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Physical Applications of Noncommutative Localization

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Abstract. Construction of quotients, or localization as it is called in mathematics, provides a powerful tool to relate different physical structures which share some underlying similarities. For example, through localizations of enveloping algebras associated with the Poincaré group we can relate much of the representation theory of the Poincaré group to the representation theory of certain deformations of it, namely SO(2,3) and SO(1,4) [1, 2]. Here we describe some additional physically interesting problems involving localization. We obtain results for the Lorentz and homogeneous Galilean groups similar to the just mentioned ones for the Poincaré group. We also describe some analogous results involving q deformations and supersymmetry, in particular, for $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1,2))$. Our analysis leads to new representations of $U_q(\mathfrak{osp}(1,2))$.

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1 Introduction

In this paper we consider problems of physical interest which involve techniques of noncommutative localization. A simple example in physics serves to make clear the need in physics for such techniques. It given by the theory of extremal projectors in the theory of angular momentum in quantum mechanics. Consider the Lie algebra $\mathfrak{so}(3)$ generated by J_+, J_- and J_0 with defining relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0, \tag{1.1}$$

$$J_\pm^* = J_\mp, \quad J_0^* = J_0. \tag{1.2}$$

Define the projector P^j onto a highest weight vector for a highest weight representation by [3]:

$$J_+ P^j = P^j J_- = 0, \quad (P^j)^2 = P^j\ . \tag{1.3}$$

Now the universal enveloping algebra $U(\mathfrak{so}(3))$ has no nontrivial divisors of zero, so there are no nontrivial solutions of these equations in $U(\mathfrak{so}(3))$. However, such facts did not prevent the physicist P.O. Löwden from obtaining solutions of Eqs. (1.3) [4]. In fact, we must go outside of the enveloping algebra in order to make precise mathematical sense out of his solutions which is part of the subject matter of noncommutative localization.

In this paper we describe physically interesting problems similar in nature to the one just given, in that they deal with extensions and quotients of enveloping algebras. We consider q generalizations of certain Lie algebras and super Lie algebras, in particular, $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1,2))$, and their relationship to $U(\mathfrak{iso}(2))$ and $U(\widetilde{\mathfrak{iso}}(2))$, respectively. Here $U(\mathfrak{iso}(2))$ is the universal enveloping algebra of the Euclidean group in the plane and $U(\widetilde{\mathfrak{iso}}(2))$ is the universal enveloping algebra of the super Euclidean Lie algebra. $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1,2))$ are the standard Drinfeld-Jimbo q deformations of the universal enveloping algebras of $\mathfrak{sl}(2)$ and its supersymmetric counterpart, $\mathfrak{osp}(1,2)$, respectively. We also consider applications involving the Lorentz and homogeneous Galilei groups similar to those described in [1] for $SO(1,4)$ and the Poincaré group and in [2] for $SO(2,3)$ and the Poincaré group.

An outline of the article is as follows. The next section is devoted to stating some necessary concepts and facts about localization of algebras which are used in the paper. Section 3 gives results on localizations of $U_q(\mathfrak{sl}(2))$ and $U(\mathfrak{iso}(2))$ and related homomorphisms, and section 4 describes supersymmetric analogs of these results for $U_q(\mathfrak{osp}(1,2))$ and $U(\widetilde{\mathfrak{iso}}(2))$. The material of sections 3 and 4 are mostly taken from [5] and we refer the reader to [5] for proofs of the results in those sections. In section 5 we use the results of sections 3 and 4 to construct representations of $U(\mathfrak{iso}(2))$ out of representations of $U_q(\mathfrak{sl}(2))$, and representations of $U(\widetilde{\mathfrak{iso}}(2))$ out of representations of $U(\mathfrak{iso}(2))$. Finally, in sections 6 and 7 we describe our results for the Lorentz and homogeneous Galilean groups.

We make the following notational conventions in this paper. Except for elements of the Cartan subalgebras for which we always use plain faced letters, quantities made out of elements of $U(\mathfrak{iso}(2))$ and $U\left(\widetilde{\mathfrak{iso}}(2)\right)$ and of their localizations are usually denoted with bold faced letters and we use plain faced letters to denote elements of $\mathfrak{sl}(2)_q$ and $\mathfrak{osp}(1|2)_q$ and their localizations. Elements of super algebras are always denoted with tildes placed over the letters. For sections 6 and 7 we adopt the convention that abstract generators of Lie algebras appear in bold type, while the actions of the generators in representations always appear in plain type.

2 Some Facts about Noncommutative Localization

Let R be a ring, with unity, and $S \neq \emptyset$ a multiplicatively closed subset of R such that $0 \notin S,\ 1_R \in S$ where I_R the identity in R. A nonzero element a in a ring R is said to be a left [resp. right] zero divisor if there exists a nonzero $b \in R$ such that $ab = 0$ [resp. $ba = 0$]. A zero divisor is an element of R which is both a left and a right zero divisor. A ring Q is said to be a left quotient ring of R with respect to S if there exists a ring homomorphism $\varphi : R \longrightarrow Q$ such that the following conditions are satisfied:

1) $\varphi(s)$ is a unit in Q for all $s \in S$ (This means that $a = \varphi(s)$ is both left and right invertible i.e. $\exists c \in Q(\text{resp. } b \in Q)$ such that $ca = I_Q(\text{resp.} ab = I_Q).)$;

2) every element of Q is in the form $(\varphi(s))^{-1}\varphi(r)$, for some $r \in R,\ s \in S$;

3) $\ker\varphi = \{r \in R :\ sr = 0,\ \text{for some s} \in S\}$.

We note that if $rs = 0$, for some $r \in R,\ s \in S$, then $s'r = 0$, for some $s' \in S$. This is because $0 = \varphi(rs) = \varphi(r)\varphi(s)$ and thus $\varphi(r) = 0$, since, by condition 1), $\varphi(s)$ is a unit of Q.

We need to multiply fractions like $(s^{-1}a)(s'^{-1}b)$ so we need to be able to move s'^{-1} to the other side of a. This leads to the *Ore condition*: $Ra \cap Rs \neq \emptyset$ for $a \in R$ and $s \in S$, a necessary and sufficient condition for the existence of localizations. Noetherian rings (which includes enveloping algebras and their q-deformations) satisfy the Ore condition [7].

The left (resp. right) quotient ring of R with respect to S, if it exists, is called the *left (resp. right) localization of* R *at* S and it is denoted by $S^{-1}R$ (resp. RS^{-1}). If $S = R$, the localization $S^{-1}R$ is the (skew) field of fractions of R i.e. the quotient field of R.

We shall make use of the following result when dealing with representations. A proof of it can be found in Dixmier [8].

Lemma. Suppose $f : R \longrightarrow R_1$ is a ring homomorphism and Q is a left (resp. right) quotient ring of R with respect to S. If $f(s)$ is a unit in R_1 for every $s \in S$, then there exists a (unique) ring homomorphism $g : Q \longrightarrow R_1$ which extends f.

3 Localizations of $\mathbf{U_q}(\mathfrak{sl}(\mathbf{2}))$ and $\mathbf{U}(\mathfrak{iso}(\mathbf{2}))$

We define the q-deformation $U_q(\mathfrak{sl}(2)) \simeq U_q(\mathfrak{so}(3,\mathbb{C}))$ of the enveloping algebra of the simple Lie algebra $\mathfrak{sl}(2)$ as the unital associative algebra with generators E, F, K, K^{-1} and relations [9]

$$KK^{-1} = K^{-1}K = I, KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \quad .$$

Equivalently, with $K = q^H$ we have generators H, $X^{\pm}$ with relations:

$$[H, X^{\pm}] \;=\; \pm 2X^{\pm}, \tag{3.1}$$

$$[X^+, X^-] \;=\; [H]_{q^2} \tag{3.2}$$

where $[x]_q = \frac{\left(q^{x/2}-q^{-x/2}\right)}{\left(q^{1/2}-q^{-1/2}\right)}$, $E = X^+$, $F = X^-$ and $[\cdot,\cdot]$ denotes commutator. The Casimir operator is

$$\begin{aligned}\Delta_q &= X^+X^- + \left(\left[\frac{H-1}{2}\right]_{q^2}\right)^2 - \frac{1}{4}\cdot I \\ &= X^-X^+ + \left(\left[\frac{H+1}{2}\right]_{q^2}\right)^2 - \frac{1}{4}\cdot I \ .\end{aligned} \tag{3.3}$$

Define a real form $U_q(\mathfrak{so}(2,1))$ with generators L_{ij} $(i, j = 1.\ 2,\ 3,\ i < j)$ specified by $X^{\pm} = L_{13} \mp iL_{23},\ L_{12} = -\frac{i}{2}\ H$. The L_{ij} are preserved under the following antilinear, anti-involution (star structure) [9]: $\omega(H) = H$, $\omega(X^{\pm}) = \ -\ X^{\mp}$.

A basis for the Euclidean Lie algebra $\mathfrak{iso}(2)$ is $\mathbf{L}_{12}$ and $\mathbf{P}_i$ $(i = 1,\ 2)$. They satisfy the following commutation relations:

$$[\mathbf{L}_{12}\ ,\ \mathbf{P}_2] = -\,\mathbf{P}_1\ \ ,\ \ [\mathbf{L}_{12}\ ,\ \mathbf{P}_1] = \ \mathbf{P}_2\ \ ,$$

$$[\mathbf{P}_1\ ,\ \mathbf{P}_2] = 0\ \ .$$

Complexified translations generators are $\mathbf{P}^{\pm} = -\mathbf{P}_1 \pm i\mathbf{P}_2$. We also define as above $H = 2i\mathbf{L}_{12}$ and it generates an $SO(2)$ subgroup whose Lie algebra is $\mathfrak{so}(2)$. We have:

$$[H, \mathbf{P}^{\pm}] \;=\; \pm 2\mathbf{P}^{\pm}\ ,\ [\mathbf{P}^+, \mathbf{P}^-] = 0\ . \tag{3.4}$$

The enveloping algebra of $\mathfrak{iso}(2)$ is $U(\mathfrak{iso}(2))$. The center $Z(U(\mathfrak{iso}(2)))$ of $U(\mathfrak{iso}(2))$ is generated by

$$Y^2 = \mathbf{P}^+\mathbf{P}^- = \mathbf{P}^-\mathbf{P}^+. \tag{3.5}$$

We shall make frequent use of the following result and it is readily established by using Eqs. (3.4) and the Maclaurin series formula: let f be any analytic function, then

$$\mathbf{P}^{\pm} f\left(\left[\frac{H}{2}\right]_q\right) = f\left(\left[\frac{H\mp 2}{2}\right]_q\right)\mathbf{P}^{\pm}\ \ . \tag{3.6}$$

We shall also make use of a similar equation for $U_q(\mathfrak{sl}(2))$; it is the same as Eqn. (3.6) but with $\mathbf{P}^{\pm}$ replaced by $X^{\pm}$.

Let

$$\mathbf{X}^{\pm} = \mathbf{L}_{13} \mp i\mathbf{L}_{23} =$$
$$= \left\{ \pm\frac{1}{Y} \left[\frac{H \mp 1}{2} \right]_{q^2} + I \right\} \mathbf{P}^{\pm} = \mathbf{P}^{\pm} \left\{ \pm\frac{1}{Y} \left[\frac{H \pm 1}{2} \right]_{q^2} + I \right\} \tag{3.7}$$

where Y is a solution of the algebraic equation $Y^2 - \mathbf{P}^+\mathbf{P}^- = 0$ in $U(\mathfrak{iso}(2))$. Note that we used Eqn. (3.6) to obtain the last equality. To make sense out of quantities like $\left[\frac{H\pm 1}{2}\right]_{q^2}$ in Eqs. (3.7) as elements of $U(\mathfrak{iso}(2))$ we consider formal series expansions in H. This requires going to an extension of $U(\mathfrak{iso}(2))$ which allows for such arbitrary formal series (cf. [10]). To keep things simple we do not make a distinction between enveloping algebras and necessary such extensions for incorporating formal series expansions. (Observe that similar observations apply to the same quantities viewed as elements of $U_q(\mathfrak{sl}(2))$.) Let $\tau(X^{\pm}) = \mathbf{X}^{\pm}$ and $\tau(H) = H$. This defines a mapping τ from $U_q(\mathfrak{sl}(2))$ into an algebraic extension of the localization of $U(\mathfrak{iso}(2))$ with denominators consisting of powers of Y.

Proposition 1. τ is a homomorphism from $U_q(\mathfrak{sl}(2))$ onto its image. In particular the $\mathbf{X}^{\pm}$ defined by Eqs. (3.7) together with H satisfy the relations, Eqs. (3.1) and (3.2), of the generators of $U_q(\mathfrak{sl}(2))$. Furthermore, let Δ_q be defined by Eqn. (3.3) but with $\mathbf{X}^{\pm}$ replacing $X^{\pm}$. Then $\Delta_q = Y^2 - \frac{1}{4} \cdot I$.

For a proof of this result we refer the reader to Ref. [6].

Now let Y be such that it commutes with all elements of $U_q(\mathfrak{sl}(2))$ and satisfies the equation

$$Y^2 = \Delta_q + \frac{1}{4} \cdot I \ . \tag{3.8}$$

Let $P^{\pm} = (D_L^{\pm})^{-1} X^{\pm} = X^{\pm} (D_R^{\pm})^{-1}$ with $D_L^{\pm} = \left(\pm\frac{1}{Y} \left[\frac{H\mp 1}{2}\right]_{q^2} + I \right)$ and $D_R^{\pm} = \left(\pm\frac{1}{Y} \left[\frac{H\pm 1}{2}\right]_{q^2} + I \right)$ and define τ' by $\tau'(\mathbf{P}^{\pm}) = P^{\pm}$ and $\tau'(H) = H$.

Proposition 2. τ' extends by linearity to a homomorphism from $\mathfrak{iso}(2)$ into a localization of $U_q(\mathfrak{sl}(2))$. In particular, $\tau'(\mathbf{P}^{\pm})$ and $\tau'(H)$ satisfy the commutation relations, Eqs. (3.4), of $\mathfrak{iso}(2)$ and, furthermore, $P^+P^- = Y^2$. For the proof of this proposition we again refer to Ref. [6].

4 Localizations of $\mathbf{U}_{\tilde{\mathbf{q}}}(\mathfrak{osp}(\mathbf{1}|\mathbf{2}))$ and $\mathbf{U}\big(\widetilde{\mathfrak{iso}(\mathbf{2})}\big)$

The $\tilde{q}$-deformation $U_{\tilde{q}}(\mathfrak{osp}(1|2)) = \mathfrak{osp}(1|2)_{\tilde{q}}$ of the enveloping algebra of the orthosymplectic Lie super algebra $\mathfrak{osp}(1|2)$ is defined as the unital associative algebra with generators e, f, k, k^{-1} and relations [9, 11]:

$$kk^{-1} = k^{-1}k = I, kek^{-1} = \tilde{q}e, kfk^{-1} = \tilde{q}^{-1}f, ef + fe = \frac{K - K^{-1}}{\tilde{q} - \tilde{q}^{-1}} \ .$$

The $\mathbb{Z}_2$ grading on $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ is $d(e) = d(f) = 1, d(k) = 0$ where $d(x)$ is the parity of x. Let $k = \tilde{q}^{\tilde{H}}$, $e = \tilde{X}^+$, $f = \tilde{X}^-$ and we obtain generators $\tilde{H}$, $\tilde{X}^{\pm}$ with relations ($\{\cdot,\cdot\}$ denote anticommutator):

$$[\tilde{H}, \tilde{X}^{\pm}] = \pm \tilde{X}^{\pm} , \tag{4.1}$$

$$\{\tilde{X}^+, \tilde{X}^-\} = [\tilde{H}]_{\tilde{q}^2} . \tag{4.2}$$

The Casimir operator of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ is $\tilde{\Delta}_{\tilde{q}} = \tilde{S}_{\tilde{q}}^{2} + 2 \cdot I$ with [11]

$$\tilde{S}_{\tilde{q}} = \frac{\tilde{q}^{1/2}k - \tilde{q}^{-1/2}k^{-1}}{\tilde{q} - \tilde{q}^{-1}} - (\tilde{q}^{1/2} + \tilde{q}^{-1/2})fe = [\tilde{H} + \frac{1}{2}]_{\tilde{q}^2} - [2]_{\tilde{q}}\tilde{X}^-\tilde{X}^+ =$$

$$-\frac{\tilde{q}^{-1/2}k - \tilde{q}^{1/2}k^{-1}}{(\tilde{q} - \tilde{q}^{-1})} + (\tilde{q}^{1/2} + \tilde{q}^{-1/2})ef = -[\tilde{H} - \frac{1}{2}]_{\tilde{q}^2} + [2]_{\tilde{q}}\tilde{X}^+\tilde{X}^- . \tag{4.3}$$

It is straightforward to show that $\tilde{S}_{\tilde{q}}$ anticommutes with $\tilde{X}^+$ and $\tilde{X}^-$ and commutes with $\tilde{H}$. A star structure (or real form) is specified as follows. Let $\tilde{\omega}$ be such that $\tilde{\omega}(\tilde{H}) = \tilde{H}$, $\tilde{\omega}(\tilde{X}^{\pm}) = -\tilde{X}^{\mp}$. $\tilde{\omega}$ is, as in the $\mathfrak{sl}(2)$ case, an antilinear, anti-involution.

A basis for the three dimensional super Euclidean Lie algebra $\widetilde{\mathfrak{iso}}(2)$ is given by $\tilde{\mathbf{L}}_{12} = -\frac{i}{2}\tilde{H}$ and $\tilde{\mathbf{P}}_i$ $(i = 1, 2)$ with commutation relations

$$[\tilde{H}, \tilde{\mathbf{P}}^{\pm}] = \pm \tilde{\mathbf{P}}^{\pm} , \ \{\tilde{\mathbf{P}}^+, \tilde{\mathbf{P}}^-\} = 0 . \tag{4.4}$$

The universal enveloping algebra of $\widetilde{\mathfrak{iso}}(2)$ is $U\big(\widetilde{\mathfrak{iso}}(2)\big)$. Let $\tilde{Y}^2$ be the element of $Z(U\big(\widetilde{\mathrm{iso}}(2)\big))$ given by

$$\tilde{Y}^2 = -i(-1)^{\tilde{H}}\, \tilde{\mathbf{P}}^+\tilde{\mathbf{P}}^- = i(-1)^{\tilde{H}}\tilde{\mathbf{P}}^-\tilde{\mathbf{P}}^+ . \tag{4.5}$$

We now establish homomorphisms from $\widetilde{\mathfrak{iso}}(2)$ and $\mathfrak{iso}(2)$ onto their images in certain spaces which we now make precise. Define τ_0 by $\tau_0(H) = -2\tilde{H}$, $\tau_0(\mathbf{P}^+) = -\tilde{\mathbf{P}}^-$, $\tau_0(\mathbf{P}^-) = e^{-i\pi\tilde{H}}\tilde{\mathbf{P}}^+$ and $\tilde{\tau}_0$ by $\tilde{\tau}_0(\tilde{H}) = -\frac{1}{2}H$, $\tilde{\tau}_0(\tilde{\mathbf{P}}^-) = -\mathbf{P}^+$, $\tilde{\tau}_0(\tilde{\mathbf{P}}^+) = e^{-\frac{i\pi}{2}H}\mathbf{P}^-$. (We again refer the reader to [10] in order to give a precise meaning to expressions like $e^{-\frac{i\pi}{2}H}$ and $e^{-i\pi\tilde{H}}$.)

Proposition 3. τ_0 and $\tilde{\tau}_0$ define Lie algebra and Lie super algebra homomorphisms from $\mathfrak{iso}(2)$ and $\widetilde{\mathfrak{iso}}(2)$ onto their images in $U\big(\widetilde{\mathfrak{iso}}(2)\big)$ and $U(\mathfrak{iso}(2))$, respectively.

In order to prove this result we extend τ_0 and $\tilde{\tau}_0$ by linearity to $\mathfrak{iso}(2)$ and $\widetilde{\mathfrak{iso}}(2)$, respectively, and verify the respective commutation relations.

Next we describe homomorphisms of $U_{\tilde q}(\mathfrak{osp}(1|2))$ and $U\big(\widetilde{\mathfrak{iso}}(2)\big)$ into extensions of localizations of $U\big(\widetilde{\mathfrak{iso}}(2)\big)$ and $U_{\tilde q}(\mathfrak{osp}(1|2))$, respectively, i.e. the analogs of Propositions 1 and 2. Let

$$\tilde{\mathbf{X}}^{\pm} = \left(\frac{1}{\tilde Y}\sqrt{e^{i\pi\tilde H}\left[\tilde H \mp \frac{1}{2}\right]_{\tilde q^2}} + \sqrt{\pm I}\right)\frac{\tilde{\mathbf{P}}^{\pm}}{\sqrt{[2]_{\tilde q}}} =$$
$$= \frac{\tilde{\mathbf{P}}^{\pm}}{\sqrt{[2]_{\tilde q}}}\left(\frac{\pm i}{\tilde Y}\sqrt{e^{i\pi\tilde H}\left[\tilde H \pm \frac{1}{2}\right]_{\tilde q^2}} + \sqrt{\pm I}\right) \tag{4.6}$$

where I is the identity in $U\big(\widetilde{\mathfrak{iso}}(2)\big)$.

Proposition 4. If $\tilde Y$ is such that it commutes with all elements of $U\big(\widetilde{\mathfrak{iso}}(2)\big)$ and satisfies Eqn. (4.5), then Eqs. (4.6) define a homomorphism $\tilde\tau$ from $U_{\tilde q}(\mathfrak{osp}(1|2))$ into a localization of $U\big(\widetilde{\mathfrak{iso}}(2)\big)$, with $\tilde\tau(\tilde X^{\pm}) = \tilde{\mathbf{X}}^{\pm}$ and $\tilde\tau(\tilde H) = \tilde H$. Furthermore, let $\tilde{\mathbf{S}}_{\tilde q}$ be defined by Eqn. (4.3) but with $\tilde{\mathbf{X}}^{\pm}$ replacing $\tilde X^{\pm}$, then $\tilde{\mathbf{S}}_{\tilde q} = -e^{-i\pi\tilde H}\tilde Y^2$.

The proof of this proposition is straightforward and is very similar to the proof of Proposition 1.

Now let $\tilde P^{\pm} = (\tilde D_L^{\pm})^{-1}\tilde X^{\pm} = \tilde X^{\pm}(\tilde D_R^{\pm})^{-1}$ with $\sqrt{[2]_{\tilde q}}\tilde D_L^{\pm} = \frac{1}{\tilde Y}\sqrt{e^{i\pi\tilde H}\left[\tilde H \mp \frac{1}{2}\right]_{\tilde q^2}} + \sqrt{\pm\, I}$ and $\sqrt{[2]_{\tilde q}}\tilde D_R^{\pm} = \frac{\pm i}{\tilde Y}\sqrt{e^{i\pi\tilde H}\left[\tilde H \pm \frac{1}{2}\right]_{\tilde q^2}} + \sqrt{\pm\, I}$ where now I is the identity in $U_{\tilde q}(\mathfrak{osp}(1|2))$.

Proposition 5. Let $'(\tilde{\mathbf{P}}^{\pm}) = \tilde{\mathtt{P}}^{\pm}$ and $'(\tilde{\mathtt{H}}) = \tilde{\mathtt{H}}$. If $\tilde Y^2$ is such that it commutes with all elements of $U_{\tilde q}(\mathfrak{osp}(1|2))$ and satisfies

$$\tilde Y^2 + e^{i\pi\tilde H}\tilde S_{\tilde q} = 0, \tag{4.7}$$

then $'$ extends to a homomorphism of $\widetilde{\mathfrak{iso}}(2)$ onto its image in a localization of $U_{\tilde q}(\mathfrak{osp}(1|2))$. In particular $\tilde P^{\pm}$ and $\tilde H$ satisfy the commutation relations (4^{bis}) of $U\big(\widetilde{\mathrm{iso}}(2)\big)$ and, furthermore, $\tilde P^{+}\tilde P^{-} = ie^{-i\pi\tilde H}\tilde Y^2$.

For a proof of this proposition we refer the reader to [5].

5 Results on Representations

We shall now make use of the Propositions to construct new representations of $U_{\tilde q}(\mathfrak{osp}(1|2))$ out of representations of $\mathfrak{iso}(2)$ and also representations of $\mathfrak{iso}(2)$ out of representations of $U_q(\mathfrak{sl}(2))$. Combining these two results we can thus obtain new representations of $U_{\tilde q}(\mathfrak{osp}(1|2))$ out of representations of $U_q(\mathfrak{sl}(2))$.

We first construct representations of $\mathfrak{iso}(2)$ from representations of $U_q(\mathfrak{sl}(2))$. Let $\mathcal{H}_{(m,\epsilon)}$ be the one dimensional vector space $\mathbb{C}e_m$ with $m = n + \epsilon, n = 0, \pm1, \pm2, \ ..$ for fixed $\epsilon = 0$ or $\frac{1}{2}$. For $\sigma \in \mathbb{C}$ and for $q \in \mathbb{C}$, $q \neq 0$ and not a root of unity, the following formulae define a representation $d\pi^{\sigma,\epsilon}$ of $U_q(\mathfrak{sl}(2))$ on the space $\oplus_{n\in\mathbb{Z}}\mathcal{H}_{(m,\epsilon)}$:

$$d\pi^{\sigma,\epsilon}(H)e_m = 2m\, e_m \ , \ d\pi^{\sigma,\epsilon}(X^{\pm})e_m = [-\sigma \pm m]_{q^2} e_{m\pm1} \ . \tag{5.1}$$

Recall the real form $U_q(\mathfrak{so}(2,1))$ of $U_q(\mathfrak{sl}(2))$ introduced in Section 2. For $|q| = 1$ and q not a root of unity we have the following [12]: (1) for $\sigma = i\rho - \frac{1}{2}$ ($\rho \in \mathbb{R}$), the representation space is $\mathcal{H}^{(\sigma,\epsilon)} = \sum_m{}^{\oplus}\mathcal{H}_{(m,\epsilon)}$ and $d\pi^{\sigma,\epsilon}$ is the (infinitesimally unitarizable) principal series of $U_q(\mathfrak{so}(2,1))$; (2) for $\sigma = \epsilon \bmod(2)$ and $\sigma = \ell$ with $\ell < -\frac{1}{2}$, a) the representation space $X_+^{-\ell,\epsilon}$ is the linear span of the above e_m with $m > -\ell$, b) the representation space $X_-^{-\ell,\epsilon}$ is the linear span of the e_m with $m < \ell$. $d\pi^{\sigma,\epsilon}$ acts irreducibly on $X_{\pm}^{-\ell,\epsilon}$. These give q deformed discrete series of $U_q(\mathfrak{so}(2,1))$.

We now use Proposition 2 to construct the representations of $\mathfrak{iso}(2)$ out of the (infinitesimally unitarizable) principal series representations of $U_q(\mathfrak{so}(2,1))$. We start with a given such $d\pi^{\sigma,\epsilon}$. A simple calculation shows that $Y^2 = ([i\rho]_{q^2})^2\ I$ on $\mathcal{H}^{(\sigma,\epsilon)}$ (I is the identity on $\mathcal{H}^{(\sigma,\epsilon)}$). It follows that the actions of $(D_R^{\pm})^{-1}$ in the representation exist as operators on the representation space.

Using Proposition 2 and Eqn. (5.1) , we can easily write down the action of the $\mathbf{P}^{\pm}$ on the e_m of the representation space $\mathcal{H}^{(\sigma,\epsilon)}$. We find:

$$d\tilde{\pi}_q^{\sigma,\epsilon}(\mathbf{P}^{\pm})\, e_m = -\left(\frac{[i\rho]_{q^2}[\sigma \mp m]_{q^2}}{[\sigma + \frac{1}{2}]_{q^2} \mp [m \pm \frac{1}{2}]_{q^2}}\right) e_{m\pm1} \ . \tag{5.2}$$

Using Propositions 3 and 4 we now construct representations of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ out of representations of $U(\mathfrak{iso}(2))$ and $U\big(\widetilde{\mathfrak{iso}}(2)\big)$. We start with the positive mass representations of the Euclidean Lie algebra. They are characterized by a real number ρ ($\rho \neq 0$)and an integer ϵ which is either 0 or $\frac{1}{2}$. They are described as follows [13]. The representation space is $\mathcal{H}^{(i\rho,\epsilon)} = \sum_m{}^{\oplus}\mathcal{H}_{(m,\epsilon)}$ where the $\mathcal{H}_{(m,\epsilon)}$ are the same one dimensional vector spaces introduced above. The actions of the generators of $U(\mathfrak{iso}(2))$ on $\mathcal{H}^{(i\rho,\epsilon)}$ are given by

$$d\pi^{\rho,\epsilon}(\mathbf{P}^{\pm})\, e_m = \ - \ (i\,\rho)\, e_{m\pm1} \tag{5.3}$$

$$d\pi^{\rho,\epsilon}(H)\, e_m = 2m\, e_m. \tag{5.4}$$

Using Proposition 3 we obtain the following representation of $U\big(\widetilde{\mathfrak{iso}}(2)\big)$ on $\mathcal{H}^{(i\rho,\epsilon)}$:

$$d\tilde{\pi}^{\rho,\epsilon}(\tilde{\mathbf{P}}^{\pm})\, e_m = \ \mp\ (i\,\rho)\, e^{-i\frac{\pi}{2}\{(m-1)\pm(m-1)\}}\, e_{m\mp1} \tag{5.5}$$

$$d\tilde{\pi}^{\rho,\epsilon}(\tilde{H})\, e_m = -m\, e_m \quad . \tag{5.6}$$

Now use Proposition 4 together with the Lemma to obtain the representation of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ on $\mathcal{H}^{(i\rho,\epsilon)}$. We claim that the conditions of the Lemma are satisfied provided zero is in the resolvent set of $d\tilde{\pi}^{\rho,\epsilon}(\tilde{Y})$. We refer the reader to [5] where it is shown that this is always the case for nonzero ρ and arbitrary ϵ.

Using Eqn. (4.6) together with Eqs. (5.5) and (5.6) we can explicitly construct the representation of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ on the above representation space. We obtain:

$$\tilde{X}^{+} e_m = \frac{1}{\sqrt{[2]_{\tilde{q}}}}(-i\rho)\left\{\frac{(-1)^{m/2}\sqrt{[m-\frac{1}{2}]_{q^2}}}{i\sqrt{i}(-1)^{\epsilon}\rho} - 1\right\}(-1)^m e_{m-1} \tag{5.7}$$

$$\tilde{X}^{-} e_m = \frac{1}{\sqrt{[2]_{\tilde{q}}}}(i\rho)\left\{\frac{(-1)^{-m/2}\sqrt{[m+\frac{1}{2}]_{q^2}}}{i\sqrt{i}(-1)^{\epsilon}\rho} + i\right\} e_{m+1} \tag{5.8}$$

with the action of $\tilde{H}$ on the representation space being given by Eqn. (5.6).

Schmüdgen has defined and classified in [14] certain series of representations of $U_q(\mathfrak{sl}(2))$ influential in Faddeev's construction of the modular double [15, 16]. According to Ponsot and Teschner, some of his representations reproduce known representations of principal or discrete series of $\mathfrak{sl}(2)$ in the classical limit $q \to 1$ [16]. Despite this fact, we easily convince ourselves that the representations of $U_q(\mathfrak{sl}(2))$ described by Eqn. (5.1) do not occur in his classification. To see this, we first observe that, if our representations occur in the classification given in [14], it is clear that they must occur in his class $(I)_{\epsilon_1,e_2,c}$ (cf. COROLLARY 5 of [14], p. 219). For representations in the class $(I)_{\epsilon_1,e_2,c}$ K is given by the self-adjoint operator $\epsilon_1 e^{2\alpha x}$ on $\mathcal{L}^2(\mathbb{R})$ where x is the operator of multiplication by x in $\mathcal{L}^2(\mathbb{R})$. (See [14] for the definiton of α.) Hence, the spectrum of K is continuous. However, for the representations of $U_q(\mathfrak{sl}(2))$ given in Eqn. (5.1) the spectrum of K is discrete. Thus it is not possible that our representations of $U_q(\mathfrak{sl}(2))$ are unitarily equivalent to those described in [14].

Ip and Tsetlin in [17] obtained by means of a tensor product construction representations of $U_q(\mathfrak{osp}(1|2))$ out of representations of $U_{q^*}(\mathfrak{sl}(2))$ for $q^* = iq$. Our construction of representations of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ from representations of $U_q(\mathfrak{sl}(2))$ differs completely from theirs. The propositions, which we invoked in order to get representations of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ out of representations of $U_q(\mathfrak{sl}(2))$, are all statements about homomorphisms of the algebras involved and do not make use of any tensor product construction. From these remarks and those of the previous paragraph it seems certain that the representations of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ which we have constructed are in fact new.

Finally, it should be clear to the reader that we may use Propositiions 2, 3 and 4 just as we did in the above to try to construct representations of $U_q(\mathfrak{osp}(1|2))$

out of the representations of $U_q(\mathfrak{sl}(2))$ given in [14]. Necessary conditions for the existence of such representations of $U_q(\mathfrak{osp}(1|2))$ is that the conditions of the Lemma are satisfied.

6 The Conformal Group and Some of its Subgroups

Next we turn to a classical example involving localization in physics, namely an analysis of the relationship between the Lorentz and homogeneous Galilei groups. In the analysis we shall make use of higher dimensional symmetry groups which contain these groups as subgroups, so we start with the conformal group and define the relevant Lie subgroups of the conformal group and their associated Lie subalgebras. Let $G = SO_0(2,4)$ be the 2-fold covering group of the conformal group in 4 dimensional Minkowski space and let $\mathfrak{g}$ be its Lie algebra. Let E_{ij} be the matrix such that the (i,j) component is equal to 1 and the other components are all equal to 0 (i,j integers such that $-1 \le i,j \le 4$). We define $\mathbf{L}_{ij} = E_{ij} - E_{ji}$ $(1 \le i < j \le 4)$, $\mathbf{L}_{0i} = E_{i,o} + E_{o,i}$ $(1 \le i \le 4)$, $\mathbf{L}_{-1,i} = E_{i,-1} + E_{-1,i}$ $(1 \le i \le 4)$ and $\mathbf{L}_{-10} = E_{-1,0} - E_{0,-1}$. The $\mathbf{L}_{ij}$ satisfy: for i,j,k all distinct,

$$[\mathbf{L}_{ij}, \mathbf{L}_{jk}] = e_j \mathbf{L}_{ik} \ \ (-e_{-1} = -e_0 = e_1 = e_2 = e_3 = e_4 = 1) \qquad (6.1)$$

otherwise the commutator is zero.

Let $\mathcal{K}'$ be the subalgebra spanned by $\mathbf{L}_{ij}$ $(1 \le i,j \le 4)$, $\mathcal{H}'$ be the subalgebra spanned by $\mathbf{L}_{ij}$ $(1 \le i,j \le 4)$ and $\mathbf{L}_{0i}$ $(1 \le i \le 4)$, $\mathcal{A}'$ be the subalgebra spanned by $\mathbf{L}_{0,4}$, $\mathcal{N}_+{}'$ ($\mathcal{N}_-{}'$) be the subalgebra spanned by $\mathbf{P}_i{}' = \frac{1}{2}(\mathbf{L}_{4,i} + \mathbf{L}_{0,i})$ $(1 \le i \le 3)$ ($\tilde{\mathbf{P}}_i{}' = \frac{1}{2}(\mathbf{L}_{4,i} - \mathbf{L}_{0,i})(1 \le i \le 3)$), $\mathcal{L} = \mathcal{L}(3)$ be the subalgebra spanned by $\mathbf{L}_{ij}$ $(1 \le i,j \le 3)$ and $\mathbf{L}_{0i}$ $(1 \le i \le 3)$ and $\mathcal{M}$ be the subalgebra spanned by $\mathbf{L}_{ij}$ $(1 \le i,j \le 3)$. Subgroups of G corresponding to $\mathcal{K}'$, $\mathcal{H}'$, $\mathcal{A}'$, $\mathcal{N}'_+$, $\mathcal{N}'_-$ and $\mathcal{L}$ are $K' \simeq SO(4)$, $H' = SO_0(1,4)$, $A' \simeq SO(1,1)$, N'_+, N'_-, $L = SO_0(1,3)$ and $M = SO(3)$, respectively. $H' = SO_0(1,4)$ is the de Sitter subgroup, and we have an Iwasawa like decomposition of H' [18]: the map $L' \times A' \times N_+^{'} \longrightarrow H'$ is an injective diffeomorphism onto an open, dense subset of H', where L' is $SO(1,3)$.

For completeness we also define subgroups of the conformal group associated to the Poincaré subgroup and the anti de Sitter subgroup. Let $\mathcal{H}$ be the subalgebra spanned by $\mathbf{L}_{ij}$ $(0 \le i,j \le 3)$, $\mathbf{L}_{-1i}$ $(0 \le i \le 3)$, $\mathcal{A}$ be the subalgebra spanned by $\mathbf{L}_{-1,0}$, $\mathcal{T}_4^+$ ($\mathcal{T}_4^-$) be the subalgebra spanned by $\mathbf{P}_i = \frac{1}{2}(\mathbf{L}_{-1,i} + \mathbf{L}_{i,4})$ $(0 \le i \le 3)$ ($\tilde{\mathbf{P}}_i = \frac{1}{2}(\mathbf{L}_{-1,i} - \mathbf{L}_{i,4})(0 \le i \le 3)$) and $\mathcal{N}_+$ ($\mathcal{N}_-$) be the subalgebra spanned by $\mathbf{P}_i = \frac{1}{2}(\mathbf{L}_{-1,i} + \mathbf{L}_{i,4})$ $(1 \le i \le 3)$ ($\tilde{\mathbf{P}}_i = \frac{1}{2}(\mathbf{L}_{-1,i} - \mathbf{L}_{i,4})(1 \le i \le 3)$). Subgroups of G corresponding to $\mathcal{H}$, $\mathcal{A}$, $\mathcal{T}_4^+$, $\mathcal{T}_4^-$, $\mathcal{N}_+$, and $\mathcal{N}_-$ are $H \simeq SO_0(2,3)$, $A \simeq SO(2)$, T_4^+, T_4^-, N_+ and N_-, respectively. The Poincaré group is $P \simeq L \times_s T_4^+$ wih Lie algebra, $\mathcal{P} = \mathcal{L} \oplus_s \mathcal{T}_4^+$. $H = SO_0(2,3)$ is the anti de Sitter subgroup, and we have an Iwasawa like decomposition of H [19]: the map $L' \times A \times N_+ \longrightarrow H$ is an injective diffeomorphism onto an open, dense subset of H.

7 The Homogeneous Galilei Group and the Lorentz Group

The homogeneous Galilei group, $G(3)$, acts on $\mathbb{R} \times \mathbb{R}^3$ as

$$G(3) \ni g : (t, x_i) \to (t', x_i') \text{ with } t' = t, x_i{}' = \sum_{j=1}^{3} R_{ij} x_j + v_i t. \tag{7.1}$$

($(R_{ij}) \in SO(3)$ and $v_i \in \mathbb{R}^3$.) The Lie algebra of $G(3)$ is $\mathcal{G}(3) = l.s.\{\mathbf{L}_{ij}, \mathbf{K}_i\}$ $(i, j = 1, 2, 3)\}$ with commutation relations

$$[\mathbf{L}_{ij}, \mathbf{L}_{jk}] = +\mathbf{L}_{ik}\ ,\ [\mathbf{L}_{ij}, \mathbf{K}_k] = -\delta_{ik}\mathbf{K}_j + \delta_{jk}\mathbf{K}_i, [\mathbf{K}_i, \mathbf{K}_j] = 0. \tag{7.2}$$

The Lie algebra of $SO_0(1,3)$ is $\mathcal{L}(3) = l.s.\{\mathbf{L}_{ij}, \mathbf{L}_{0i}\}$ $(i, j = 1, 2, 3)\}$ with commutation relations

$$[\mathbf{L}_{ij}, \mathbf{L}_{jk}] = +\mathbf{L}_{ik}\ ,\ [\mathbf{L}_{ij}, \mathbf{L}_{0k}] = -\delta_{ik}\mathbf{L}_{0j} + \delta_{jk}\mathbf{L}_{0i}, [\mathbf{L}_{0i}, \mathbf{L}_{0j}] = +\mathbf{L}_{ij}. \tag{7.3}$$

To obtain our desired results we need to make use of the (Segal) Inönü-Wigner contraction [20, 21] of $so(1,3)$ into $\mathcal{G}(3)$. Consider the underlying vector space V of both $\mathcal{G}(3)$ and $\mathcal{L}(3) \simeq so(1,3)$. Let $(\phi_\lambda)_{\lambda \in \boldsymbol{R}^+}$ $(\phi_\lambda \in GL(V))$ be a continuous function such that ϕ_1 is the identity in $GL(V)$. Define a new bracket on V by

$$[x, y]_\lambda \ =\ \phi_\lambda^{-1}[\phi_\lambda(x), \phi_\lambda(y)]\ \ \forall\ \lambda\ \in\ (0, 1]\ ,\ \ \forall\ x,\ y\ \in\ V \tag{7.4}$$

where $[\ ,\ ]$ means bracket in $\mathcal{L}(3)$. Set $\mathcal{G}_\lambda(3) = (V, [\cdot,\cdot]_\lambda)$ and assume that $\mathcal{G}_1 = \mathcal{L}(3)$ so that ϕ_1 defines an isomorphism of $\mathcal{L}(3)$ onto its image.

For

$$\phi_\lambda(\mathbf{L}_{0i}) = \lambda\mathbf{L}_{0i}, \phi_\lambda(\mathbf{L}_{ij}) = \mathbf{L}_{ij} \tag{7.5}$$

then it is easy to see that

$$\lim_{\lambda \to 0}\ [x,\ y]_\lambda\ :=\ [x,\ y]_0 \tag{7.6}$$

exists for all $x,\ y\ \in\ V$ and it defines a new Lie algebra which is isomorphic to $\mathcal{G}(3)$. Thus $\mathcal{G}(3) \simeq (V,\ [\cdot,\cdot]_0)$, is a contraction of the Lie algebra $so(1,3)$. In physical terms, $\lambda = \frac{1}{c}$ where c is the speed of light.

Now let $V^4 \simeq SO_0(1,4)/SO_0(1,3)$ denote de Sitter space. We recall the weight two scalar representation [22] of $SO_0(2,4)$ and consider its realization on V^4 [23]. This representation remains irreducible when restricted to

$SO_0(1,4)$ [23] and we may decompose this representation into irreducible representations of $SO_0(1,4)$. It amounts to the well-known problem of the decomposition of the left regular representation of $SO_0(1,4)$ on V^4 into irreducibles [27]. It turns out that the representation decomposes into a continuous part and a discrete part with the continuous part of the spectrum being parameterized by a positive real number, ρ, with multiplicity two. Explicitly we have the following decomposition [24–28]:

$$\mathcal{L}^2(V^4) \simeq \int_0^\infty \oplus d\rho \left\{ \mathfrak{H}_{-\frac{3}{2}+i\rho} \oplus \mathfrak{H}_{-\frac{3}{2}+i\rho} \right\} \oplus \sum_{\ell=-1,0,1,2,\ ...} \mathfrak{H}_{-3-\ell} \tag{7.7}$$

where the $\mathfrak{H}_{-\frac{3}{2}+i\rho}$ and $\mathfrak{H}_{-3-\ell}$ are irreducible representation spaces described in the just mentioned references.

We may use the Hannabuss decomposition to introduce horospherical coordinates associated with V^4. They consist of the ordered pairs $(\lambda = e^{-\tau}, \vec{x})$ where τ represents time and $\vec{x} \in \mathbb{R}^3$ represents a point in space. We can transfer the $SO_0(1,4)$ irreducible representations of the continuous spectrum via the light cone in 5 dimensional Minkowski space to actions on functions on $\mathbb{R}^3$ (conformal action) and we obtain for the action of the generators of $SO_0(1,4)_R$ on $\mathfrak{H}_{-\frac{3}{2}+i\rho}$ in this realization the following [29, 30]:

$$d\pi_\rho(\mathbf{L}_{40}) = (S + d)\,, \tag{7.8}$$

$$d\pi_\rho(\mathbf{L}_{4i}) = R\left(-D\frac{\partial}{\partial x_i} + \frac{1}{2R^2}x_i\,(S+d)\right)\,, \tag{7.9}$$

$$d\pi_\rho(\mathbf{L}_{0i}) = R\left(F\frac{\partial}{\partial x_i} + \frac{1}{2R^2}x_i\,(S+d)\right), \tag{7.10}$$

$$d\pi_\rho(\mathbf{L}_{ij}) = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i} \tag{7.11}$$

where $S = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$, $F = 1 - \frac{x^2}{4R^2}$, $D = 1 + \frac{x^2}{4R^2}$, $x^2 = x_1^2 + x_2^2 + x_3^2$, and $d = -\frac{3}{2R} + i\rho$. (By $SO_0(1,4)_R$ we mean the physical de Sitter group acting on the de Sitter hyperboloid with the radius R of the de Sitter space not set equal to 1.) We leave it to the reader to show that these formulae indeed define a representation of $SO_0(1,4)_R$.

Now $d\pi_\rho$ restricts to a representation of $SO_0(1,3)$ and we can contract this representation of the Lorentz group to a representation of $G(3)$. We have [29, 31]:

$$\begin{array}{ccc} & c \longrightarrow \infty & \\ SO_0(1,3) & \longrightarrow & G(3) \\ \frac{1}{c}d\pi_\rho(\mathbf{L}_{0i}) & \longrightarrow & G_i = m x_i \\ d\pi_\rho(\mathbf{L}_{ij}) & \longrightarrow & L_{ij} = d\pi_m(\mathbf{L}_{ij}) \end{array}$$

with m such that $\lim_{c\to\infty} \frac{\rho}{c} = mR$ and $d\pi_m$ denotes the contracted representation. The G_i are three commuting position operators so we can use the following formula (Gell-Mann formula) [1] to get back to $SO_0(1,3)$:

$$L_{0i} = \frac{i\Lambda}{2Y}\left[d\pi_m(\Delta), G_i\right] + G_i \tag{7.12}$$

where $\Delta = \frac{1}{2}\sum_{i,j=1}^{3} L_{ij}L_{ij}$ and $Y^2 = \Sigma_{i=1}^{3} G_i^2$ and where Λ is the deformation parameter (c.f. the $SO_0(1,4)$ case in [1]). With this formula as are starting point we can carry out an analysis for $G(3)$ and $SO_0(1,3)$ in a way analogous to the way we did it for the Poincaré and de Sitter groups in [1] and [2]. It enables us to relate at least some of the representation theory of $G(3)$ to that of $SO_0(1,3)$ just as we did for the Poincaré and de Sitter groups.

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