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Non-Associative Algebras, Yang-Baxter equations, and Quantum Computers

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Abstract. Non-associative algebras is a research direction gaining much attention these days. New developments show that associative algebras and some not-associative structures can be unified at the level of Yang-Baxter structures. In this paper, we present a unification for associative algebras, Jordan algebras and Lie algebras. The (quantum) Yang-Baxter equation and related structures are interesting topics, because they have applications in many areas of physics, mathematics, and computer science. Several new interpretations and results are presented in this paper.

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1 Introduction

According to [1], there are two important classes of non-associative structures: *Lie structures* (introduced in 1870 by the Norwegian mathematician Sophus Lie in his study of the groups of transformations) and *Jordan structures* (introduced in 1932-1933 by the quantum mechanics German specialist Pasqual Jordan in his algebraic formulation of quantum mechanics). Jordan structures also appear in quantum group theory, and exceptional Jordan algebras play an important role in recent fundamental physical theories, namely, in the theory of super-strings (see [2]).

Non-associative algebras are currently a research direction in fashion (see [3], and the references therein). New developments show that associative algebras and Lie algebras can be unified at the level of Yang-Baxter structures (see [1]). In this paper, we present a unification for associative algebras, Jordan algebras and Lie algebras.

The Yang-Baxter equation (see [4, 5]) first appeared in theoretical physics and statistical mechanics. It can be interpreted in terms of combinatorial logical circuits, and, in logic, it represents some kind of compatibility condition, when working with many logical sentences in the same time. This equation is also related to the theory of universal quantum gates and to the quantum computers (see, for example, [6]).

The organization of our paper is the following. The next section introduces the mathematical terminology needed in this paper, and it presents results about the Yang-Baxter equation. Section 3 deals with the unification of Jordan algebras, Lie algebras and associative algebras. Section 4 is a conclusions section and an update on quantum computers.

2 The QYBE

An introduction to the (quantum) Yang-Baxter equation (QYBE) could be found in the paper [4]. Several special sessions on it followed at the open-access journal AXIOMS, explaining its role in many areas of physics, mathematics, and computer science.

We will work over the field k, and the tensor products will be defined over k. For V a k-space, we denote by $\tau : V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$, and by $I : V \to V$ the identity map of the space V. For $R : V \otimes V \to V \otimes V$ a k-linear map, let $R^{12} = R \otimes I$, $R^{23} = I \otimes R$, $R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition 2.1 *A Yang-Baxter operator is defined as an invertible k-linear map $R : V \otimes V \to V \otimes V$ which satisfies the equation:*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{1}$$

R satisfies (1) if and only if $R \circ \tau$ (respectively $\tau \circ R$) satisfies the QYBE:

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} \tag{2}$$

Remark 2.2 *For A be a (unitary) associative k-algebra, and $\alpha, \beta, \gamma \in k$, [7] defined the k-linear map*

$$R^A_{\alpha,\beta,\gamma} : A \otimes A \to A \otimes A, \;\; R^A_{\alpha,\beta,\gamma}(a \otimes b) = \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b \,,$$

which is a Yang-Baxter operator if and only if one of the following holds:
(i) $\alpha = \gamma \neq 0, \;\; \beta \neq 0$; (ii) $\beta = \gamma \neq 0, \;\; \alpha \neq 0$; (iii) $\alpha = \beta = 0, \;\; \gamma \neq 0$.

Remark 2.3 *Using Turaev's general scheme to derive an invariant of oriented links from a Yang-Baxter operator (see [8]), $R^A_{\alpha,\beta,\gamma}$ leads to the Alexander polynomial of knots (see [9]).*

Remark 2.4 *In dimension two, $R^A_{\alpha,\beta,\alpha} \circ \tau$, can be expressed as:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-q & q & 0 \\ \eta & 0 & 0 & -q \end{pmatrix} \tag{3}$$

where $\eta \in \{0,\ 1\}$, and $q \in k - \{0\}$. For $\eta = 0$ and $q = 1$, it can be related to the universal quantum gate CNOT.

Definition 2.5 *A Lie super-algebra is a (non-associative) Z_2-graded algebra, or super-algebra, over a field k with the Lie super-bracket, satisfying the two conditions:*

$$[x, y] = -(-1)^{|x||y|}[y, x$$

$$(-1)^{|z||x|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0$$

where x, y and z are pure in the Z_2-grading. Here, $|x|$ denotes the degree of x (either 0 or 1). The degree of $[x, y]$ is the sum of degree of x and y modulo 2.

Remark 2.6 *For $(L, [,])$ a Lie super-algebra over k, $z \in Z(L) = \{z \in L : [z, x] = 0\ \ \forall\, x \in L\}$, $|z| = 0$ and $\alpha \in k$, [10] (and [11]) defined*

$$\phi^L_\alpha \ :\ L \otimes L \ \longrightarrow\ L \otimes L,\ \ x \otimes y \mapsto \alpha [x, y] \otimes z + (-1)^{|x||y|} y \otimes x\ ,$$

which is a YB operator.

Remark 2.7 *The Remarks 2.2 and 2.6 lead to some kind of unification of associative algebras and structures that are not associative at the level of Yang-Baxter structures (see [1, 12]. For example, the first isomorfism theorem for groups (algebras) and the first isomorfism theorem for Lie algebras, can be unified as an isomorphism theorem for Yang-Baxter structures (see [13]). The fact that Theorem 7.2.3 from [13] (The fundamental isomorphism theorem for YB structures) unifies not only associative algebras and coalgebras, but also Lie (super)algebras is a new result.*

Remark 2.8 *Following a question of Prof. Tatiana Gateva-Ivanova, we can construct an algebra structure associated to the operator*

$$R = R^A_{1,1,1} : A \otimes A \to A \otimes A, \quad R(a \otimes b) = ab \otimes 1 + 1 \otimes ab - a \otimes b,$$

if we use Theorem 3.1 (i) from [5].

For $a, b \in T^1(A) = A$, we have:

$$\mu(a \otimes b) = ab \otimes 1 + 1 \otimes ab - a \otimes b \ \in T^2(A)\,.$$

For $a\otimes a' \in T^2(A) = A\otimes A$ and $b \in T^1(A) = A$, we have:

$$\begin{aligned}\mu((a\otimes a')\otimes b) =& R^{12} \circ R^{23}(a\otimes a'\otimes b)\\ =& aa'b\otimes 1\otimes 1 + 1\otimes aa'b\otimes 1 - a\otimes a'b\otimes 1 + 1\otimes a\otimes a'b\\ & - aa'\otimes 1\otimes b - 1\otimes aa'\otimes b + a\otimes a'\otimes b \in T^3(A).\end{aligned}$$

For $a \in T^1(A) = A$ and $b\otimes b' \in T^2(A)$, we have:

$$\mu(a\otimes(b\otimes b') = R^{23} \circ R^{12}(a\otimes b\otimes b') \in T^3(A).$$

In the same manner, we can compute other products.

3 Non-associative algebras

Jordan algebras emerged in the early thirties, and their applications are in physics, differential geometry, ring geometries, quantum groups, analysis, biology, etc (see [2, 14]).

One of the main results of [15] is the following theorem, which explains when the Jordan identity implies associativity. It is an intrinsic result.

Theorem 3.1 *Let V be a vector space spanned by a and b, which are linearly independent. Let $\theta : V \otimes V \to V,\ \ \theta(x \otimes y) = xy$, be a linear map which is a commutative operation with the property*

$$a^2 = b\ , \quad b^2 = a\ . \tag{4}$$

Then: $(V,\ \theta)$ is a Jordan algebra $\iff$ $(V,\ \theta)$ is a non-unital commutative (associative) algebra.

The next remark finds a relationship between Jordan algebras, Lie algebras and associative algebras. In this case, we have an extrinsic result about non-associative structures.

Remark 3.2 *For the vector space V, let $\eta : V \otimes V \to V,\ \ \eta(x \otimes y) = xy$, be a linear map which satisfies:*

$$(ab)c + (bc)a + (ca)b \ = \ a(bc) + b(ca) + c(ab)\ \ ; \tag{5}$$

$$(a^2b)a \ = \ a^2(ba)\ \ . \tag{6}$$

Then, (V, η) is a structure which unifies (non-unital) associative algebras, Lie algebras and Jordan algebras.

Indeed, the associativity and the Lie identity are unified by relation (5). Also, the commutativity of a Jordan algebra implies (5). But, the Jordan identity, (6), which appears in the definition of Jordan algebras, holds true in any associative algebra and Lie algebra.

This unifying approach is different from the unification proposed in Remark 2.7.

4 Conclusions and Applications

The current paper emerged after the International Conference "Mathematics Days in Sofia", July 7-10, 2014, Sofia, Bulgaria. It presents our scientific contributions for that conference, as well as new results and directions of research.

Prof. Radu Iordanescu contributed with a book on Jordan algebras ([2]) to the library of the Institute of Mathematics and Informatics from Sofia.

The other authors of this paper presented a poster on combinatorial logical circuits and solutions to the set-theoretical Yang-Baxter equation from Boolean algebras (following the works [16, 17]). Dr. Violeta Ivanova (who gave the talk [18]) was interested in the applications of these problems in computer science. The Yang-Baxter equation can be interpreted in terms of combinatorial logical circuits, and, in logic, it represents some kind of compatibility condition, when working with many logical sentences in the same time. This equation is also related to the theory of universal quantum gates and to the quantum computers (see, for example, [6]).

The first quantum computer (which uses principles of quantum mechanics) was sold to the aerospace and security of defense company Lockheed Martin. It can address issues related to number theory and optimization, which require large computational power. An example is the Shor's algorithm, a quantum algorithm that determines quickly and effectively the prime factors of a large number. With enough qubii, such a computer could use the Shor's algorithm to break algorithms encryption used today.

Florin F. Nichita gave a talk on the Yang-Baxter equation (see [4, 5, 19–21]). In his talk he referred to several papers of Prof. Vladimir Dobrev (see [22–24]) and to the work of Prof. Tatiana Gateva-Ivanova (one of her questions, arising at that time, was partially answered in this paper). As an observation, our Yang-Baxter operator $R^A_{\alpha,\beta,\alpha}$ is related to a universal quantum gate.

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