambda\sigma} u_{\sigma\nu}^{++\star} + u_{\mu\lambda}^{+-} g_{-}^{*\lambda\sigma} u_{\sigma\nu}^{+-\star} = 0_2 \quad (77)$$

and the $\star$ -conjugate of (77). Obviously, the entries for $u^{+-\star}$ and $u^{-+\star}$ must account for the interchange of operator actions borne by (8) and (21) whence, in the computational basis, we have

$$u_{\mu\nu}^{+-} = \langle e_{(\mu)}^{+} | u^{+-} | e_{(\nu)}^{-} \rangle_{g^{-}}, \quad u_{\mu\nu}^{+-\star} = \langle e_{(\mu)}^{-} | u^{+-\star} | e_{(\nu)}^{+} \rangle_{g^{+}} \quad (78)$$

and

$$u_{\mu\nu}^{-+} = \langle e_{(\mu)}^{-} | u^{-+} | e_{(\nu)}^{+} \rangle_{g^{+}}, \quad u_{\mu\nu}^{-+\star} = \langle e_{(\mu)}^{+} | u^{-+\star} | e_{(\nu)}^{-} \rangle_{g^{-}}. \quad (79)$$

Whenever u is taken to bear unitarity as well, its decomposition as provided by (73) and (74) turns out to be such that the constituents u^{+-} and u^{-+} amount both to zero operators. Under this circumstance, we should take into account the conditions

$$u^{++\star} = u^{++\dagger}, \quad u^{--\star} = u^{--\dagger}, \quad (80)$$

with

$$\begin{aligned} (u_{\mu\nu}^{++}) &\in GL(2, \mathbb{C}) \cap U(2) \ni (u_{\mu\nu}^{--}), \\ \det(u_{\mu\nu}^{++}) &= \exp[i\phi] = \det(u_{\mu\nu}^{--})^{-1}, \end{aligned} \quad (81)$$

and ϕ being some real number. Therefore, any unitary element of $SU(2, 2)$ possesses the block-diagonal form

$$(u_{\mu\nu}) = \begin{pmatrix} (u_{\lambda\sigma}^{++}) & 0_2 \\ 0_2 & (u_{\lambda\sigma}^{--}) \end{pmatrix}. \quad (82)$$

From (22) and (72), we can see that the matrix $(I_{\mu\nu})$ bears $SU(2, 2)$ -invariance by definition, while $(\Delta_{\mu\nu})$ and $(\Delta^{*\mu\nu})$ must bear invariance under transformations belonging to the unitary intersection $SU(2, 2) \cap U(4)$.

4 Traces

The trace of an invertible operator A in $\mathfrak{C}$ is a linear attribute defined as

$$\text{Tr } A = \langle e_{(\mu)} | A | e_{(\lambda)} \rangle_g g^{*\lambda\mu} = A_{\mu}^{\mu}, \quad (83)$$

with its adjoint counterpart being

$$\text{Tr } A^{*} = \langle e^{*(\mu)} | A^{*} | e^{*(\lambda)} \rangle_{g^{*}} g_{\lambda\mu} = A^{*\mu}_{\mu}. \quad (84)$$

More explicitly, we have

$$\text{Tr } A = A_{00} + A_{11} - A_{22} - A_{33}, \quad (85)$$

or, equivalently,

$$\text{Tr } A = a_{00} + a_{11} + a_{22} + a_{33}. \quad (86)$$

Equations (25), (50) and (52) thus supply us with the restricted expressions

$$\text{Tr } A^\pm = A_{\mu\lambda}^\pm g_{\pm}^{*\lambda\mu} = g_{\mu\lambda}^\pm A_{\pm}^{*\lambda\mu}, \quad \text{Tr } A_\pm^* = A_{\pm}^{*\mu\lambda} g_{\lambda\mu}^\pm = g_{\pm}^{*\mu\lambda} A_{\lambda\mu}^\pm, \quad (87)$$

which, by virtue of (45), can be reset as

$$\text{Tr } A^\pm = a_{\mu\lambda}^\pm \Delta_{\pm}^{*\lambda\mu}, \quad \text{Tr } A_\pm^* = a_{\pm}^{*\mu\lambda} \Delta_{\lambda\mu}^\pm. \quad (88)$$

Then, for the decompositions (51), we get

$$\text{Tr } |e_{(\mu)}^\pm\rangle A_\pm^{*\mu\nu} \langle e_{(\nu)}^\pm| = \text{Tr } A^\pm \quad (89)$$

and

$$\text{Tr } |e_\pm^{*(\mu)}\rangle A_{\mu\nu}^\pm \langle e_\pm^{*(\nu)}| = \text{Tr } A_\pm^*. \quad (90)$$

By recalling (27) and allowing for an operator product AB along with the entries

$$\langle e_{(\mu)}^\pm | A^\pm B^\pm | e_{(\nu)}^\pm \rangle_{g^\pm} = A_{\mu\lambda}^\pm g_{\pm}^{*\lambda\sigma} B_{\sigma\nu}^\pm \quad (91)$$

and

$$\langle e_\pm^{*(\mu)} | A_\pm^* B_\pm^* | e_\pm^{*(\nu)} \rangle_{g_\pm^*} = A_{\pm}^{*\mu\lambda} g_{\lambda\sigma}^\pm B_{\pm}^{*\sigma\nu}, \quad (92)$$

we obtain

$$\text{Tr } (A^\pm B^\pm) = A_{\mu\lambda}^\pm g_{\pm}^{*\lambda\sigma} B_{\sigma\rho}^\pm g_{\pm}^{*\rho\mu} = A_{\mu\lambda}^\pm B_{\pm}^{*\lambda\mu} \quad (93)$$

and

$$\text{Tr } (A_\pm^* B_\pm^*) = A_{\pm}^{*\mu\lambda} g_{\lambda\sigma}^\pm B_{\pm}^{*\sigma\rho} g_{\rho\mu}^\pm = A_{\pm}^{*\mu\lambda} B_{\lambda\mu}^\pm. \quad (94)$$

One could expect that the definition of traces of operators subjected to the pseudo-Hermitian conjugation should be formally similar to (83). This is really the case as can be seen by invoking Eqs. (60) and (61) along with (62) to perform the computation

$$\text{Tr } A^{\pm\star} = g_\mu^{\pm\lambda} a_{\lambda\sigma}^{\pm\dagger} g_{\pm}^{*\sigma\mu} = g_\mu^{\pm\lambda} A_{\lambda}^{\pm\dagger\sigma} \Delta_{\sigma\rho}^\pm g_{\pm}^{*\rho\mu} = g_\mu^{\pm\lambda} A_{\lambda}^{\pm\dagger\sigma} g_{\sigma}^{\pm\mu} = A_{\mu}^{\pm\star\mu}. \quad (95)$$

The formal properties of traces of operators in Cartan's space are the same as the ones which occur in the Hilbert-space framework [5, 22]. A typical unrestricted calculation is, in effect, carried out as follows:

$$\langle e_{(\mu)} | A | e_{(\lambda)} \rangle_g g^{*\lambda\mu} = \overline{\langle e_{(\lambda)} | A^\star | e_{(\mu)} \rangle_g g^{*\mu\lambda}} = \overline{A_{\lambda\mu}^\star g^{*\mu\lambda}}, \quad (96)$$

which brings about the relation

$$\text{Tr } A^\pm = \overline{\text{Tr } A^\pm \star}. \quad (97)$$

In a similar way, we obtain

$$\text{Tr } A_\pm^* = \overline{\text{Tr } A_\pm^{*\star}}, \quad (98)$$

whence, using (97), say, with AB in place of A , yields

$$\text{Tr } (A^\pm B^\pm) = \overline{\text{Tr } (B^\pm \star A^\pm \star)}. \quad (99)$$

We notice that Eq. (99) can be straightforwardly obtained from the combination of (97) with the prescription

$$\text{Tr } (A^\pm B^\pm) = \text{Tr } (A^\pm B^\pm) \star \star = \text{Tr } (B^\pm \star A^\pm \star)^\star, \quad (100)$$

which enables us to carry out the development¹⁴

$$\begin{aligned} \text{Tr } (A^\pm B^\pm) &= \overline{\text{Tr } (A^\pm B^\pm) \star} = \overline{\text{Tr } (B^\pm \star A^\pm \star)} = \overline{B_{\mu\lambda}^\pm \star A_\pm^{*\star\lambda\mu}} \\ &= B_{\mu\lambda}^{\pm T} A_\pm^{*T\lambda\mu} = B_{\lambda\mu}^\pm A_\pm^{*\mu\lambda} = \text{Tr } (B^\pm A^\pm), \end{aligned} \quad (101)$$

whose adjoint version reads off

$$\text{Tr } (A_\pm^* B_\pm^*) = \text{Tr } (B_\pm^* A_\pm^*). \quad (102)$$

If B is reset as $A^\star$ in (99), we will conclude that

$$\text{ImTr } (A^\pm A^\pm \star) = 0. \quad (103)$$

In the case of a pseudo-unitary A , we thus obtain the particular value

$$\text{Tr } (A^\pm A^\pm \star) = \text{Tr } (A^\pm \star A^\pm) = \text{Tr } I^\pm = g_{\mu\lambda}^\pm g_{\pm}^{*\lambda\mu} = 2, \quad (104)$$

whilst in the pseudo-Hermitian case, by (97) and (103), we get

$$\text{ImTr } A^\pm = 0, \text{ImTr } (A^\pm)^2 = 0. \quad (105)$$

Furthermore, if the operator A carried by (104) also holds unitarity, then we may write

$$\text{Tr } A^\pm \star = \text{Tr } A^{\pm\dagger}, \quad (106)$$

which produces the specialized configuration

$$A_{\mu\lambda}^\pm \star g_{\pm}^{*\lambda\mu} = a_{\mu\lambda}^{\pm\dagger} \Delta_{\pm}^{*\lambda\mu}. \quad (107)$$

¹⁴It is worth remarking that the $\star$ -adjointness commutes with the $\star$ -conjugation.

5 Measurement Processes

We shall in this Section transcribe into the framework of Cartan's space the description of the measurement processes we had referred to before. It will suffice to allow for the unstarred picture of Section 2 as far as our transcription is concerned, since adjoint formulations in Cartan's space are physically equivalent.

According to the usual definition of complete sets of commuting observables [1, 2, 5], any quantum system may have in some given frame only one complete set of compatible observables and these can always admit common eigenvectors even within the context of indefinite inner-product spaces, as mentioned in Section 1. We consider an ensemble constituted by pairs of companion systems

$$\{(\mathcal{A}^+, \mathcal{A}^-), (\mathcal{B}^+, \mathcal{B}^-), (\mathcal{C}^+, \mathcal{C}^-), \dots, (\mathcal{D}^+, \mathcal{D}^-)\}, \quad (108)$$

with the complete sets for each pair (Σ^+, Σ^-) carrying observables that supposedly possess non-degenerate twofold discrete spectra and can be measured separately. The complete sets for different pairs should carry observables that act on identifiable copies of Cartan's space such as, for instance, $\mathfrak{C}^+(\mathcal{A}^+)$ and $\mathfrak{C}^-(\mathcal{B}^-)$, whereas the space of states for the ensemble should be looked upon as the tensor product of all copies, namely¹⁵

$$\mathfrak{C}^+(\mathcal{A}^+) \otimes \mathfrak{C}^-(\mathcal{A}^-) \otimes \dots \otimes \mathfrak{C}^+(\mathcal{D}^+) \otimes \mathfrak{C}^-(\mathcal{D}^-). \quad (109)$$

The common eigenbras of the observables for $\Sigma^\pm$ are taken as the elements of the computational basis of the copy $\mathfrak{C}^\pm(\Sigma^\pm)$ but, as the computational bases of all copies are indistinguishable among themselves, we will for simplicity denote the common eigenbras for the complete sets of all systems of (108) as¹⁶

$$\{< e_{(0)}^+ |, < e_{(1)}^+ | \}, \{< e_{(0)}^- |, < e_{(1)}^- | \}. \quad (110)$$

A state bra for $\Sigma^\pm$ is identified with a normalized element of $\mathfrak{C}^\pm(\Sigma^\pm)$, and thus written as

$$< s^\pm | = S_\pm^\mu < e_{(\mu)}^\pm |, \quad (111)$$

with Eqs. (41) and (42) yielding the amplitudes

$$S_\pm^0 = \pm < s^\pm | e_{(0)}^\pm >_{g^\pm}, \quad S_\pm^1 = \pm < s^\pm | e_{(1)}^\pm >_{g^\pm} \quad (112)$$

and the normalization condition

$$< s^\pm | g^\pm || s^\pm >_{g^\pm} = \text{Tr} | s^\pm > < s^\pm | g^\pm = S_\pm^\mu \Delta_{\mu\nu}^\pm \overline{S_\pm^\nu} = 1, \quad (113)$$

¹⁵Tensor products of copies of Cartan's space involve juxtapositions of suitable numbers of $g^\pm$ and $g_\pm^*$. A detailed description of composite systems in Cartan's space can be found in Section 4 of Ref. [13].

¹⁶By this point, we recall that (30) entails that the entry labels borne by any restricted configurations must just take the values 0 and 1. The notation of (110) may formally facilitate the evaluation of inner products between states for different systems of (108).

such that we can write the density equations

$$\langle s^\pm | \rho(\Sigma^\pm) = \langle s^\pm |, \rho^\star(\Sigma^\pm) | s^\pm \rangle = | s^\pm \rangle, \quad (114)$$

where¹⁷

$$\rho(\Sigma^\pm) = g^\pm | s^\pm \rangle \langle s^\pm | = | s^\pm \rangle \langle s^\pm | g^\pm = \rho^\star(\Sigma^\pm). \quad (115)$$

We adopt the definition put into practice in Ref. [13] by which any observables belonging to the complete set for either Σ^+ or Σ^- amount to pseudo-Hermitian restrictions on $\mathfrak{C}^\pm(\Sigma^\pm)$ that formally resemble the operator pieces carried by Eq. (25). Then, by (57), such an observable $S^\pm$ must obey the Hermiticity relations

$$S^\pm = S^{\pm\star} = S^{\pm\dagger}, \quad (116)$$

which guarantee [19, 21] its spectral reality, whereas

$$D(S^\pm) = R(S^\pm) = \mathfrak{C}^\pm(\Sigma^\pm). \quad (117)$$

The spectra

$$\text{spec } S^+ = \{s_0^+, s_1^+\}, \text{ spec } S^- = \{s_0^-, s_1^-\}, \quad (118)$$

come straightaway from the eigenvalue equations

$$\langle e_{(0)}^\pm | S^\pm = s_0^\pm \langle e_{(0)}^\pm |, \langle e_{(1)}^\pm | S^\pm = s_1^\pm \langle e_{(1)}^\pm |, \quad (119)$$

which produce the spectral entries

$$\langle e_{(0)}^\pm | S^\pm || e_{(0)}^\pm \rangle_{g^\pm} = \pm s_0^\pm \quad (120)$$

and

$$\langle e_{(1)}^\pm | S^\pm || e_{(1)}^\pm \rangle_{g^\pm} = \pm s_1^\pm, \quad (121)$$

together with the diagonal array

$$(S_{\mu\nu}^\pm) = \begin{pmatrix} \pm s_0^\pm & 0 \\ 0 & \pm s_1^\pm \end{pmatrix} \quad (122)$$

and the expansion

$$\langle s^\pm | S^\pm = S_\pm^0 s_0^\pm \langle e_{(0)}^\pm | + S_\pm^1 s_1^\pm \langle e_{(1)}^\pm |. \quad (123)$$

Consequently, the spectral decompositions (51) yield the definite traces

$$\text{Tr } S^\pm = \text{Tr } | e_{(\mu)}^\pm \rangle S_\pm^{*\mu\nu} \langle e_{(\nu)}^\pm | = s_0^\pm + s_1^\pm, \quad (124)$$

¹⁷According to Ref. [13], the local evolution of any state is controlled by a suitably defined unitary operator restriction which has to be required to depend explicitly only upon the proper time.

whence, for the pertinent expectation value of $S^\pm$, we have

$$\langle S^\pm \rangle_{s^\pm} = \pm s_\mu^\pm | S_\pm^\mu |^2 = \text{Tr}(\rho(\Sigma^\pm) S^\pm). \quad (125)$$

The full apparatuses used for performing measurements on quantum systems generally consist of ordered sequences of physical devices that pick up and transmit state vectors out of incoming ones, in agreement with prescribed dynamical selections. In the works exhibited in Refs. [2, 3], each such device is represented by a Hermitian operator in the Hilbert space of a non-relativistic microcosmic ensemble whose dynamical attributes possess finite numbers of eigenstates. Incoming states of similar systems are thus sorted by the utilized measurement devices into outgoing subensembles that are distinguishable from each other by means of selected eigenstates.

We require any measurement device to be represented by a decomposable operator restriction on an appropriate subspace of the tensor product (109). Selective measurements on the systems (108) are accomplished from the implementation of measurement operators that adequately absorb amplitudes of state vectors upon performing state selections and reductions.

The simplest measurement process involves a selective measurement on some $\Sigma^\pm$ which is in either case performed by a single device that accepts one of the eigenbras carried by (110) while correspondingly rejecting the other. For any such process, the representative measurement operator appears as one of the restrictions

$$\pi_{(0)}^+(\Sigma^+), \pi_{(1)}^+(\Sigma^+), \pi_{(0)}^-(\Sigma^-), \pi_{(1)}^-(\Sigma^-), \quad (126)$$

which, by definition, bears pseudo-Hermiticity and reduces an incoming state in accordance with the prescription

$$\langle s^\pm | \pi_{(\mu)}^\pm(\Sigma^\pm) = \langle s_{(\mu)}^\pm |, \quad (127)$$

where¹⁸

$$\langle s_{(0)}^\pm | = S_\pm^0 \langle e_{(0)}^\pm |, \quad \langle s_{(1)}^\pm | = S_\pm^1 \langle e_{(1)}^\pm |. \quad (128)$$

We can then write

$$\langle s^\pm | = \langle s_{(0)}^\pm | + \langle s_{(1)}^\pm | \quad (129)$$

and

$$\langle s^\pm | s^\pm \rangle_{g^\pm} = \langle s_{(0)}^\pm | s_{(0)}^\pm \rangle_{g^\pm} + \langle s_{(1)}^\pm | s_{(1)}^\pm \rangle_{g^\pm}, \quad (130)$$

such that $\pi_{(\mu)}^\pm(\Sigma^\pm)$ also causes the reduced-state annihilations

$$\langle s_{(1)}^\pm | \pi_{(0)}^\pm(\Sigma^\pm) = \langle 0^\pm |, \quad \langle s_{(0)}^\pm | \pi_{(1)}^\pm(\Sigma^\pm) = \langle 0^\pm |, \quad (131)$$

while (123) becomes

$$\langle s^\pm | S^\pm = s_0^\pm \langle s_{(0)}^\pm | + s_1^\pm \langle s_{(1)}^\pm |. \quad (132)$$

¹⁸The eigenvalue character of (119) will still hold if we substitute $\langle e_{(\mu)}^\pm |$ for $\langle s_{(\mu)}^\pm |$.

Equations (127) and (128) thus give the completeness decomposition

$$\pi_{(\mu)}^{\pm}(\Sigma^{\pm}) = |e_{(\rho)}^{\pm} \rangle g_{\pm}^{*\rho\lambda} \pi_{(\mu)\lambda\sigma}^{\pm}(\Sigma^{\pm}) g_{\pm}^{*\sigma\tau} \langle e_{(\tau)}^{\pm} |, \quad (133)$$

along with the representations¹⁹

$$(\pi_{(0)\lambda\sigma}^{\pm}(\Sigma^{\pm})) = \begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\pi_{(1)\lambda\sigma}^{\pm}(\Sigma^{\pm})) = \begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad (134)$$

and

$$\pi_{(\mu)\lambda\sigma}^{\pm}(\Sigma^{\pm}) = \langle e_{(\lambda)}^{\pm} | \pi_{(\mu)}^{\pm}(\Sigma^{\pm}) | e_{(\sigma)}^{\pm} \rangle_{g^{\pm}}, \quad (135)$$

whence, fitting together the first of (87) with (134), yields

$$\text{Tr } \pi_{(\mu)}^{\pm}(\Sigma^{\pm}) = 1, \quad (136)$$

whilst the operators (126) hold the projection property²⁰

$$\pi_{(0)}^{\pm}(\Sigma^{\pm}) \pi_{(0)}^{\pm}(\Sigma^{\pm}) = \pi_{(0)}^{\pm}(\Sigma^{\pm}), \quad \pi_{(1)}^{\pm}(\Sigma^{\pm}) \pi_{(1)}^{\pm}(\Sigma^{\pm}) = \pi_{(1)}^{\pm}(\Sigma^{\pm}). \quad (137)$$

The state reductions (127) and the occurrences of the spectral values as set by (118), are interrelated via the Born probabilities

$$\langle s^{\pm} | \pi_{(0)}^{\pm}(\Sigma^{\pm}) g^{\pm} \parallel s^{\pm} \rangle_{g^{\pm}} = |S_{\pm}^0|^2 \quad (138)$$

and

$$\langle s^{\pm} | \pi_{(1)}^{\pm}(\Sigma^{\pm}) g^{\pm} \parallel s^{\pm} \rangle_{g^{\pm}} = |S_{\pm}^1|^2, \quad (139)$$

with one of Eqs. (60) having been employed to write (138) and (139). Hence, the condition (113) can be reinstated as

$$\begin{aligned} \langle s^{\pm} | g^{\pm} \parallel s^{\pm} \rangle_{g^{\pm}} = & \langle \langle s^{\pm} | \pi_{(0)}^{\pm}(\Sigma^{\pm}) \parallel s^{\pm} \rangle \rangle \\ & + \langle \langle s^{\pm} | \pi_{(1)}^{\pm}(\Sigma^{\pm}) \parallel s^{\pm} \rangle \rangle, \end{aligned} \quad (140)$$

or, alternatively, as

$$\text{Tr } |e_{(\lambda)}^{\pm} \rangle \pi_{\pm}^{*(\mu)\lambda\sigma}(\Sigma^{\pm}) \langle e_{(\sigma)}^{\pm} | = 1, \quad (141)$$

but the normalizability of $\langle s^{\pm} |$ gets lost when the reductions prescribed by $\pi_{(\mu)}^{\pm}(\Sigma^{\pm})$ are brought into effect. Therefore, the significant reduced states produced by the measurement operators (126), are given by

$$\frac{1}{|S_{\pm}^0|} \langle s_{(0)}^{\pm} |, \quad \frac{1}{|S_{\pm}^1|} \langle s_{(1)}^{\pm} |. \quad (142)$$

¹⁹Matrix representations equal their adjoint counterparts in accordance with Eqs. (48)-(52).

²⁰The operators $\pi_{(\mu)}^{\pm}$ may afford a relativistic version of POVM operators.

The process just described can be extended to the extent that a measurement operator $\pi^\pm(s_\mu^\pm, s_\nu^\pm)$ is used²¹ in lieu of those of (126), which annihilates an incoming reduced state $\langle s_{(\mu)}^\pm |$ and produces an outgoing reduced state $\langle s_{(\nu)}^\pm |$. It is clear that the situation of general interest demands the label distinction $\mu \neq \nu$, such that the operator $\pi^\pm(s_\mu^\pm, s_\nu^\pm)$ generally should perform the simultaneous interchanges

$$0^\pm \longleftrightarrow 1^\pm. \quad (143)$$

Of course, as regards this latter process, some incoming state could have been prepared beforehand by a device from (126), whereas the respective outgoing state would emerge as another prepared state from the operability of $\pi^\pm(s_\mu^\pm, s_\nu^\pm)$. We have the operator-eigenvalue association

$$\pi^\pm(s_\mu^\pm, s_\nu^\pm) \longmapsto \pi^\pm(\pi_{(\mu)}^\pm(\Sigma^\pm), \pi_{(\nu)}^\pm(\Sigma^\pm)), \quad (144)$$

along with the annihilation-production prescriptions

$$\langle s_{(0)}^\pm | \pi^\pm(s_1^\pm, s_0^\pm) = \langle 0^\pm |, \quad \langle s_{(1)}^\pm | \pi^\pm(s_0^\pm, s_1^\pm) = \langle 0^\pm | \quad (145)$$

and

$$\langle s_{(0)}^\pm | \pi^\pm(s_0^\pm, s_1^\pm) = \langle s_{(1)}^\pm |, \quad \langle s_{(1)}^\pm | \pi^\pm(s_1^\pm, s_0^\pm) = \langle s_{(0)}^\pm |. \quad (146)$$

Hence, as Eq. (128) ensures that any reduced states carrying different labels must bear $g^\pm$ -orthogonality, we may write the non-vanishing products

$$\langle s_{(0)}^\pm | \pi^\pm(s_0^\pm, s_1^\pm) \parallel s_{(1)}^\pm \rangle_{g^\pm} = \langle s_{(1)}^\pm | s_{(1)}^\pm \rangle_{g^\pm} = \pm |S_\pm^1|^2 \quad (147)$$

and

$$\langle s_{(1)}^\pm | \pi^\pm(s_1^\pm, s_0^\pm) \parallel s_{(0)}^\pm \rangle_{g^\pm} = \langle s_{(0)}^\pm | s_{(0)}^\pm \rangle_{g^\pm} = \pm |S_\pm^0|^2, \quad (148)$$

which agree with both (130) and

$$\begin{aligned} \langle s^\pm | s^\pm \rangle_{g^\pm} &= \langle s^\pm | \pi^\pm(s_0^\pm, s_1^\pm) \parallel s^\pm \rangle_{g^\pm} \\ &+ \langle s^\pm | \pi^\pm(s_1^\pm, s_0^\pm) \parallel s^\pm \rangle_{g^\pm}. \end{aligned} \quad (149)$$

Equations (145)-(148) thus give the decomposition

$$\pi^\pm(s_\mu^\pm, s_\nu^\pm) = |e_{(\lambda)}^\pm \rangle \pi_{\pm}^{*\lambda\sigma}(s_\mu^\pm, s_\nu^\pm) \langle e_{(\sigma)}^\pm |, \quad (150)$$

along with the entries

$$\langle e_{(0)}^\pm | \pi^\pm(s_0^\pm, s_1^\pm) \parallel e_{(1)}^\pm \rangle_{g^\pm} = \pm \frac{S^1}{S^0} \quad (151)$$

²¹Our non-degeneracy assumption allows us to unambiguously identify measurement selections and reductions by means of eigenvalues.

and

$$\langle e_{(1)}^\pm | \pi^\pm(s_1^\pm, s_0^\pm) | e_{(0)}^\pm \rangle_{g^\pm} = \pm \frac{S^0}{S^1}, \quad (152)$$

which constitute the representations

$$(\pi_{\mu\nu}^\pm(s_0^\pm, s_1^\pm)) = \begin{pmatrix} 0 & \pm \frac{S^1}{S^0} \\ 0 & 0 \end{pmatrix}, \quad (\pi_{\mu\nu}^\pm(s_1^\pm, s_0^\pm)) = \begin{pmatrix} 0 & 0 \\ \pm \frac{S^0}{S^1} & 0 \end{pmatrix}. \quad (153)$$

The $\star$ -conjugates of (145) and (146) should then absorb an interchange of reduced-state labels, that obeys the associated arrangements

$$\pi^{\pm\star}(s_0^\pm, s_1^\pm) | s_{(0)}^\pm \rangle = | 0^\pm \rangle, \quad \pi^{\pm\star}(s_1^\pm, s_0^\pm) | s_{(1)}^\pm \rangle = | 0^\pm \rangle \quad (154)$$

and

$$\pi^{\pm\star}(s_1^\pm, s_0^\pm) | s_{(0)}^\pm \rangle = | s_1^\pm \rangle, \quad \pi^{\pm\star}(s_0^\pm, s_1^\pm) | s_{(1)}^\pm \rangle = | s_0^\pm \rangle, \quad (155)$$

that is to say

$$\langle s_{(0)}^\pm | \pi^\pm(s_0^\pm, s_1^\pm) | s_{(1)}^\pm \rangle_{g^\pm} = \langle s_{(1)}^\pm | \pi^{\pm\star}(s_1^\pm, s_0^\pm) | s_{(0)}^\pm \rangle_{g^\pm} \quad (156)$$

and

$$\langle s_{(1)}^\pm | \pi^\pm(s_1^\pm, s_0^\pm) | s_{(0)}^\pm \rangle_{g^\pm} = \langle s_{(0)}^\pm | \pi^{\pm\star}(s_0^\pm, s_1^\pm) | s_{(1)}^\pm \rangle_{g^\pm}, \quad (157)$$

which surely conform to the $\star$ -version of (153).

Let us now allow for a process related to a sequence of pseudo-Hermitian measurement operators²²

$$\Pi_{(\mu)}^\pm(\mathcal{B}^\pm) \Pi_{(\lambda)}^\pm(\mathcal{A}^\pm) \Pi_{(\sigma)}^\pm(\mathcal{B}^\pm), \quad (158)$$

with each of which being defined so as to enter the decomposition

$$\Pi^\pm(\Sigma^\pm) = | s_{(\alpha)}^\pm \rangle g_{\pm}^{*\alpha\beta} \langle s_{(\beta)}^\pm | = \Pi_{(0)}^\pm(\Sigma^\pm) + \Pi_{(1)}^\pm(\Sigma^\pm), \quad (159)$$

where²³

$$\Pi_{(0)}^\pm(\Sigma^\pm) = \pm | s_{(0)}^\pm \rangle \langle s_{(0)}^\pm | = \pm \pi_{(0)}^\pm(\Sigma^\pm) | s^\pm \rangle \langle s^\pm | \pi_{(0)}^\pm(\Sigma^\pm) \quad (160)$$

and

$$\Pi_{(1)}^\pm(\Sigma^\pm) = \pm | s_{(1)}^\pm \rangle \langle s_{(1)}^\pm | = \pm \pi_{(1)}^\pm(\Sigma^\pm) | s^\pm \rangle \langle s^\pm | \pi_{(1)}^\pm(\Sigma^\pm), \quad (161)$$

²²Any sequence of operators to be considered hereafter should conventionally be taken from left to right. This convention is different from the one adopted in Ref. [2]. Thus the attributes "incoming" and "outgoing" always get interchanged when the ordering convention is modified.

²³The operators $| s_{(0)}^\pm \rangle \langle s_{(1)}^\pm |$ and $| s_{(1)}^\pm \rangle \langle s_{(0)}^\pm |$ bear tracelessness and have been ruled out here.

with the kernel letters Σ , s thus standing for either $\mathcal{B}$, b or $\mathcal{A}$, a . Presumably, the initial incoming state for the process is prepared by $\Pi_{(\mu)}^{\pm}(\mathcal{B}^{\pm})$ as $\langle b_{(\mu)}^{\pm} a^{\pm} |$ from an impinging outer-product state like either $\langle b^{+} a^{+} |$ or $\langle b^{-} a^{-} |$. Also, (159) is taken to involve the same reduced states as those occurring in (127), whence we have the representations

$$(\Pi_{(0)\gamma\delta}^{\pm}(\Sigma^{\pm})) = \begin{pmatrix} \pm |S_{\pm}^0|^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\Pi_{(1)\gamma\delta}^{\pm}(\Sigma^{\pm})) = \begin{pmatrix} 0 & 0 \\ 0 \pm |S_{\pm}^1|^2 \end{pmatrix}, \quad (162)$$

along with the traces

$$\begin{aligned} \text{Tr } \Pi_{(\eta)}^{\pm}(\Sigma^{\pm}) &= |S_{\pm}^{\gamma} g_{\gamma\eta}^{\pm}|^2, \\ \text{Tr } \Pi^{\pm}(\Sigma^{\pm}) &= \text{Tr } (\Pi_{(0)}^{\pm}(\Sigma^{\pm}) + \Pi_{(1)}^{\pm}(\Sigma^{\pm})) = 1. \end{aligned} \quad (163)$$

Equation (158) can therefore be rewritten as

$$|b_{(\mu)}^{\pm} \rangle g_{\pm}^{*\mu\alpha} \langle b_{(\alpha)}^{\pm} | a_{(\lambda)}^{\pm} \rangle_{g^{\pm}} g_{\pm}^{*\lambda\beta} \langle a_{(\beta)}^{\pm} | b_{(\sigma)}^{\pm} \rangle_{g^{\pm}} g_{\pm}^{*\sigma\rho} \langle b_{(\rho)}^{\pm} |, \quad (164)$$

while (159) establishes that no crossed terms do really occur in (164) for each value of the label carried by the prepared incoming state. We then obtain the individual configurations from (164)

$$\text{for } \mu = 0: \langle b_{(0)}^{\pm} | a_{(0)}^{\pm} \rangle_{g^{\pm}} |^2 | b_{(0)}^{\pm} \rangle_{g^{\pm}} g_{\pm}^{*00} \langle b_{(0)}^{\pm} | \quad (165)$$

and

$$\text{for } \mu = 1: \langle b_{(1)}^{\pm} | a_{(1)}^{\pm} \rangle_{g^{\pm}} |^2 | b_{(1)}^{\pm} \rangle_{g^{\pm}} g_{\pm}^{*11} \langle b_{(1)}^{\pm} |, \quad (166)$$

which suggest writing out the equality [2]

$$\Pi_{(\mu)}^{\pm}(\mathcal{B}^{\pm}) \Pi_{(\lambda)}^{\pm}(\mathcal{A}^{\pm}) \Pi_{(\sigma)}^{\pm}(\mathcal{B}^{\pm}) = \langle b_{(\mu)}^{\pm} | a_{(\lambda)}^{\pm} \rangle_{g^{\pm}} |^2 \Pi_{(\sigma)}^{\pm}(\mathcal{B}^{\pm}), \quad (167)$$

provided that $\mu = \lambda = \sigma$. The operational effect of (158) differs from that of (137), with $\Sigma^{\pm}$ being thereabout identified with $\mathcal{B}^{\pm}$, because of the disturbance induced by the $\mathcal{A}^{\pm}$ -measurement [2] via the presence of $\Pi_{(\lambda)}^{\pm}(\mathcal{A}^{\pm})$ in the second stage of the process, which actually permits a fraction of $\langle b_{(\mu)}^{\pm} |$ to be passed on, in addition to producing the $\mathcal{A}^{\pm}$ -reduction $\langle a_{(\lambda)}^{\pm} |$. Subsequently, the device $\Pi_{(\sigma)}^{\pm}(\mathcal{B}^{\pm})$ annihilates the reduced state $\langle b_{(\mu)}^{\pm} |$ unless $\mu = \sigma$, and allows for a fraction of $\langle a_{(\lambda)}^{\pm} |$ to be transferred such that, when $\mu = \sigma$, the state $\langle b_{(\sigma)}^{\pm} a_{(\lambda)}^{\pm} |$ will amount to what is transmitted through the whole apparatus, with the relevant statistical transmission weights being

$$\langle b_{(0)}^{\pm} | a_{(0)}^{\pm} \rangle_{g^{\pm}} \langle a_{(0)}^{\pm} | b_{(0)}^{\pm} \rangle_{g^{\pm}} = \langle b_{(0)}^{\pm} | a_{(0)}^{\pm} \rangle_{g^{\pm}} |^2 \quad (168)$$

and

$$\langle b_{(1)}^{\pm} | a_{(1)}^{\pm} \rangle_{g^{\pm}} \langle a_{(1)}^{\pm} | b_{(1)}^{\pm} \rangle_{g^{\pm}} = \langle b_{(1)}^{\pm} | a_{(1)}^{\pm} \rangle_{g^{\pm}} |^2. \quad (169)$$

If we reexpress (158) by replacing $\Pi_{(\lambda)}^{\pm}(\mathcal{A}^{\pm})$ with $\pi_{(\lambda)}^{\pm}(\mathcal{A}^{\pm})$, then the corresponding modified version of (164) would hide the $\mathcal{A}^{\pm}$ -states according to the Born-like description

$$\Pi_{(\mu)}^{\pm}(\mathcal{B}^{\pm})\pi_{(\lambda)}^{\pm}(\mathcal{A}^{\pm})\Pi_{(\sigma)}^{\pm}(\mathcal{B}^{\pm}) = \langle b_{(\mu)}^{\pm} | b_{(\lambda)}^{\pm} \rangle_{g^{\pm}} \Pi_{(\sigma)}^{\pm}(\mathcal{B}^{\pm}), \quad (170)$$

with $\mu = \lambda = \sigma$ once again.

The sequence²⁴

$$\Pi_{(\mu)}^{\pm}(\mathcal{A}^{\pm})\Pi_{(\lambda)}^{\pm}(\mathcal{B}^{\pm})\Pi_{(\sigma)}^{\pm}(\mathcal{C}^{\pm}), \quad (171)$$

constitutes an extension of (158). Some manipulations similar to those performed for the case of (158), give the arrangements

$$\begin{aligned} \Pi_{(0)}^{\pm}(\mathcal{A}^{\pm})\Pi_{(0)}^{\pm}(\mathcal{B}^{\pm})\Pi_{(0)}^{\pm}(\mathcal{C}^{\pm}) &= \\ &= \langle a_{(0)}^{\pm} | b_{(0)}^{\pm} \rangle_{g^{\pm}} \langle b_{(0)}^{\pm} | c_{(0)}^{\pm} \rangle_{g^{\pm}} M^{\pm}(a_0^{\pm}, c_0^{\pm}) \end{aligned} \quad (172)$$

and

$$\begin{aligned} \Pi_{(1)}^{\pm}(\mathcal{A}^{\pm})\Pi_{(1)}^{\pm}(\mathcal{B}^{\pm})\Pi_{(1)}^{\pm}(\mathcal{C}^{\pm}) &= \\ &= \langle a_{(1)}^{\pm} | b_{(1)}^{\pm} \rangle_{g^{\pm}} \langle b_{(1)}^{\pm} | c_{(1)}^{\pm} \rangle_{g^{\pm}} M^{\pm}(a_1^{\pm}, c_1^{\pm}), \end{aligned} \quad (173)$$

along with the definitions

$$\begin{aligned} M^{\pm}(a_0^{\pm}, c_0^{\pm}) &= \pm | a_{(0)}^{\pm} \rangle \langle c_{(0)}^{\pm} |, \\ M^{\pm}(a_1^{\pm}, c_1^{\pm}) &= \pm | a_{(1)}^{\pm} \rangle \langle c_{(1)}^{\pm} | \end{aligned} \quad (174)$$

and the extended decomposition

$$\begin{aligned} M^{\pm}(\mathcal{A}^{\pm}, \mathcal{C}^{\pm}) &= | a_{(\alpha)}^{\pm} \rangle \langle g_{\pm}^{*\alpha\beta} c_{(\beta)}^{\pm} | \\ &= M^{\pm}(a_0^{\pm}, c_0^{\pm}) + M^{\pm}(a_1^{\pm}, c_1^{\pm}). \end{aligned} \quad (175)$$

We thus have the representations

$$(M_{\mu\nu}^{\pm}(a_0^{\pm}, c_0^{\pm})) = \begin{pmatrix} \pm C_{\pm}^0 \overline{A_{\pm}^0} & 0 \\ 0 & 0 \end{pmatrix}, \quad (M_{\mu\nu}^{\pm}(a_1^{\pm}, c_1^{\pm})) = \begin{pmatrix} 0 & 0 \\ 0 & \pm C_{\pm}^1 \overline{A_{\pm}^1} \end{pmatrix}, \quad (176)$$

which recover (162) in the case of (158), while Eqs. (174)-(176) accordingly yield

$$\begin{aligned} \text{Tr}(M^{\pm}(a_0^{\pm}, c_0^{\pm})) &= \pm \langle c_{(0)}^{\pm} | a_{(0)}^{\pm} \rangle_{g^{\pm}}, \\ \text{Tr}(M^{\pm}(a_1^{\pm}, c_1^{\pm})) &= \pm \langle c_{(1)}^{\pm} | a_{(1)}^{\pm} \rangle_{g^{\pm}}, \end{aligned} \quad (177)$$

²⁴The defining pseudo-Hermiticity of $\Pi_{(\mu)}^{\pm}(\Sigma^{\pm})$ tells us that the arrangement (171) is invariant under the combination of the $\star$ -conjugation with a kernel-letter interchange.

whereas

$$\text{Tr } M^\pm(\mathcal{A}^\pm, \mathcal{E}^\pm) = \text{Tr } (M^\pm(a_0^\pm, c_0^\pm) + M^\pm(a_1^\pm, c_1^\pm)) = C_\pm^\sigma \Delta_{\sigma\mu}^\pm \overline{A_\pm^\mu}. \quad (178)$$

Equations (172) and 173 represent a compound apparatus whose first device prepares a reduced state $\langle a_{(\mu)}^\pm b^\pm c^\pm |$ from an incident outer product $\langle a^\pm b^\pm c^\pm |$, and allows for it to be transmitted to the second device which, in turn, accepts a fraction of the prepared $\mathcal{A}^\pm$ -states and produces a reduced state $\langle b_{(\lambda)}^\pm |$. The third device then annihilates the $\mathcal{A}^\pm$ -states and accepts a fraction of the $\mathcal{B}^\pm$ -states coming from the second stage of the measurement process, while creating thereafter an outgoing reduced state $\langle b_{(\lambda)}^\pm c_{(\sigma)}^\pm |$ from the action of $\Pi_{(\sigma)}^\pm(\mathcal{E}^\pm)$. The overall statistical relations that characterize the accepted and transmitted $\mathcal{B}^\pm$ -states are supplied by

$$\langle a_{(0)}^\pm | b_{(0)}^\pm \rangle_{g^\pm} \langle c_{(0)}^\pm | c_{(0)}^\pm \rangle_{g^\pm}, \quad \langle a_{(1)}^\pm | b_{(1)}^\pm \rangle_{g^\pm} \langle c_{(1)}^\pm | c_{(1)}^\pm \rangle_{g^\pm}. \quad (179)$$

The intermediate-stage device of (171) may be taken to play a formal catalytic role during the occurrence of the process if it is replaced with $\pi_{(\lambda)}^\pm(\mathcal{B}^\pm)$, in which case the completeness of (133) would afford by means of (177) the following alternative relations for the incoming and outgoing states produced by the whole process

$$\langle a_{(0)}^\pm | c_{(0)}^\pm \rangle_{g^\pm}, \quad \langle a_{(1)}^\pm | c_{(1)}^\pm \rangle_{g^\pm}. \quad (180)$$

Hence, the process as a whole could be described by a device like $M^\pm(a_\mu^\pm, c_\sigma^\pm)$ whose overall operability transmits a fraction of a reduced state $\langle a_{(\mu)}^\pm |$ and creates a reduced state $\langle c_{(\sigma)}^\pm |$.

As we have seen, the device represented by (171) may transmit information on a fraction of an incoming reduced state while creating an outgoing reduced state after eventually hiding a middle-stage state for an observable that partakes of the process under a dynamical condition of spectral completeness. A measurement associated to a sequence like

$$M^\pm(a_\mu^\pm, b_\nu^\pm) M^\pm(c_\lambda^\pm, d_\sigma^\pm), \quad (181)$$

gives rise to characteristic statistical relationships between incoming and outgoing reduced states which pertain to the operability of devices of the same type as (174). The apparatus symbolized by (181) serves, in effect, to perform a compound measurement whose first device prepares a state $\langle a_{(\mu)}^\pm |$ and creates a reduced state $\langle b_{(\nu)}^\pm |$ from an incoming state $\langle a^\pm b^\pm c^\pm d^\pm |$, whereas the second device accepts a fraction of $\langle a_{(\mu)}^\pm b_{(\nu)}^\pm |$ and produces the reduced state $\langle c_{(\lambda)}^\pm d^\pm |$ while ultimately supplying an outgoing reduced state $\langle a_{(\mu)}^\pm b_{(\nu)}^\pm c_{(\lambda)}^\pm d_{(\sigma)}^\pm |$. Then, calling for (175) gives

$$M^\pm(a_\mu^\pm, b_\nu^\pm) M^\pm(c_\lambda^\pm, d_\sigma^\pm) = \pm \langle b_{(\nu)}^\pm | c_{(\lambda)}^\pm \rangle_{g^\pm} M^\pm(a_\mu^\pm, d_\sigma^\pm), \quad (182)$$

with $\mu = \nu = \lambda = \sigma$. It follows that (181) can, in the last instance, be viewed as a selective measurement operator like $M^\pm(a_\mu^\pm, d_\sigma^\pm)$, with $\langle b_{(\nu)}^\pm | c_{(\lambda)}^\pm \rangle_{g^\pm}$ carrying in this way information on an intermediate transmission amount. Thus, by replacing kernel letters from (176), we obtain

$$\begin{aligned} \langle e_{(0)}^\pm | M^\pm(a_0^\pm, b_0^\pm) M^\pm(c_0^\pm, d_0^\pm) | e_{(0)}^\pm \rangle_{g^\pm} = \\ = \langle b_{(0)}^\pm | c_{(0)}^\pm \rangle_{g^\pm} D_\pm^0 \overline{A_\pm^0} \end{aligned} \quad (183)$$

and

$$\begin{aligned} \langle e_{(1)}^\pm | M^\pm(a_1^\pm, b_1^\pm) M^\pm(c_1^\pm, d_1^\pm) | e_{(1)}^\pm \rangle_{g^\pm} = \\ = \langle b_{(1)}^\pm | c_{(1)}^\pm \rangle_{g^\pm} D_\pm^1 \overline{A_\pm^1}, \end{aligned} \quad (184)$$

together with the characteristic statistical trace

$$\text{Tr } M^\pm(a_\mu^\pm, b_\nu^\pm) M^\pm(c_\lambda^\pm, d_\sigma^\pm) = \langle b_{(\nu)}^\pm | c_{(\lambda)}^\pm \rangle_{g^\pm} \langle d_{(\sigma)}^\pm | a_{(\mu)}^\pm \rangle_{g^\pm}, \quad (185)$$

with $\mu = \nu = \lambda = \sigma$ as before.

A particular couple of similar operators emerges from (181) when we identify in (108) $\mathcal{A}^\pm$ with $\mathcal{B}^\pm$ and $\mathcal{C}^\pm$ with $\mathcal{D}^\pm$, while suitably allowing for in either case some measurement operators coming from (158). Hence, by accounting for the decomposition (159) for each of $\mathcal{A}^\pm$ and $\mathcal{C}^\pm$ which, by (175), provides Born constituents like

$$\Pi_{(0)}^\pm(\Sigma^\pm) = | s_{(0)}^\pm \rangle_{g^\pm} \langle s_{(0)}^{\pm 00} | = M^\pm(s_0^\pm, s_0^\pm) \quad (186)$$

and

$$\Pi_{(1)}^\pm(\Sigma^\pm) = | s_{(1)}^\pm \rangle_{g^\pm} \langle s_{(1)}^{\pm 11} | = M^\pm(s_1^\pm, s_1^\pm), \quad (187)$$

after recalling (182), we arrive at the correspondences

$$\mathcal{A}^\pm \equiv \mathcal{B}^\pm: M^\pm(a_\mu^\pm, b_\nu^\pm) M^\pm(c_\lambda^\pm, d_\sigma^\pm) \mapsto \Pi_{(\mu)}^\pm(\mathcal{A}^\pm) M^\pm(c_\lambda^\pm, d_\sigma^\pm) \quad (188)$$

and

$$\mathcal{C}^\pm \equiv \mathcal{D}^\pm: M^\pm(a_\mu^\pm, b_\nu^\pm) M^\pm(c_\lambda^\pm, d_\sigma^\pm) \mapsto M^\pm(a_\mu^\pm, b_\nu^\pm) \Pi_{(\lambda)}^\pm(\mathcal{C}^\pm). \quad (189)$$

Equation (188) thus yields for $\mu = 0$:

$$\Pi_{(0)}^\pm(\mathcal{A}^\pm) M^\pm(c_0^\pm, d_0^\pm) = \pm \langle a_{(0)}^\pm | c_{(0)}^\pm \rangle_{g^\pm} M^\pm(a_0^\pm, d_0^\pm) \quad (190)$$

and for $\mu = 1$:

$$\Pi_{(1)}^\pm(\mathcal{A}^\pm) M^\pm(c_1^\pm, d_1^\pm) = \pm \langle a_{(1)}^\pm | c_{(1)}^\pm \rangle_{g^\pm} M^\pm(a_1^\pm, d_1^\pm), \quad (191)$$

whereas (189) leads to for $\mu = 0$:

$$M^\pm(a_0^\pm, b_0^\pm) \Pi_{(0)}^\pm(\mathcal{C}^\pm) = \pm \langle b_{(0)}^\pm | c_{(0)}^\pm \rangle_{g^\pm} M^\pm(a_0^\pm, c_0^\pm) \quad (192)$$

and for $\mu = 1$:

$$M^\pm(a_1^\pm, b_1^\pm) \Pi_{(1)}^\pm(\mathcal{C}^\pm) = \pm \langle b_{(1)}^\pm | c_{(1)}^\pm \rangle_{g^\pm} M^\pm(a_1^\pm, c_1^\pm). \quad (193)$$

For (188), for instance, we have the values

$$\text{Tr}(\Pi_{(0)}^\pm(\mathcal{A}^\pm) M^\pm(c_0^\pm, d_0^\pm)) = \langle d_{(0)}^\pm | a_{(0)}^\pm \rangle_{g^\pm} \langle a_{(0)}^\pm | c_{(0)}^\pm \rangle_{g^\pm} \quad (194)$$

and

$$\text{Tr}(\Pi_{(1)}^\pm(\mathcal{A}^\pm) M^\pm(c_1^\pm, d_1^\pm)) = \langle d_{(1)}^\pm | a_{(1)}^\pm \rangle_{g^\pm} \langle a_{(1)}^\pm | c_{(1)}^\pm \rangle_{g^\pm}, \quad (195)$$

which bring back the statistical shape of (180) whenever $\Pi_{(\mu)}^\pm(\mathcal{A}^\pm)$ is replaced by $\pi_{(\mu)}^\pm(\mathcal{A}^\pm)$.

6 Observational Correlations

In the g -realization of $SU(2, 2)$, the representation of the orthochronous proper Poincaré group consists [13, 15, 28, 29] of the set $\mathcal{P}_+^\uparrow$ of all ten-parameter matrices of the form

$$U(\mathcal{P}_+^\uparrow) = \frac{1}{2} \begin{pmatrix} a + (I_2 + iW)a^{-1\dagger} & a - (I_2 + iW)a^{-1\dagger} \\ -a + (I_2 - iW)a^{-1\dagger} & -a - (I_2 - iW)a^{-1\dagger} \end{pmatrix}. \quad (196)$$

In Eq. (196), a denotes an element of $SL(2, \mathbb{C})$ such that it corresponds to an orthochronous proper Lorentz transformation, and W is the van der Waerden Hermitian (2×2) -matrix associated to a time-like or space-like Minkowskian translation. The representation of the orthochronous proper Lorentz group is thus constituted by the set $\mathcal{L}_+^\uparrow$ of all six-parameter matrices of the type²⁵

$$U(\mathcal{L}_+^\uparrow) = \frac{1}{2} \begin{pmatrix} a + a^{-1\dagger} & a - a^{-1\dagger} \\ -a + a^{-1\dagger} & -a - a^{-1\dagger} \end{pmatrix}. \quad (197)$$

The correlations between the physically meaningful information that comes from quantum measurements performed in different spacetime frames, are taken to be supplied by the dynamical intersection

$$SU(4) \supset \mathcal{P}_{Dyn}^{+\uparrow} = \mathcal{P}_+^\uparrow \cap U(4). \quad (198)$$

Any changes of spacetime frames are induced by the action of $\mathcal{P}_{Dyn}^{+\uparrow}$ on some admissible bases, with $\mathcal{P}_{Dyn}^{+\uparrow}$ accordingly consisting of the totality of seven-parameter Poincaré matrices of the form

$$U(\mathcal{P}_{Dyn}^{+\uparrow}) = \frac{1}{\sqrt{2}} \begin{pmatrix} (I_2 + iW)\beta & 0_2 \\ 0_2 & \beta^\dagger(I_2 - iW) \end{pmatrix}, \quad (199)$$

²⁵Matrix blocks that enter any realizations of $SU(2, 2)$ typically bear a skew-Hermiticity property [24, 28, 29].

where $\beta \in U(2)$ and W has now to be normalized as $W^2 = I_2$. Dropping W from $U(\mathcal{P}_{Dyn}^{+\dagger})$ calls for the suppression of the unimodularization factor from (199), and thence produces the four-parameter unitary pattern

$$\begin{pmatrix} \beta & 0_2 \\ 0_2 & \beta^\dagger \end{pmatrix} \in \mathcal{L}_+^\dagger \cap \mathcal{P}_{Dyn}^{+\dagger}. \quad (200)$$

A restriction represented by an array like (199) appears as

$$u(\mathcal{P}_{Dyn}^{+\dagger}) = \begin{pmatrix} u_{Dyn}^+ & 0 \\ 0 & u_{Dyn}^- \end{pmatrix}, \quad u_{Dyn}^{\pm\star} = (u_{Dyn}^\pm)^{-1} = u_{Dyn}^{\pm\dagger}, \quad (201)$$

with the prototype entry

$$u_{\mu\nu}^\pm(Dyn) = \langle e_{(\mu)}^\pm | u_{Dyn}^\pm | e_{(\nu)}^\pm \rangle_{g^\pm}, \quad (202)$$

which, by (75) and (76), must satisfy

$$u_{\mu\lambda}^\pm(Dyn) g_\pm^{*\lambda\sigma} u_{\sigma\nu}^{\pm\star}(Dyn) = g_{\mu\nu}^\pm. \quad (203)$$

We assume that the copy of Cartan's space for any spacetime frame admits a decomposition just like $\mathfrak{e}^\pm$ such that Eq. (11) must bear $\mathcal{P}_{Dyn}^{+\dagger}$ -invariance (see (225) below). Hence, the computational bases for any two frames are subject to a relation that looks like

$$\langle e_{(\mu)}^{\prime\pm} | = \langle e_{(\mu)}^\pm | u_{Dyn}^\pm, \quad (204)$$

whence the behaviours under $\mathcal{P}_{Dyn}^{+\dagger}$ of both $g_{\mu\nu}^\pm$ and $\Delta_{\mu\nu}^\pm$ can be exhibited by

$$\langle e_{(\mu)}^{\prime\pm} | e_{(\nu)}^{\prime\pm} \rangle_{g^{\prime\pm}} = g_{\mu\nu}^{\prime\pm} = \langle e_{(\mu)}^\pm | u_{Dyn}^\pm | u_{Dyn}^\pm | e_{(\nu)}^\pm \rangle_{g^\pm} = g_{\mu\nu}^\pm \quad (205)$$

and

$$\langle \langle e_{(\mu)}^{\prime\pm} | e_{(\nu)}^{\prime\pm} \rangle \rangle = \Delta_{\mu\nu}^{\prime\pm} = \langle \langle e_{(\mu)}^\pm | u_{Dyn}^\pm | u_{Dyn}^\pm | e_{(\nu)}^\pm \rangle \rangle = \Delta_{\mu\nu}^\pm, \quad (206)$$

which apply in form equally well to $g_\pm^{*\mu\nu}$ and $\Delta_\pm^{*\mu\nu}$ too. Thus, we also have

$$g_{\mu\nu}^{\prime\pm} = u_{\mu\lambda}^\pm(Dyn) g_\pm^{*\lambda\sigma} u_{\sigma\nu}^{\pm\star}(Dyn) = g_{\mu\nu}^\pm \quad (207)$$

and

$$\Delta_{\mu\nu}^{\prime\pm} = u_{\mu\lambda}^\pm(Dyn) \Delta_\pm^{*\lambda\sigma} u_{\sigma\nu}^{\pm\dagger}(Dyn) = \Delta_{\mu\nu}^\pm, \quad (208)$$

whilst the behaviour of (46) can be specified from

$$| e_{(\mu)}^{\prime\pm} \rangle_{g_\pm^{\prime*\mu\nu}} \langle e_{(\nu)}^{\prime\pm} | = | e_{(\mu)}^\pm \rangle_{u_\pm^{*\star\mu\lambda}(Dyn) g_{\lambda\sigma}^\pm u_\pm^{*\sigma\nu}(Dyn)} \langle e_{(\nu)}^\pm |. \quad (209)$$

The pattern for the adjoint version of (207) then yields the invariance property

$$| e_{(\mu)}^{\prime\pm} \rangle_{g_\pm^{\prime*\mu\nu}} \langle e_{(\nu)}^{\prime\pm} | = | e_{(\mu)}^\pm \rangle_{g_\pm^{*\mu\nu}} \langle e_{(\nu)}^\pm |. \quad (210)$$

Any state vector like (111) is naturally invariant under $\mathcal{P}_{Dyn}^{+\dagger}$. This assertion relies partially upon the fact that the effective action of $\mathcal{P}_{Dyn}^{+\dagger}$ requires that

$$\langle s'^{\pm} | s^{\pm} \rangle_{g^{\pm}} = \langle s^{\pm} | s^{\pm} \rangle_{g^{\pm}}, \quad (211)$$

whence we have to encompass the $\star$ -conjugate configurations

$$\langle s'^{\pm} | = S_{\pm}^{\mu} u_{\mu\sigma}^{\pm\star}(Dyn) u_{\pm}^{*\sigma\lambda}(Dyn) \langle e_{(\lambda)}^{\pm} | = \langle s^{\pm} | \quad (212)$$

and

$$| s'^{\pm} \rangle = | e_{(\rho)}^{\pm} \rangle u_{\pm}^{*\star\rho\tau}(Dyn) u_{\tau\nu}^{\pm}(Dyn) \overline{S_{\pm}^{\nu}} = | s^{\pm} \rangle, \quad (213)$$

which take up (52) as well as the auxiliary relations

$$u_{\mu}^{\pm\star\nu}(Dyn) = g_{\mu\lambda}^{\pm} \overline{u^{\pm\lambda\sigma}}(Dyn) g_{\pm}^{*\sigma\nu} \quad (214)$$

and²⁶

$$u_{\mu}^{\pm\star\sigma}(Dyn) u_{\sigma}^{\pm\lambda}(Dyn) = u_{\mu\sigma}^{\pm\star}(Dyn) u_{\pm}^{*\sigma\lambda}(Dyn). \quad (215)$$

Therefore, the defining invariance of $g^{\pm}$ assures that both (114) and (115) are invariant, while the normalization condition (113) can be appropriately regarded as an invariant dynamical feature albeit each of the involved amplitudes can not at all. We have, for instance,

$$S_{\pm}^{\lambda} = S_{\pm}^{\mu} u_{\mu\sigma}^{\pm\star}(Dyn) g_{\pm}^{*\sigma\lambda}. \quad (216)$$

It follows that any reduced states like those defined by (128) do not hold $\mathcal{P}_{Dyn}^{+\dagger}$ -invariance, but their sum does since (204) and (216) may recover (212), say, from

$$\begin{aligned} \langle s'_{(0)}^{\pm} | + \langle s'_{(1)}^{\pm} | &= \\ &= S_{\pm}^{\mu} u_{\mu\lambda}^{\pm\star}(Dyn) (g_{\pm}^{*\lambda 0} g_{0\rho}^{\pm} + g_{\pm}^{*\lambda 1} g_{1\rho}^{\pm}) u_{\pm}^{*\rho\sigma}(Dyn) \langle e_{(\sigma)}^{\pm} |, \end{aligned} \quad (217)$$

whence both of (129) and (130) behave invariantly.

The behaviours under $\mathcal{P}_{Dyn}^{+\dagger}$ of dynamical variables and spectra depend closely on the physical nature of the magnitudes being observed. For bringing out the immediately relevant situations, we allow for the primed-frame entry

$$S_{\mu\nu}^{\pm} = \langle e'_{(\mu)}^{\pm} | S'^{\pm} | e'_{(\nu)}^{\pm} \rangle_{g'^{\pm}}, \quad (218)$$

to write down the correlations

$$S'^{\pm} = S^{\pm} \implies S_{\mu\nu}^{\pm} = u_{\mu\lambda}^{\pm}(Dyn) S_{\pm}^{*\lambda\sigma} u_{\sigma\nu}^{\pm\star}(Dyn) \quad (219)$$

²⁶It is shown in Ref. [18] that $g_{\mu\lambda}^{\pm} g_{\pm}^{*\lambda\nu} = \Delta_{\mu\lambda}^{\pm} \Delta_{\pm}^{*\lambda\nu}$. When combined with (207), this property guarantees the legitimacy of (212) and (213).

and²⁷

$$S'_{\mu\nu}^{\pm} = S_{\mu\nu}^{\pm} \implies S'^{\pm} = u_{Dyn}^{\pm\star} S^{\pm} u_{Dyn}^{\pm}. \quad (220)$$

The spectrum (122) is required to bear $\mathcal{P}_{Dyn}^{+\dagger}$ -invariance in accordance with the behaviour specified by (220) such that the eigenvalues carried by (120), (121) and (124) are subject to

$$s'_0{}^{\pm} = s_0{}^{\pm}, \quad s'_1{}^{\pm} = s_1{}^{\pm}, \quad (221)$$

whereas Eq. (125) undergoes the transformation

$$< S'^{\pm} >_{s'^{\pm}} = \pm s_{\mu}^{\pm} \mid S_{\pm}^{\lambda} u_{\lambda\sigma}^{\pm\star} (Dyn) g_{\pm}^{*\sigma\mu} \mid^2. \quad (222)$$

Formal traces of observables, as given by (87) and (88), hold $\mathcal{P}_{Dyn}^{+\dagger}$ -invariance in the case of either of Eqs. (219) and (220). The case of (220) is treated as

$$\text{Tr } S'^{\pm} = \text{Tr } (u_{Dyn}^{\pm\star} S^{\pm} u_{Dyn}^{\pm}) = \text{Tr } (u_{Dyn}^{\pm} u_{Dyn}^{\pm\star} S^{\pm}) = \text{Tr } S^{\pm}, \quad (223)$$

and it agrees with (221). Owing to the adjoint version of (207), use can be made of (52) to perform the following calculation for (219):

$$\begin{aligned} S'_{\mu\nu}{}^{\pm} g'^{\nu\mu} &= u_{\mu\lambda}^{\pm} g_{\pm}^{*\lambda\tau} S_{\tau\sigma}^{\pm} g_{\pm}^{*\sigma\rho} g_{\rho\theta}^{\pm} u_{\pm}^{*\theta\mu} \\ &= S_{\tau\sigma}^{\pm} g_{\pm}^{*\sigma\rho} g_{\rho\theta}^{\pm} u_{\pm}^{*\theta\mu} u_{\mu\lambda}^{\pm} g_{\pm}^{*\lambda\tau} = S_{\tau\sigma}^{\pm} g_{\pm}^{*\sigma\tau}, \end{aligned} \quad (224)$$

with the parenthesized Dyn having been suppressed just for once. It should be obvious that Eqs. (219) and (220) still apply formally to operators other than observables. In particular, the invariant character of (11) implies that the representative matrices for the operators $P^{\pm}$ defined by Eq. (15) must bear $\mathcal{P}_{Dyn}^{+\dagger}$ -invariance in accordance with (220), namely,

$$P'_{\mu\nu}{}^{\pm} = P_{\mu\nu}^{\pm} \implies P'^{\pm} = u_{Dyn}^{\pm\star} P^{\pm} u_{Dyn}^{\pm}. \quad (225)$$

Any measurement processes should run covariantly with respect to the action of $\mathcal{P}_{Dyn}^{+\dagger}$, which means that any process must admit invariant operator patterns. This requirement is certainly fulfilled by the implementation of Eq. (219) and, therefore, also agrees with the non-invariant character due to (216) and its complex conjugate of any representations that carry individual amplitudes alone.

The state reduction produced by the operator $\pi_{(\mu)}^{\pm}(\Sigma^{\pm})$ that occurs on the right-hand side of Eq. (127), gives rise to a loss of invariance which may be posed as

$$\pi_{(\mu)}^{\pm}(\Sigma^{\pm}): \text{invariant } < s^{\pm} \mid \longrightarrow \text{non-invariant } < s_{(\mu)}^{\pm} \mid, \quad (226)$$

but an invariance recovery may be attained from (217) according to

$$< s^{\pm} \mid (\pi_{(0)}^{\pm}(\Sigma^{\pm}) + \pi_{(1)}^{\pm}(\Sigma^{\pm})) = \text{invariant}, \quad (227)$$

²⁷The diagonalizability of diagonal arrays is generally lost when (219) comes about. In either of the situations of (219) and (220), the compatibility of observables remains invariant.

which reflects the behaviour of (133), namely,

$$|e_{(\rho)}^{\pm} > g_{\pm}^{*\rho\lambda} \pi_{(\mu)\lambda\sigma}^{\pm}(\Sigma^{\pm}) g_{\pm}^{*\sigma\tau} < e_{(\tau)}^{\pm} | = \text{invariant}. \quad (228)$$

Hence, the invariance of (113) gets translated into the statement

$$\begin{aligned} < s^{\pm} | \pi_{(0)}^{\pm}(\Sigma^{\pm}) g^{\pm} || s^{\pm} >_{g^{\pm}} + \\ &+ < s^{\pm} | \pi_{(1)}^{\pm}(\Sigma^{\pm}) g^{\pm} || s^{\pm} >_{g^{\pm}} = \text{invariant}, \end{aligned} \quad (229)$$

while the representations (134) must behave as

$$\pi_{(\mu)\lambda\sigma}^{\pm}(\Sigma'^{\pm}) = u_{\lambda\rho}^{\pm}(Dyn) \pi_{\pm}^{*(\mu)\rho\tau}(\Sigma^{\pm}) u_{\tau\sigma}^{\pm\star}(Dyn). \quad (230)$$

In addition, the invariance property exhibited by (224) applies to (134), that is to say,

$$\text{Tr}(\pi_{(0)}^{\pm}(\Sigma'^{\pm}) + \pi_{(1)}^{\pm}(\Sigma'^{\pm})) = \text{Tr}(\pi_{(0)}^{\pm}(\Sigma^{\pm}) + \pi_{(1)}^{\pm}(\Sigma^{\pm})). \quad (231)$$

When combined together with (204) and its dual, Eq. (230) shows that the decomposition (141) is invariant.

The $\mathcal{P}_{Dyn}^{+\dagger}$ -behaviour of (113) implies that Eqs. (147) and (148) should be subjected to

$$\begin{aligned} < s_{(0)}^{\pm} | \pi^{\pm}(s_0^{\pm}, s_1^{\pm}) || s_{(1)}^{\pm} >_{g^{\pm}} + \\ &+ < s_{(1)}^{\pm} | \pi^{\pm}(s_1^{\pm}, s_0^{\pm}) || s_{(0)}^{\pm} >_{g^{\pm}} = \text{invariant}, \end{aligned} \quad (232)$$

whilst (211) ensures that the relation (149) must bear invariance. Requiring that the operator decomposition (150) should behave invariantly then yields the following transformation law for the representations (153):

$$\pi_{\lambda\sigma}^{\pm}(s_{\mu}^{\pm}, s_{\nu}^{\pm}) = u_{\lambda\rho}^{\pm}(Dyn) g_{\pm}^{*\rho\tau} \pi_{\tau\gamma}^{\pm}(s_{\mu}^{\pm}, s_{\nu}^{\pm}) g_{\pm}^{*\gamma\theta} u_{\theta\sigma}^{\pm\star}(Dyn). \quad (233)$$

The behaviours of any operator components like (160), (161) and (174) get specified when we bring up the transformation laws for dual reduced states. By invoking (217), we get, for instance,

$$< s'_{(0)}^{\pm} | = S_{\pm}^{\mu} u_{\mu\lambda}^{\pm\star}(Dyn) g_{\pm}^{*\lambda 0} g_{\pm}^{\pm} u_{\pm}^{*\rho\sigma}(Dyn) < e_{(\sigma)}^{\pm} | \quad (234)$$

and

$$| s'_{(0)}^{\pm} > = | e_{(\sigma)}^{\pm} > u_{\pm}^{*\star\rho}(Dyn) g_{\rho 0}^{\pm} g_{\pm}^{*0\lambda} u_{\lambda\mu}^{\pm}(Dyn) \overline{S_{\pm}^{\mu}}, \quad (235)$$

such that the decompositions (159) and (175) fulfill an invariant-operator requirement, while (115) and (178) bear invariance because of the applicability of (212), (213) and (224). As a consequence, any completion of measurements like the one which leads to (170) and (180), is invariant. Likewise, Eqs. (130) and

(211) together imply that any appropriate sum of statistical weights is a dynamical invariant, in compatibility with (224). For (177), for instance, we have

$$< c_{(0)}^{\pm} | a_{(0)}^{\pm} >_{g^{\pm}} + < c_{(1)}^{\pm} | a_{(1)}^{\pm} >_{g^{\pm}} = \text{invariant}, \quad (236)$$

whence (182) is subjected to

$$M^{\pm}(a_0^{\pm}, b_0^{\pm})M^{\pm}(c_0^{\pm}, d_0^{\pm}) + M^{\pm}(a_1^{\pm}, b_1^{\pm})M^{\pm}(c_1^{\pm}, d_1^{\pm}) = \text{invariant}, \quad (237)$$

and Eqs. (190)-(193) thus behave typically as

$$\Pi_{(0)}^{\pm}(\mathcal{A}^{\pm})M^{\pm}(c_0^{\pm}, d_0^{\pm}) + \Pi_{(1)}^{\pm}(\mathcal{A}^{\pm})M^{\pm}(c_1^{\pm}, d_1^{\pm}) = \text{invariant}. \quad (238)$$

7 Concluding Remarks and Outlook

The physical role that had been originally assigned to Cartan's space is reinforced by the extraordinary mathematical fact posed by Segal's theorem [30] which states that the restricted conformal symmetry of special relativity is algebraically associated to the maximal spacetime extension of the Poincaré group. The combination of this fact with the Pauli-Weisskopf theorem [31], according to which the occurrence in Nature of particles and antiparticles does not depend upon any spin values, considerably strengthens our belief that the twofold framework proposed in Ref. [13] can supply not only a self-consistent unified mechanical description of particles and antiparticles, but also arrangements of relativistic observational correlations between dynamical variables and measurements of states for fermions and bosons.

Apart from the unitary representations of the Galilei group [32–34], which are generally infinite-dimensional, the only observational correlations that can be sorted out by the standard non-relativistic quantum mechanical framework just emanate from Pauli's $SU(2)$ -spin theory [35]. Therefore, such correlations bear strictly a combination of locality with a spin character, whence any preparations and measurements of spin one-half states that are eventually performed locally by an observer can at most be systematically manipulated by the same observer by carrying out $SO(3)$ -transformations. A noteworthy implication of this result is that the underlying operationality of ordinary quantum computational processes makes it inviable to provide any observational correlations between outcomes of measurements carried through in spatially separated frames. So, one of the questions that might arise here is whether the (physical) character of quantum computation [36] should bear Poincaré invariance or not if operational computing tasks were conducted within the realm of special relativity.

It has become evident from our work that it is mainly the performance of complete measurements which may hold a manifestly invariant character in space-time. Hence, with respect to this, we can say that relativistic quantum measurement processes should account for an adequate definition of total measurements

regardless of the spin values of the systems to be measured. We have claimed that the formal mechanisms involved in the traditional measurement processes may be neatly fitted in with the framework of Ref. [13], whence the covariant symbolism of Sections 5 and 6 should accordingly yield the feasibility of elegantly supplying a refined description of Wu-Shaknov entangled states of particle-antiparticle pairs [37–41]. From our perspective, observational correlations must be necessarily incorporated into the construction of any quantum computing operations that could explicitly allow for outcomes of measurement devices set upon different spacetime frames [7].

As we had presupposed before, copies of Cartan’s space constitute the only known spaces of state vectors in which a legitimate relativistic quantum information theory may be developed side by side with a covariant measurement-based quantum computation approach. In this connection, all the standard computational gates can be reconstructed covariantly by defining suitable gate operators in $\mathfrak{C}^\pm$ and utilizing the representation theory developed in Ref. [13] along with the dynamical behaviours described by Eqs. (219) and (220). Since practical entangled states naively carry a reduced character, their behaviours under arbitrary spacetime displacements must be controlled by W -matrices like the one involved in Eq. (199). We can then expect that the popular non-locality property of non-relativistic quantum mechanics should be thoroughly revisited when any plausible relativistic framework is actually taken into account. These remarks will probably be entertained elsewhere in a forthcoming paper.

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