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Spontaneous Supersymmetry Breaking and Instanton Sum in 2D Type IIA Superstring Theory*

F. Sugino

Okayama Institute for Quantum Physics, Kyoyama 1-9-1, Kita-ku, Okayama 700-0015, Japan

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Abstract. We consider a double-well supersymmetric matrix model and its interpretation as a nonperturbative definition of two-dimensional type IIA superstring theory in the presence of a nontrivial Ramond-Ramond background. The interpretation is based on symmetries in both sides of the matrix model and the IIA string theory, and confirmed by direct comparison of various correlation functions . The full nonperturbative free energy of the matrix model in its double scaling limit is represented by the Tracy-Widom distribution in random matrix theory. We show that instanton contributions in the matrix model survive in the double scaling limit and trigger spontaneous supersymmetry breaking. It implies that the target-space supersymmetry is spontaneously broken due to nonperturbative effects in the IIA string theory.

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1 Introduction

Solvable matrix models for two-dimensional quantum gravity or noncritical string theory had been vigorously investigated around 1990, focusing on nonperturbative effects in string theory [1]. While this approach has been successful for bosonic string theory, little has been known for superstring theory, in particular which possesses target-space supersymmetry (SUSY). It would be interesting to consider (solvable) matrix models describing superstring theory with target-space SUSY. In this article, we discuss correspondence between a simple zero-dimensional SUSY double-well matrix model and two-dimensional type IIA superstring theory on a nontrivial Ramond-Ramond (RR) background. Then, nonperturbative effect of the matrix model is computed in its double scaling limit.

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As a result, we find that SUSY is spontaneously broken due to instantons in the matrix model. According to the correspondence, this suggests spontaneous SUSY breaking at the nonperturbative level in the type IIA superstring theory. We hope our analysis is helpful to understand nonperturbative dynamics of matrix models of super Yang-Mills type for critical superstring theory [2–4].

This article is mainly based on the work [5–8].

2 Double-Well SUSY Matrix Model

The following simple matrix model has been analyzed in [9]:

$$S = N\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2 - \mu^2) + \bar{\psi}(\phi\psi + \psi\phi)\right], \tag{1}$$

where B and ϕ are $N \times N$ hermitian matrices, and ψ and $\bar{\psi}$ are $N \times N$ Grassmann-odd matrices. The action S is invariant under SUSY transformations generated by Q and $\bar{Q}$:

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \tag{2}$$

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0, \tag{3}$$

which leads to the nilpotency: $Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0$. After integrating out B, we have a scalar potential of a double-well shape: $\frac{1}{2}(\phi^2 - \mu^2)^2$. A large-$N$ saddle point equation for the eigenvalue distribution of the matrix ϕ: $\rho(x) \equiv \frac{1}{N}\mathrm{tr}\,\delta(x - \phi)$ reads

$$\int dy\,\rho(y)\,\mathrm{P}\frac{1}{x-y} + \int dy\,\rho(y)\,\mathrm{P}\frac{1}{x+y} = x^3 - \mu^2 x. \tag{4}$$

Its solution with filling fraction (ν_+, ν_-) is given by

$$\rho(x) = \begin{cases} \dfrac{\nu_+}{\pi}\, x\,\sqrt{(x^2-a^2)(b^2-x^2)} & (a < x < b) \\ \dfrac{\nu_-}{\pi}\,|x|\,\sqrt{(x^2-a^2)(b^2-x^2)} & (-b < x < -a), \end{cases} \tag{5}$$

where $a = \sqrt{\mu^2 - 2}$ and $b = \sqrt{\mu^2 + 2}$. The filling fractions satisfying $\nu_+ + \nu_- = 1$ indicate that $\nu_+ N$ ($\nu_- N$) eigenvalues are around the right (left) minimum of the double-well. The solution exists for $\mu^2 > 2$. The large-N free energy and the expectation values $\left\langle \frac{1}{N}\mathrm{tr}\,B^n \right\rangle$ $(n = 1, 2, \cdots)$ evaluated at the solution turn out to all vanish [9]. This strongly suggests that the solution preserves SUSY. Thus, we conclude that the SUSY minima are infinitely degenerate and parametrized by (ν_+, ν_-) in the simple large-N limit (the planar limit).

On the other hand, there exists a solution having support of a single interval $x \in [-c, c]$ for $\mu^2 < 2$ [10]:

$$\rho(x) = \frac{1}{2\pi}\left(x^2 - \mu^2 + \frac{c^2}{2}\right)\sqrt{c^2 - x^2} \tag{6}$$

with $c = \sqrt{\frac{2}{3}}\left(\mu^2 + \sqrt{\mu^4 + 12}\right)^{1/2}$. This solution gives nonzero values of $\left\langle \frac{1}{N}\mathrm{tr}B \right\rangle$ and of the large-N free energy, showing that SUSY is broken. We observed that the third derivative of the free energy with respect to μ^2 is not continuous at $\mu^2 = 2$. Thus, the transition between the SUSY phase ($\mu^2 > 2$) and the SUSY broken phase ($\mu^2 < 2$) is of the third order.

The rest of this paper is organized as follows. In the next section, we will compute various correlation functions at the saddle point (5) and find new logarithmic critical behavior as $\mu^2 \to 2 + 0$. Based on the result, we will discuss correspondence between the matrix model and two-dimensional type IIA superstring theory on a nontrivial Ramond-Ramond (RR) background in sections 4 and 5. The two-dimensional IIA superstring theory is defined on the target space (Liouville direction)$\times$(S^1 with self-dual radius) [11–14], where target space SUSY exists only at the self-dual radius of the circle. In section 6, the full nonperturbative expression of the matrix-model free energy is given in its double scaling limit. It is represented as the Tracy-Widom distribution in random matrix theory [15]. We can see instanton contributions to the free energy from its weak coupling expansion, due to which the SUSY is dynamically broken. According to the correspondence, it implies that spontaneous breaking of the target-space SUSY occurs in the IIA string theory. Furthermore, strong coupling expansion of the free energy shows that the third order phase transition in the planar limit becomes a smooth crossover in the double scaling limit. In the string theory side, it suggests that some non-SUSY string theory is smoothly connected to the IIA superstring theory at the nonperturbative level. In section 7, the results presented so far are summarized and some of future directions are discussed.

Before proceeding to the next section, we would like to give some remarks on our matrix model.

- Our matrix model is interpreted as the $O(n)$ model on a random surface [16] with $n = -2$, whose critical behavior is described by the $c = -2$ topological gravity [17]. The partition function after B, ψ and $\bar{\psi}$ are integrated out is expressed as a Gaussian one-matrix model by the Nicolai mapping $H = \phi^2$, where the H-integration is over the *positive definite* hermitian matrices, not over all the hermitian matrices. The difference of the integration region has only effects which are nonperturbative in $1/N$, and the model can be regarded as the standard Gaussian matrix model at each order of genus expansion [18].

- The Nicolai mapping changes the operators $\frac{1}{N}\mathrm{tr}\,\phi^{2n}$ ($n = 1, 2, \cdots$) to regular operators $\frac{1}{N}\mathrm{tr}\,H^n$. Hence, the behavior of their correlators is expected to be described by the Gaussian one-matrix model (the $c = -2$ topological gravity) at least perturbatively in $1/N$. However, the operators $\frac{1}{N}\mathrm{tr}\,\phi^{2n+1}$ ($n = 0, 1, 2, \cdots$) are mapped to $\pm\frac{1}{N}\mathrm{tr}\,H^{n+1/2}$ that are singular at the origin. They are not observables in the $c = -2$ topological gravity, while they are natural observables as well as $\frac{1}{N}\mathrm{tr}\,\phi^{2n}$ in the original setting (1). In the next section, we will see that correlation functions among operators

$$\frac{1}{N}\mathrm{tr}\,\phi^{2n+1}, \quad \frac{1}{N}\mathrm{tr}\,\psi^{2n+1}, \quad \frac{1}{N}\mathrm{tr}\,\bar{\psi}^{2n+1} \qquad (n=0,1,2,\cdots) \tag{7}$$

exhibit logarithmic singular behavior of powers of $\ln(\mu^2-2)$ at the planar topology, that cannot be seen in the $c=-2$ topological gravity.

3 Correlation Functions

3.1 Planar one-point functions

The planar one-point function $\left\langle \frac{1}{N}\mathrm{tr}\,\phi^n\right\rangle_0$ $(n=1,2,\cdots)$ are computed as

$$\begin{aligned}\left\langle \frac{1}{N}\mathrm{tr}\,\phi^n\right\rangle_0 &= \int dx\, x^n\rho(x) \\ &= (\nu_+ + (-1)^n\nu_-)(2+\mu^2)^{n/2}F\left(-\frac{n}{2},\frac{3}{2},3;\frac{4}{2+\mu^2}\right),\end{aligned} \tag{8}$$

where the suffix "0" in the left-hand side indicates the planar contribution. For n even, the expression reduces to a polynomial of μ^2 giving nonsingular behavior as expected from the $c=-2$ topological gravity. On the other hand, when μ^2 is odd, it exhibits logarithmic singular behavior as $\mu^2\to 2+0$:

$$\left\langle \frac{1}{N}\mathrm{tr}\,\phi^{2k+1}\right\rangle_0 \sim (\nu_+ - \nu_-)\frac{2^{k+2}}{\pi}\frac{(2k+1)!!}{(k+2)!}\,\omega^{k+2}\ln\omega \tag{9}$$

with $\omega \equiv \frac{1}{4}(\mu^2-2)$. The symbol "$\sim$" denotes equality up to additive less singular terms. Matrix models can be seen as a sort of "lattice models" for string theory. In the hypergeometric function $F\left(-\frac{n}{2},\frac{3}{2},3;\frac{1}{1+\omega}\right)$ for n being odd, the logarithmic singular terms can be regarded as universal parts relevant to "continuum physics", whereas polynomials of ω appearing in the less singular terms as nonuniversal "lattice artifacts".

3.2 Planar higher-point functions (bosons)

The two-point functions of the operators Φ_{2k+1} $(k=0,1,2,\cdots)$ *:

$$\Phi_{2k+1} \equiv \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} + \text{(mixing)} \tag{10}$$

*The second term "(mixing)" in (10) consists of lower even powers of ϕ. Its explicit form is given in ref. [5].

behave as

$$\langle \Phi_{2k+1}\Phi_{2\ell+1}\rangle_{C,0} \sim -(\nu_+ - \nu_-)^2 \frac{1}{2\pi^2}\frac{1}{k+\ell+1}\frac{(2k+1)!}{(k!)^2}\frac{(2\ell+1)!}{(\ell!)^2} \times \omega^{k+\ell+1}(\ln\omega)^2. \quad (11)$$

The suffix "C" means that connected parts are taken. Some of the three-point functions are computed in [5]. These results suggest the form of higher-point functions as

$$\left\langle \prod_{i=1}^{n} \Phi_{2k_i+1} \right\rangle_{C,0} \sim (\nu_+ - \nu_-)^n \,(\text{const.})\, \omega^{2-\gamma+\sum_{i=1}^n (k_i-1)}\,(\ln\omega)^n \qquad (12)$$

with $\gamma = -1$. Besides the power of logarithm $(\ln\omega)^n$, it has the standard scaling behavior with the string susceptibility $\gamma = -1$ (the same as in the $c = -2$ topological gravity) and the gravitational scaling dimension k of Φ_{2k+1}, if we identify ω with "the cosmological constant" coupled to the lowest dimensional operator on a random surface [19–21].

3.3 Planar two-point functions (fermions)

The simplest two-point correlator of fermions is computed as

$$\begin{aligned}\left\langle \frac{1}{N}\mathrm{tr}\,\psi\,\frac{1}{N}\mathrm{tr}\,\bar\psi \right\rangle_{C,0} &= \frac{1}{2}\int_\Omega dx\,\frac{1}{x}\,\rho(x)\\ &= (\nu_+ - \nu_-)\,\frac{1}{2}\,(4(1+\omega))^{-1/2}\,F\left(\frac{1}{2},\frac{3}{2},3;\frac{1}{1+\omega}\right)\\ &= (\nu_+ - \nu_-)\left[\frac{4}{3\pi} + \frac{1}{\pi}\,\omega\ln\omega + \mathcal{O}(\omega)\right] \qquad (\omega \to +0), \quad (13)\end{aligned}$$

exhibiting singular behavior of $\ln\omega$.

Next, for $\left\langle \frac{1}{N}\mathrm{tr}\,\psi^3\,\frac{1}{N}\mathrm{tr}\,\bar\psi^3\right\rangle_{C,0}$, we should consider operator mixing similar to the bosonic case (10). Let us take a new basis as

$$\begin{aligned}\Psi_1 &\equiv \tfrac{1}{N}\mathrm{tr}\,\psi, & \bar\Psi_1 &\equiv \frac{1}{N}\mathrm{tr}\,\bar\psi,\\ \Psi_3 &\equiv \tfrac{1}{N}\mathrm{tr}\,\psi^3 + (\text{mixing}), & \bar\Psi_3 &\equiv \frac{1}{N}\mathrm{tr}\,\bar\psi^3 + (\text{mixing}),\\ \Psi_5 &\equiv \tfrac{1}{N}\mathrm{tr}\,\psi^5 + (\text{mixing}), & \bar\Psi_5 &\equiv \frac{1}{N}\mathrm{tr}\,\bar\psi^5 + (\text{mixing}),\\ &\cdots, & &\cdots, \qquad (14)\end{aligned}$$

where "mixing" means operators of lower powers of ψ or $\bar\psi$ to be added so that

$$\left\langle \Psi_{2k+1}\bar\Psi_{2\ell+1}\right\rangle_{C,0} \sim \delta_{k,\ell}\,v_k\,(\nu_+ - \nu_-)^{2k+1}\omega^{2k+1}\ln\omega \qquad (15)$$

with v_k constants holds for $k, \ell = 0, 1$ [5]. Here, $v_0 = \frac{1}{\pi}$ and $v_1 = \frac{6}{\pi}$.

The result (15) tells us that Ψ_{2k+1} and $\bar{\Psi}_{2k+1}$ have the gravitational scaling dimension k same as Φ_{2k+1} besides the logarithmic factor.

4 2D Type IIA Superstring

The type II superstring theory discussed in refs. [11–13] has the target space $(\varphi, x) \in$ (Liouville direction) $\times$ (S^1 with self-dual radius). The holomorphic energy-momentum tensor on the string world-sheet is

$$T = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\psi_x \partial \psi_x - \frac{1}{2}(\partial \varphi)^2 + \partial^2 \varphi - \frac{1}{2}\psi_\ell \partial \psi_\ell \qquad (16)$$

excluding ghosts' part. ψ_x and ψ_ℓ are superpartners of x and φ, respectively. Target-space supercurrents in the type IIA theory

$$q_+(z) = e^{-\frac{1}{2}\phi(z) - \frac{i}{2}H(z) - ix(z)}, \qquad \bar{q}_-(\bar{z}) = e^{-\frac{1}{2}\bar{\phi}(\bar{z}) + \frac{i}{2}\bar{H}(\bar{z}) + i\bar{x}(\bar{z})} \qquad (17)$$

exist only on the S^1 target space of the self-dual radius. ϕ ($\bar{\phi}$) is the holomorphic (anti-holomorphic) bosonized superconformal ghost, and the fermions are bosonized as $\psi_\ell \pm i\psi_x = \sqrt{2}\, e^{\mp iH}$, $\bar{\psi}_\ell \pm i\bar{\psi}_x = \sqrt{2}\, e^{\mp i\bar{H}}$. In addition, we should care about cocycle factors in order to realize the anticommuting nature between q_+ and $\bar{q}_-$. Supercurrents with the cocycle factors are

$$\hat{q}_+(z) = e^{\pi\beta(\frac{1}{2}p_{\bar{\phi}} - i\frac{1}{2}p_{\bar{h}} - ip_{\bar{x}})}\, q_+(z), \qquad \hat{\bar{q}}_-(\bar{w}) = e^{-\pi\beta(\frac{1}{2}p_\phi + i\frac{1}{2}p_h + ip_x)}\, \bar{q}_-(\bar{w}), \qquad (18)$$

where $\beta \in \mathbf{Z} + \frac{1}{2}$, and p_ϕ, p_h and p_x ($p_{\bar{\phi}}$, $p_{\bar{h}}$ and $p_{\bar{x}}$) are momentum modes of holomorphic part (anti-holomorphic part) of free bosons [6]. Then the supercharges

$$\hat{Q}_+ = \oint \frac{dz}{2\pi i}\, \hat{q}_+(z), \qquad \hat{\bar{Q}}_- = \oint \frac{d\bar{z}}{2\pi i}\, \hat{\bar{q}}_-(\bar{z}) \qquad (19)$$

are nilpotent $\hat{Q}_+^2 = \hat{\bar{Q}}_-^2 = \{\hat{Q}_+, \hat{\bar{Q}}_-\} = 0$, which indeed matches the property of the supercharges Q and $\bar{Q}$ in the matrix model.

The spectrum except special massive states is represented by the NS "tachyon" * vertex operator (in (-1) picture):

$$T_k = e^{-\phi + ikx + p_\ell \varphi}, \qquad \bar{T}_{\bar{k}} = e^{-\bar{\phi} + i\bar{k}\bar{x} + p_\ell \bar{\varphi}}, \qquad (20)$$

and by the R vertex operator (in $(-\frac{1}{2})$ picture):

$$V_{k,\,\epsilon} = e^{-\frac{1}{2}\phi + \frac{i}{2}\epsilon H + ikx + p_\ell \varphi}, \qquad \bar{V}_{\bar{k},\,\bar{\epsilon}} = e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{\epsilon}\bar{H} + i\bar{k}\bar{x} + p_\ell \bar{\varphi}} \qquad (21)$$

*In two dimensions, "tachyon" turns out to be not truely tachyonic but massless.

with $\epsilon, \bar{\epsilon} = \pm 1$. Cocycle factors for vertex operators are introduced as [6]

$$\begin{aligned}
&\hat{T}_k(z) = e^{\pi\beta(p_{\bar{\phi}}+ikp_{\bar{x}})}\, T_k(z), \quad \hat{\bar{T}}_{\bar{k}}(\bar{z}) = e^{-\pi\beta(p_\phi+i\bar{k}p_x)}\, \bar{T}_{\bar{k}}(\bar{z}),\\
&\hat{V}_{k,\,\epsilon}(z) = e^{\pi\beta(\frac{1}{2}p_{\bar{\phi}}+i\frac{\epsilon}{2}p_{\bar{h}}+ikp_{\bar{x}})}\, V_{k,\,\epsilon}(z),\\
&\hat{\bar{V}}_{\bar{k},\,\bar{\epsilon}}(\bar{z}) = e^{-\pi\beta(\frac{1}{2}p_\phi+i\frac{\bar{\epsilon}}{2}p_h+i\bar{k}p_x)}\, \bar{V}_{\bar{k},\,\bar{\epsilon}}(\bar{z}).
\end{aligned} \tag{22}$$

Locality with the supercurrents, mutual locality, superconformal invariance (including the Dirac equation constraint) and the level matching condition determine physical vertex operators. As discussed in [13], there are two consistent sets of physical vertex operators - "momentum background" and "winding background". Let us consider the "winding background". * The physical spectrum in the "winding background" is given by

$$\begin{aligned}
&\text{(NS, NS)}: &&\hat{T}_k\,\hat{\bar{T}}_{-k} &&(k \in \mathbf{Z}+\tfrac{1}{2}),\\
&\text{(R+, R}-\text{)}: &&\hat{V}_{k,\,+1}\,\hat{\bar{V}}_{-k,\,-1} &&(k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),\\
&\text{(R}-\text{, R+)}: &&\hat{V}_{-k,\,-1}\,\hat{\bar{V}}_{k,\,+1} &&(k = 0, 1, 2, \cdots),\\
&\text{(NS, R}-\text{)}: &&\hat{T}_{-k}\,\hat{\bar{V}}_{-k,\,-1} &&(k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),\\
&\text{(R+, NS)}: &&\hat{V}_{k,\,+1}\,\hat{\bar{T}}_{k} &&(k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),
\end{aligned} \tag{23}$$

where we take a branch of $p_\ell = 1-|k|$ satisfying the locality bound $p_\ell \le Q/2 = 1$ [22]. We can see that the vertex operators

$$\hat{V}_{\frac{1}{2},\,+1}\,\hat{\bar{V}}_{-\frac{1}{2},\,-1}, \qquad \hat{T}_{-\frac{1}{2}}\,\hat{\bar{V}}_{-\frac{1}{2},\,-1}, \qquad \hat{V}_{\frac{1}{2},\,+1}\,\hat{\bar{T}}_{\frac{1}{2}}, \qquad \hat{T}_{-\frac{1}{2}}\,\hat{\bar{T}}_{\frac{1}{2}} \tag{24}$$

form a quartet under $\hat{Q}_+$ and $\hat{\bar{Q}}_-$:

$$\begin{aligned}
[\hat{Q}_+, \hat{V}_{\frac{1}{2},\,+1}\,\hat{\bar{V}}_{-\frac{1}{2},\,-1}] &= \hat{T}_{-\frac{1}{2}}\,\hat{\bar{V}}_{-\frac{1}{2},\,-1}, \;\{\hat{Q}_+, \hat{T}_{-\frac{1}{2}}\,\hat{\bar{V}}_{-\frac{1}{2},\,-1}\} = 0,\\
\{\hat{Q}_+, \hat{V}_{\frac{1}{2},\,+1}\,\hat{\bar{T}}_{\frac{1}{2}}\} &= \hat{T}_{-\frac{1}{2}}\,\hat{\bar{T}}_{\frac{1}{2}}, \qquad [\hat{Q}_+, \hat{T}_{-\frac{1}{2}}\,\hat{\bar{T}}_{\frac{1}{2}}] = 0,
\end{aligned} \tag{25}$$

$$\begin{aligned}
[\hat{\bar{Q}}_-, \hat{V}_{\frac{1}{2},\,+1}\,\hat{\bar{V}}_{-\frac{1}{2},\,-1}] &= -\hat{V}_{\frac{1}{2},\,+1}\,\hat{\bar{T}}_{\frac{1}{2}}, \;\{\hat{\bar{Q}}_-, \hat{V}_{\frac{1}{2},\,+1}\,\bar{T}_{\frac{1}{2}}\} = 0,\\
\{\hat{\bar{Q}}_-, \hat{T}_{-\frac{1}{2}}\,\hat{\bar{V}}_{-\frac{1}{2},\,-1}\} &= \hat{T}_{-\frac{1}{2}}\,\hat{\bar{T}}_{\frac{1}{2}}, \qquad [\hat{\bar{Q}}_-, \hat{T}_{-\frac{1}{2}}\,\bar{T}_{\frac{1}{2}}] = 0.
\end{aligned} \tag{26}$$

Notice that (25) and (26) are isomorphic to (2) and (3), respectively. It leads to correspondence of single-trace operators in the matrix model to integrated vertex

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* We can repeat the parallel argument for "momentum background" in the type IIB theory, which is equivalent to the "winding background" in the type IIA theory through T-duality with respect to the S^1 direction.

operators in the type IIA theory:

$$\begin{aligned}
\Phi_1 &= \frac{1}{N}\mathrm{tr}\,\phi \;\Leftrightarrow\; \mathcal{V}_\phi(0) \equiv g_s^2 \int d^2z\, \hat{V}_{\frac{1}{2},+1}(z)\, \hat{\tilde{V}}_{-\frac{1}{2},-1}(\bar{z}), \\
\Psi_1 &= \frac{1}{N}\mathrm{tr}\,\psi \;\Leftrightarrow\; \mathcal{V}_\psi(0) \equiv g_s^2 \int d^2z\, \hat{T}_{-\frac{1}{2}}(z)\, \hat{\tilde{V}}_{-\frac{1}{2},-1}(\bar{z}), \\
\bar{\Psi}_1 &= \frac{1}{N}\mathrm{tr}\,\bar{\psi} \;\Leftrightarrow\; \mathcal{V}_{\bar{\psi}}(0) \equiv g_s^2 \int d^2z\, \hat{V}_{\frac{1}{2},+1}(z)\, \hat{\tilde{T}}_{\frac{1}{2}}(\bar{z}), \\
\frac{1}{N}&\mathrm{tr}\,(-iB) \;\Leftrightarrow\; \mathcal{V}_B(0) \equiv g_s^2 \int d^2z\, \hat{T}_{-\frac{1}{2}}(z)\, \hat{\tilde{T}}_{\frac{1}{2}}(\bar{z}),
\end{aligned} \tag{27}$$

where the bare string coupling g_s put in the right-hand sides is to count the number of external lines of amplitudes in the IIA theory. Furthermore, it can be naturally extended as

$$\begin{aligned}
\Phi_{2k+1} &= \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} + (\text{mixing}) \\
&\qquad \Leftrightarrow \mathcal{V}_\phi(k) \equiv g_s^2 \int d^2z\, \hat{V}_{k+\frac{1}{2},+1}(z)\, \hat{\tilde{V}}_{-k-\frac{1}{2},-1}(\bar{z}), \\
\Psi_{2k+1} &= \frac{1}{N}\mathrm{tr}\,\psi^{2k+1} + (\text{mixing}) \\
&\qquad \Leftrightarrow \mathcal{V}_\psi(k) \equiv g_s^2 \int d^2z\, \hat{T}_{-k-\frac{1}{2}}(z)\, \hat{\tilde{V}}_{-k-\frac{1}{2},-1}(\bar{z}), \\
\bar{\Psi}_{2k+1} &= \frac{1}{N}\mathrm{tr}\,\bar{\psi}^{2k+1} + (\text{mixing}) \\
&\qquad \Leftrightarrow \mathcal{V}_{\bar{\psi}}(k) \equiv g_s^2 \int d^2z\, \hat{V}_{k+\frac{1}{2},+1}(z)\, \hat{\tilde{T}}_{k+\frac{1}{2}}(\bar{z})
\end{aligned} \tag{28}$$

for higher $k(= 1, 2, \cdots)$.

Note that (R$-$, R$+$) operators are singlets under the target-space SUSYs $\hat{Q}_+, \hat{\tilde{Q}}_-$, and appear to have no counterpart in the matrix model side. Since the expectation value of operators measuring an RR charge $\langle \Phi_{2k+1} \rangle_0$ does not vanish as seen in (9), the matrix model is considered to correspond to the type IIA theory on a nontrivial background of the (R$-$, R$+$) fields. We may introduce the (R$-$, R$+$) background in the form of vertex operators, when the strength of the background $(\nu_+ - \nu_-)$ is small.

5 Correspondence between the Matrix Model and the Type IIA Theory

Correlation functions among integrated vertex operators in the type IIA theory on the trivial background are given by

$$\left\langle \prod_i \mathcal{V}_i \right\rangle = \frac{1}{\text{Vol.(CKV)}} \int \mathcal{D}(x, \varphi, H, \text{ghosts})\, e^{-S_{\mathrm{CFT}}} e^{-S_{\mathrm{int}}} \prod_i \mathcal{V}_i, \tag{29}$$

where "Vol.(CKV)" means the volume of the space generated by the conformal Killing vectors on the sphere,

$$S_{\rm CFT} = \frac{1}{2\pi}\int d^2z\,\left[\partial x\bar\partial x + \partial\varphi\bar\partial\varphi + \frac{1}{2}\sqrt{\hat g}\hat R\varphi + \partial H\bar\partial H + (\text{ghosts})\right],$$
$$S_{\rm int} = \mu_1\mathcal{V}_B^{(0,0)}(0) \equiv \mu_1\int d^2z\,\hat T^{(0)}_{-\frac{1}{2}}(z)\hat{\bar T}^{(0)}_{\frac{1}{2}}(\bar z). \tag{30}$$

The 0-picture (NS, NS) "tachyon" is given by

$$\hat T^{(0)}_{-\frac{1}{2}}(z) = e^{\pi\beta(ip_{\bar h}-i\frac{1}{2}p_{\bar x})}\,\frac{i}{\sqrt2}\,e^{iH-i\frac{1}{2}x+\frac{1}{2}\varphi}(z),$$
$$\hat{\bar T}^{(0)}_{\frac{1}{2}}(\bar z) = e^{-\pi\beta(-ip_h+i\frac{1}{2}p_x)}\,\frac{i}{\sqrt2}\,e^{-i\bar H+i\frac{1}{2}\bar x+\frac{1}{2}\bar\varphi}(\bar z). \tag{31}$$

We consider correlation functions in the IIA theory on a nontrivial (R−, R+) background as a form

$$\left\langle\!\!\left\langle \prod_i \mathcal{V}_i \right\rangle\!\!\right\rangle \equiv \left\langle \left(\prod_i\mathcal{V}_i\right) e^{W_{\rm RR}}\right\rangle. \tag{32}$$

The background $W_{\rm RR}$ is invariant under the target-space SUSYs:

$$W_{\rm RR} = (\nu_+ - \nu_-)\sum_{k\in\mathbf{Z}} a_k\,\mu_1^{k+1}\mathcal{V}_k^{\rm RR},$$
$$\mathcal{V}_k^{\rm RR} \equiv \begin{cases}\displaystyle\int d^2z\,\hat V_{k,\,-1}(z)\hat{\bar V}_{-k,\,+1}(\bar z) & (p_\ell = 1-|k|, k\le 0)\\ \displaystyle\int d^2z\,\hat V^{(\rm nonlocal)}_{-k,\,-1}(z)\hat{\bar V}^{(\rm nonlocal)}_{k,+1}(\bar z) & (p_\ell = 1+|k|,\ k\ge 1)\end{cases} \tag{33}$$

with a_k being numerical constants. Although the nonlocal operators in (33) with $p_\ell > 1$ do not satisfy the Dirac equation constraint on the trivial background, these operators are necessary to make correspondence with the matrix model as we will see later. Since the RR background possibly change the on-shell condition, it would be acceptable. We treat the RR background for $(\nu_+ - \nu_-)$ small as

$$\left\langle\!\!\left\langle \prod_i \mathcal{V}_i \right\rangle\!\!\right\rangle \equiv \left\langle \left(\prod_i\mathcal{V}_i\right) e^{W_{\rm RR}}\right\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle \left(\prod_i\mathcal{V}_i\right)(W_{\rm RR})^n\right\rangle, \tag{34}$$

and the picture is adjusted by hand so that the total picture is equal to -2.

In computation of amplitudes in the type IIA theory, we consider the so-called $s = 0$ amplitude in the Liouville theory, which is interpreted as a bulk amplitude insensitive to details of the Liouville wall [23]. It is somewhat similar to considering the leading nontrivial contribution for small $(\nu_+ - \nu_-)$, because higher orders of $(\nu_+ - \nu_-)$ seems to detect a cigar geometry deformed from the two-dimensional target space (Liouville direction) $\times$ (S^1 with self-dual radius) [12]. The direction to the Liouville wall corresponds to the direction to the tip of the cigar. Computation in the Liouville theory [6] yields

$$\left\langle \mathcal{V}_B(0)\,\mathcal{V}_\phi(k)\,\mathcal{V}^{\rm RR}_\ell \right\rangle = -g_s^4\,\delta_{k,\ell}\,(2\ln\mu_1)\,e^{i2\pi\beta(-k^2-\frac{1}{2}k+\frac{1}{4})}, \qquad (35)$$

$$\begin{aligned}\left\langle \mathcal{V}_\phi(k_1), \mathcal{V}_\phi(k_2)\,\mathcal{V}^{\rm RR}_{\ell_1}\,\mathcal{V}^{\rm RR}_{\ell_2} \right\rangle &= g_s^4\,\left(\delta_{\ell_1,k_1+k_2}\delta_{\ell_2,-1} + (\ell_1 \leftrightarrow \ell_2)\right) \\ &\times c_L(2\ln\mu_1)^2\,\frac{\pi}{2}\left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)^2 e^{-i\pi\beta\{\sum_{i=1}^2(k_i+\frac{1}{2})^2+\sum_{i=1}^2\ell_i^2\}}. \qquad (36)\end{aligned}$$

In the derivation of (36), we encountered the integral

$$\int d^2z\,z^\alpha\bar{z}^{\bar\alpha}(1-z)^\beta(1-\bar{z})^{\bar\beta} = \pi\frac{\Gamma(\bar\alpha+1)\Gamma(\bar\beta+1)}{\Gamma(\bar\alpha+\bar\beta+2)}\frac{\Gamma(-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)} \qquad (37)$$

with

$$\alpha = \bar\alpha = k_1+k_2, \qquad \beta = \bar\beta = -k_1-1, \qquad (k_1,k_2 = 0,1,2,\cdots). \qquad (38)$$

This expression is indefinite. We computed it by regularizing as

$$\alpha \to \alpha+\epsilon, \qquad \bar\alpha \to \bar\alpha+\epsilon, \qquad \beta \to \beta+\epsilon, \qquad \bar\beta \to \bar\beta+\epsilon, \qquad (39)$$

where $\epsilon = \frac{1}{c_L V_L}$. $V_L \equiv 2\ln\frac{1}{\mu_1}$ is the Liouville volume, and c_L is a numerical constant. The point of the regularization is preserving mutual locality of vertex operators due to the homogeneous shifts.

Let us identify the coupling μ_1 of the Liouville interaction $S_{\rm int}$ in (30) with the "cosmological constant" ω by an appropriate shift of the Liouville coordinate. Then, it leads to the identification

$$N\,{\rm tr}(-iB) \cong \frac{1}{4}\mathcal{V}_B^{(0,0)}(0), \qquad (40)$$

which is consistent to the last line in (27) (up to the choice of the picture) with

$$\frac{1}{N} \cong g_s. \qquad (41)$$

We also introduce coefficients c_k, d_k, $\bar{d}_k$ to precisely express the correspondence in (27) and (28) as

$$\Phi_{2k+1} \cong c_k\mathcal{V}_\phi(k), \qquad \Psi_{2k+1} \cong d_k\mathcal{V}_\psi(k), \qquad \bar\Psi_{2k+1} \cong \bar{d}_k\mathcal{V}_{\bar\psi}(k). \qquad (42)$$

We put the overall normalization factor $\mathcal{N}$ in identifying the amplitudes in the matrix-model side and those in the IIA theory side:

$$\langle N{\rm tr}(-iB)\,\Phi_{2k+1}\rangle_{C,0} \cong \mathcal{N}g_s^{-2}\left\langle\!\!\left\langle \frac{1}{4}\mathcal{V}_B^{(0,0)}(0)\,c_k\mathcal{V}_\phi(k) \right\rangle\!\!\right\rangle. \qquad (43)$$

The left-hand side is calculated by using (9):

$$(\text{LHS}) = -\frac{1}{4}\partial_\omega \left\langle \Phi_{2k+1}\right\rangle_0 \sim -(\nu_+ - \nu_-)\frac{2^k}{\pi}\frac{(2k+1)!!}{(k+1)!}\,\omega^{k+1}\ln\omega. \tag{44}$$

On the other hand, under a suitable choice of the picture, leading nontrivial contribution for $(\nu_+ - \nu_-)$ small to the right-hand side is

$$\begin{aligned}
&\frac{1}{4}\mathcal{N}g_s^{-2}c_k \left\langle \mathcal{V}_B(0)\,\mathcal{V}_\phi(k)\,W_{\rm RR}\right\rangle \\
&\qquad = \frac{1}{4}\mathcal{N}g_s^{-4}c_k(\nu_+ - \nu_-)\sum_{\ell\in\mathbf{Z}} a_\ell\,\omega^{\ell+1}\left\langle \mathcal{V}_B(0)\,\mathcal{V}_\phi(k)\,\mathcal{V}_\ell^{\rm RR}\right\rangle \\
&\qquad = -\frac{1}{2}(\nu_+ - \nu_-)\mathcal{N}c_k\,a_k\,\omega^{k+1}(\ln\omega)\,e^{i2\pi\beta(-k^2-\frac{1}{2}k+\frac{1}{4})},
\end{aligned} \tag{45}$$

where (35) was used. The identification (43) leads to

$$\mathcal{N}\,\hat{c}_k\hat{a}_k\,e^{i\pi\beta\frac{3}{4}} = \frac{2}{\pi}\frac{(2k+1)!}{k!(k+1)!} \tag{46}$$

with

$$\hat{c}_k \equiv c_k\,e^{-i\pi\beta(k+\frac{1}{2})^2}, \qquad \hat{a}_k \equiv a_k\,e^{-i\pi\beta k^2}. \tag{47}$$

Next, let us consider the correspondence

$$\left\langle \Phi_{2k_1+1}\Phi_{2k_2+1}\right\rangle_{C,0} \cong \mathcal{N}g_s^{-2}\left\langle\!\left\langle\, c_{k_1}\mathcal{V}_\phi(k_1)\,c_{k_2}\mathcal{V}_\phi(k_2)\,\right\rangle\!\right\rangle. \tag{48}$$

Leading nontrivial contribution to the right-hand side is obtained from (36) as

$$\begin{aligned}
&\mathcal{N}g_s^{-2}c_{k_1}\,c_{k_2}\left\langle \mathcal{V}_\phi(k_1)\,\mathcal{V}_\phi(k_2)\,\frac{1}{2!}(W_{\rm RR})^2\right\rangle \\
&= \frac{1}{2}\mathcal{N}g_s^{-2}c_{k_1}\,c_{k_2}\,(\nu_+ - \nu_-)^2 \\
&\qquad\times \sum_{\ell_1,\ell_2\in\mathbf{Z}} a_{\ell_1}\,a_{\ell_2}\,\omega^{\ell_1+\ell_2+2}\left\langle \mathcal{V}_\phi(k_1)\,\mathcal{V}_\phi(k_2)\,\mathcal{V}_{\ell_1}^{\rm RR}\,\mathcal{V}_{\ell_2}^{\rm RR}\right\rangle \\
&= (\nu_+ - \nu_-)^2\mathcal{N}g_s^2 c_L\,\hat{c}_{k_1}\,\hat{c}_{k_2}\,\hat{a}_{k_1+k_2}\,\hat{a}_{-1}\,2\pi\left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)^2 \\
&\qquad\times\omega^{k_1+k_2+1}\,(\ln\omega)^2,
\end{aligned} \tag{49}$$

while the result of the left-hand side is given by (11). Comparing these, we find the same dependence on $\nu_\pm$ and ω for any k_1 and k_2. In addition, we have an equation for coefficients:

$$\left(\frac{\hat{c}_{k_1}}{(2k_1+1)!}\right)\left(\frac{\hat{c}_{k_2}}{(2k_2+1)!}\right)(\hat{a}_{k_1+k_2}(k_1+k_2)!(k_1+k_2+1)!) = -\frac{1}{4\pi^3}\frac{1}{\mathcal{N}c_L\hat{a}_{-1}}, \tag{50}$$

which is solved as

$$\hat{c}_k = \hat{c}_0\, e^{\gamma k}\,(2k+1)!, \qquad \hat{a}_k = \frac{\hat{a}_0\, e^{-\gamma k}}{k!(k+1)!} \qquad (k=0,1,2,\cdots) \tag{51}$$

with γ being a numerical constant and

$$\hat{c}_0^2 \hat{a}_0 = -\frac{1}{4\pi^3}\frac{1}{\mathcal{N} c_L\, \hat{a}_{-1}}. \tag{52}$$

Remarkably, (46) is consistent to (51). It serves a nontrivial check of the correspondence.

Furthermore, the correspondence of the amplitudes containing fermions

$$\begin{aligned}
\left\langle \Psi_1 \bar{\Psi}_1 \right\rangle_{C,0} &\cong \mathcal{N} g_s^{-2} \left\langle\!\!\left\langle\, d_0 \mathcal{V}_\psi(0)\, \bar{d}_0 \mathcal{V}_{\bar{\psi}}(0) \,\right\rangle\!\!\right\rangle, \\
\left\langle \Psi_3 \bar{\Psi}_3 \right\rangle_{C,0} &\cong \mathcal{N} g_s^{-2} \left\langle\!\!\left\langle\, d_1 \mathcal{V}_\psi(1)\, \bar{d}_1 \mathcal{V}_{\bar{\psi}}(1) \,\right\rangle\!\!\right\rangle
\end{aligned} \tag{53}$$

yields

$$d_0 \bar{d}_0 = \frac{1}{4} c_0, \qquad d_1 \bar{d}_1 = -\frac{3}{\pi^2}\frac{c_0}{a_0^2}. \tag{54}$$

It leads to the precise correspondence between the supercharges:

$$Q \cong \frac{d_0}{c_0}\,\hat{Q}_+, \qquad \bar{Q} \cong \frac{\bar{d}_0}{c_0}\,\hat{\bar{Q}}_-. \tag{55}$$

So far, the correspondence seems consistent at the level of planar or tree amplitudes. The consistency has also been checked in the torus partition function [6].

6 Nonperturbative SUSY Breaking in the Matrix Model

In this section, we obtain the full nonperturbative free energy of the matrix model as the Tracy-Widom distribution in random matrix theory in the double scaling limit

$$N \to \infty, \quad \omega \to 0 \quad \text{with } s \equiv 4N^{2/3}\omega \text{ fixed.} \tag{56}$$

In its weakly coupled region (s: large), instanton effects can be seen in the matrix model which are nonperturbative in $1/N$. Although such effects are of the order e^{-N} and vanish in the simple large N limit, interestingly we will see that they are nonvanishing in the double scaling limit (56).

The partition function of the matrix model given by the action (1) is expressed as

$$\begin{aligned}
Z &= \int d^{N^2}\phi\, e^{-N\frac{1}{2}\mathrm{tr}(\phi^2-\mu^2)^2}\ \det(\phi\otimes \mathbf{1} + \mathbf{1}\otimes\phi) \\
&= \tilde{C}_N \int \Big(\prod_{i=1}^N d\lambda_i\Big)\, \triangle(\lambda)^2 \prod_{i,j=1}^N (\lambda_i+\lambda_j)\, e^{-N\sum_{i=1}^N \frac{1}{2}(\lambda_i^2-\mu^2)^2},
\end{aligned} \tag{57}$$

after integrating out matrices other than ϕ. Here, $\mathbf{1}$ is an $N \times N$ unit matrix, λ_i $(i = 1, \cdots, N)$ are eigenvalues of ϕ, and $\triangle(\lambda)$ denotes the Vandermonde determinant $\triangle(\lambda) = \prod_{i>j}(\lambda_i - \lambda_j)$. $\tilde{C}_N$ is an numerical factor depending only on N given by

$$\frac{1}{\tilde{C}_N} = \int \Big(\prod_{i=1}^{N} d\lambda_i\Big) \triangle(\lambda)^2 \, e^{-N\sum_{i=1}^{N}\frac{1}{2}\lambda_i^2} = (2\pi)^{\frac{N}{2}} \frac{\prod_{k=0}^{N} k!}{N^{\frac{N^2}{2}}}. \tag{58}$$

Contributions to the partition function are divided by sectors labeled by the filling fraction (ν_+, ν_-) as

$$Z = \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!} \, Z_{(\nu_+,\nu_-)} \tag{59}$$

with

$$Z_{(\nu_+,\nu_-)} \equiv \tilde{C}_N \int_0^\infty \left(\prod_{i=1}^{\nu_+ N} d\lambda_i\right) \int_{-\infty}^0 \left(\prod_{j=\nu_+ N+1}^{N} d\lambda_j\right) \left(\prod_{n=1}^{N} 2\lambda_n\right) \times \left\{\prod_{n>m} (\lambda_n^2 - \lambda_m^2)^2\right\} e^{-N\sum_{i=1}^{N}\frac{1}{2}(\lambda_i^2-\mu^2)^2}. \tag{60}$$

Here, it is easy to see

$$Z_{(\nu_+,\nu_-)} = (-1)^{\nu_- N} Z_{(1,0)}, \tag{61}$$

which leads to the vanishing partition function:

$$Z = (1 + (-1))^N Z_{(1,0)} = 0. \tag{62}$$

In order for expectation values normalized the partition function to be well-defined, we regularize the partition function by introducing a factor $e^{-i\alpha\nu_- N}$ with small α in front of $Z_{(\nu_+,\nu_-)}$. The regularized partition function becomes

$$Z_\alpha \equiv \sum_{\nu_- N=0}^{N} \frac{N!}{(\nu_+ N)!(\nu_- N)!} \, e^{-i\alpha\nu_- N} Z_{(\nu_+,\nu_-)} = (1 - e^{-i\alpha})^N \, Z_{(1,0)}. \tag{63}$$

Notice that calculations in perturbation theory of $1/N$ in sections 2 and 3 concern the partition function in a single sector ($Z_{(\nu_+,\nu_-)}$), in which such a regularization was not needed. On the other hand, since nonperturbative contributions to be computed here possibly communicate among various sectors of filling fractions, we should consider the total partition function (59) and its vanishing value requires the regularization.

The expectation value of $\frac{1}{N}\mathrm{tr}(iB)$ under the regularization (63) is expressed as

$$\begin{aligned}\left\langle \frac{1}{N}\mathrm{tr}\,(iB)\right\rangle_\alpha &= \frac{1}{N^2}\,\frac{1}{Z_\alpha}\,\frac{\partial}{\partial(\mu^2)}Z_\alpha \\ &= \frac{1}{N^2}\,\frac{1}{Z_{(1,0)}}\,\frac{\partial}{\partial(\mu^2)}Z_{(1,0)} = \left\langle \frac{1}{N}\mathrm{tr}\,(iB)\right\rangle^{(1,0)} \end{aligned} \qquad (64)$$

due to a cancellation of the factor $(1-e^{-i\alpha})^N$ in (63) between the numerator and the denominator. The regularized expectation value $\left\langle \frac{1}{N}\mathrm{tr}\,(iB)\right\rangle_\alpha$ is independent of α and well-defined in the limit $\alpha \to 0$, and thus serves as an order parameter for spontaneous SUSY breaking.

6.1 Tracy-Widom distribution

Under the change of variables $x_i = -\lambda_i^2 + \mu^2$, the partition function $Z_{(1,0)}$ defined in (60) reduces to Gaussian matrix integrals

$$Z_{(1,0)} = \tilde{C}_N \int_{-\infty}^{\mu^2} \left(\prod_{i=1}^N dx_i\right) \triangle(x)^2\, e^{-N\sum_{i=1}^N \frac{1}{2}x_i^2}. \qquad (65)$$

It seems almost trivial, but a nontrivial effect arises from the upper bound of the integration region. Techniques in random matrix theory [15] give a closed form for the partition function in the double scaling limit (56):

$$F(s) = -\ln Z_{(1,0)} = \int_s^\infty (x-s)q(x)^2 dx, \qquad (66)$$

where $q(x)$ satisfies a Painlevé II differential equation

$$q(x)'' = xq(x) + 2q(x)^3 \qquad (67)$$

with the boundary condition

$$q(x) \to \mathrm{Ai}(x) \qquad (x \to +\infty). \qquad (68)$$

Such a solution is unique and known as the Hastings-McLeod solution [24]. Since eq. (56) indicates that the string coupling constant $g_s \sim 1/N$ is proportional to $s^{-3/2}$, the region of $s \gg 1$ ($0 < s \ll 1$) describes the weakly (strongly) coupled IIA strings.

6.2 Weak coupling expansion

The partition function is given by the Fredholm determinant of the Airy kernel [15]

$$Z_{(1,0)} = \mathrm{Det}(1 - \hat{K}_{\mathrm{Ai}}|_{[s,\infty)}), \qquad (69)$$

where the operator $\hat{K}_{\rm Ai}|_{[s,\infty)}$ can be represented as the integration kernel on the interval $[s,\infty)$:

$$K_{\rm Ai}(x,y) \equiv \frac{{\rm Ai}(x){\rm Ai}'(y)-{\rm Ai}'(x){\rm Ai}(y)}{x-y}. \tag{70}$$

From the above fact, it turns out that the weak coupling expansion (large-s expansion) of the free energy is expressed as an instanton sum

$$F=-\ln Z_{(1,0)}=\sum_{k=1}^{\infty}F_{k-{\rm inst.}} \tag{71}$$

with

$$\begin{aligned} F_{k-{\rm inst.}} &= \frac{1}{k}\int_s^{\infty} dt_1\dots dt_k\, K_{\rm Ai}(t_1,t_2)\,K_{\rm Ai}(t_2,t_3)\cdots K_{\rm Ai}(t_k,t_1) \\ &\sim \frac{1}{k}\left(\frac{1}{16\pi s^{3/2}}\,e^{-\frac{4}{3}s^{3/2}}\right)^k\left[1+a_1^{(k)}s^{-3/2}+a_2^{(k)}s^{-3}+\cdots\right]. \end{aligned} \tag{72}$$

Some of the coefficients are

$$\begin{aligned} &a_1^{(1)}=-\frac{35}{24},\quad a_2^{(1)}=\frac{3745}{1152},\quad a_3^{(1)}=-\frac{805805}{82944},\cdots \\ &a_1^{(2)}=-\frac{35}{12},\quad a_2^{(2)}=\frac{619}{72},\quad a_3^{(2)}=-\frac{592117}{20736},\cdots \\ &a_1^{(3)}=-\frac{35}{8},\quad a_2^{(3)}=\frac{2059}{128},\quad a_3^{(3)}=-\frac{184591}{3072},\cdots \\ &a_1^{(4)}=-\frac{35}{6},\quad a_2^{(4)}=\frac{3701}{144},\quad a_3^{(4)}=-\frac{1112077}{10368},\cdots \\ &\cdots\cdots. \end{aligned} \tag{73}$$

The contribution to the free energy has no perturbative part and starts from nonperturbative effects of the instanton action $\frac{4}{3}s^{3/2}\propto N$ and its fluctuations expanded by $s^{-3/2}\propto N^{-1}$. It seems plausible that the nonperturbative contributions are provided by D-brane like objects. The order parameter of the SUSY breaking (with the wave function renormalization factor $N^{4/3}$) $N^{4/3}\cdot\left\langle\frac{1}{N}{\rm tr}(iB)\right\rangle^{(1,0)}=-F'(s)$ remains nonzero, implying that the target-space SUSY in the two-dimensional IIA theory is spontaneously broken by D-brane like objects. Corresponding Nambu-Goldstone fermions are identified with $\frac{1}{N}{\rm tr}\bar{\psi}$ and $\frac{1}{N}{\rm tr}\psi$ associated with the breaking of Q and $\bar{Q}$, respectively [7].

6.3 Strong coupling expansion

The Taylor series expansion of (66) around $s=0$ is

$$\begin{aligned} F(s)=0.0311059853-0.0690913807s+0.0673670913s^2 \\ -0.0361399144s^3+\cdots, \end{aligned} \tag{74}$$

which gives strong coupling expansion of the IIA superstring theory. The strong coupling limit is regular and finite. In particular, the expression is smooth around $s = 0$ and there is no obstruction to be continued to the $s < 0$ region (i.e. $\mu^2 < 2$), whereas in section 2 we had seen the third order phase transition across the point $\mu^2 = 2$ in the planar limit. Thus, the singularity in the planar limit becomes completely smeared out in the double scaling limit. In the string-theory perspective, singular behavior at the string tree level is smoothed out by quantum effects. Similar phenomenon can be seen in the unitary one-matrix model [25].

In the region of $s < 0$, the free energy has a perturbative series in $(-s)^{-3} \propto N^{-2}$, which seems to allow an interpretation as non-SUSY string theory. Thus, the matrix model is expected to describe both of the IIA superstrings and some non-SUSY strings in a unified manner at least concerning the free energy.

7 Summary and Discussion

We have computed planar correlation functions in the double-well SUSY matrix model, and discussed its correspondence to two-dimensional type IIA superstring theory on (R$-$,R$+$) background. This is an interesting example of matrix models for superstrings with target-space SUSY, in which various amplitudes not restricted to those protected by SUSY are explicitly calculable.

It is interesting to examine the correspondence at deeper level in higher genus or higher point amplitudes and in amplitudes containing special massive operators. It is also important to discuss the correspondence in the off-shell formulation such as the hybrid formalism [14].

Next, we have explicitly presented the full nonperturbative expression of the matrix-model free energy in a closed form. The result in the weakly coupled regime shows that the SUSY is spontaneously broken by nonperturbative effects due to instantons. In particular, the instanton effects survive in the double scaling limit, which implies that SUSY breaking takes place by nonperturbative dynamics in the target space of the type IIA superstring theory. It is interesting to investigate dynamics of D-branes in the type IIA theory and to reproduce the instanton contributions from the string theory side.

The free energy in the strongly coupled limit is smooth, which means that the singular behavior of the third order phase transition in the planar limit (at the string tree level) becomes smeared out in the double scaling limit (including quantum effects in the IIA strings). The regularity at $s = 0$ might suggest the existence of an S-dual theory, and the region $s < 0$ seems to describe a non-SUSY string theory. It would be intriguing to identify such S-dual theory and non-SUSY string theory and to understand the moduli space of noncritical string theories given by the matrix model.

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