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A Generalization of Calabi-Yau Fourfolds Arising from M-Theory Compactifications

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Abstract. Using a reconstruction theorem, we prove that the supersymmetry conditions for a certain class of flux backgrounds are equivalent with a tractable subsystem of relations on differential forms which encodes the full set of contraints arising fom Fierz identities and from the differential and algebraic conditions on the internal part of the supersymmetry generators. The result makes use of the formulation of such problems through Kähler-Atiyah bundles, which we developed in previous work. Applying this to the most general $\mathcal{N} = 2$ flux compactifications of 11-dimensional supergravity on 8-manifolds, we can extract the conditions constraining such backgrounds and give an overview of the resulting geometry, which generalizes that of Calabi-Yau fourfolds.

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1 Introduction

The geometry of the most general $\mathcal{N} = 2$ compactifications of M-theory on eight-manifolds is of interest, for example, for resolving certain issues in F-theory. The analysis of this class of backgrounds was initiated in our previous papers using the methods of geometric algebra, which prove to be extremely useful computationally. We give a brief announcement of new results regarding this class of compactifications.

After recally some basic facts from [1–3] for the case $\mathcal{N} = 1$ and from [4, 5] for the case $\mathcal{N} = 2$, we give a reconstruction theorem for eight-domensional Majorana spinors within the frame of geometric algebra, emphasizing its use for encoding the full set of contraints arising fom Fierz identities and from the differential and algebraic conditions on the internal part of the supersymmetry generators into constraints on differential forms, for any amount of supersymmetry preserved by the effective action. We exemplify this for $\mathcal{N} = 1$ and $\mathcal{N} = 2$. We also briefly announce some results regarding the geometry of such backgrounds. Our notations and conventions are explained in [1].

2 Introduction to Compactifications of M-Theory to AdS$_3$

Consider M-theory warped compactifications down to AdS_3 (or Minkowski) spaces with cosmological constant $\Lambda = -8\kappa^2$, where $\kappa \geq 0$ (with $\kappa = 0$ giving the Minkowski limit). The 11-dimensional background is of the form $\tilde{M} = N \times M$, where the external space N is an oriented 3-manifold diffeomorphic with $\mathbb{R}^3$ and carrying the AdS_3 metric, while the internal space M is an oriented spinnable Riemannian 8-manifold with metric g. The metric on $\tilde{M}$, denoted by $\tilde{g}$, is a warped product:

$$\mathrm{d}\tilde{s}^2_{11} = e^{2\Delta}\mathrm{d}s^2_{11} \quad \text{where} \quad \mathrm{d}s^2_{11} = \mathrm{d}s^2_3 + g_{mn}\mathrm{d}x^m\mathrm{d}x^n \ ,$$

where the warp factor Δ is a smooth function defined on M and $\mathrm{d}s^2_3$ is the squared length element on N.

The supergravity action in 11 dimension involves the metric $\tilde{g}$, the 3-form potential $\tilde{C}$ and the gravitino $\tilde{\Psi}$:

$$S_{11} = \frac{1}{2}\int d^{11}y \left[\tilde{R}\star 1 - \frac{1}{2}\tilde{G}\wedge\star\tilde{G} - \frac{1}{6}\tilde{G}\wedge\tilde{G}\wedge\tilde{C}\right] + \text{terms involving } \tilde{\Psi} \ .$$

For the field strength $\tilde{G}$, the following compactification ansatz is used:

$$\tilde{G} = e^{3\Delta}G \quad \text{with} \quad G = \mathrm{vol}_3 \wedge f + F \ ,$$

where vol_3 is the volume form of N, while the 1-form f and the 4-form F are the fluxes on M.

The supersymmetric flux backgrounts of interest arise when the gravitino and its supersymmetry variation vanish:

$$\delta_{\tilde{\eta}}\tilde{\Psi}_A = \tilde{D}_A\tilde{\eta} = 0 \ , \quad A = 0, ..., 10 \ , \tag{1}$$

where $\tilde{D}_A$ is the supercovariant connection [1,2]. For the supersymmetry generator $\tilde{\eta}$, which is a Majorana spinor, one uses the ansatz:

$$\tilde{\eta} = e^{\frac{\Delta}{2}}\eta \quad \text{with} \quad \eta = \varphi \otimes \xi \ ,$$

where ξ and φ are Majorana spinors on M and N respectively.

Assuming that φ is a Killing spinor on the AdS_3 space, the supersymmetry condition (1) decomposes into a set of constraints for the internal supersymmetry generator ξ which we call the constrained generalized Killing (CGK) spinor equations:

$$D_m\xi = Q\xi = 0 \ . \tag{2}$$

Here, D_m is a certain linear connection on S and $Q \in \Gamma(M, \mathrm{End}(S))$ is a globally-defined endomorphism of the vector bundle S. As in [2, 3], *we do not require that ξ has definite chirality* (i.e. ξ need *not* satisfy the Weyl condition). The expressions for D_m and Q depend on the metric and fluxes and can be found explicitely in the literature, particularly in [1].

The space of solutions of (2) is a finite-dimensional $\mathbb{R}$-linear subspace $\mathcal{K}(D, Q)$ of the space $\Gamma(M, S)$ of smooth sections of S. Of course, this subspace is trivial for generic metrics g and fluxes F and f on M, since the generic compactification of the type we consider breaks all supersymmetry. The interesting problem is to find those metrics and fluxes on M for which some fixed amount of supersymmetry is preserved in three dimensions, i.e. for which the space $\mathcal{K}(D, Q)$ has some given non-vanishing dimension, which we denote by s. The case $s = 1$ (which corresponds to $\mathcal{N} = 1$ supersymmetry in three dimensions) was studied in [2, 3] is recalled below in the context of the the reconstruction theorem. An important observation made in [1, 4] is that the pairing $\mathscr{B}$ on S is D_m-flat:

$$\partial_m \mathscr{B}(\xi, \xi') = \mathscr{B}(D_m\xi, \xi') + \mathscr{B}(\xi, D_m\xi') \ , \ \forall \xi, \xi' \in \Gamma(M, S) \ , \tag{3}$$

so that D_m is an $O(16)$-connection. It can be proved that requiring our background to preserve s independent supersymmetry generators (i.e. requiring that $\dim \mathcal{K}(D, Q) = s$) is *equivalent* to requiring that the system (2) admits s solutions $\xi_1, \ldots, \xi_s$ which are $\mathscr{B}$-orthonormal at every point of M.

What we want to do is to solve the system of differential and algebraic constraints resulting from (2) for $\mathcal{N}$=2 (or higher), together with all the Fierz identities involved, with the purpose of finding all the information about the geometry of the background space. Before we do that, let us introduce a simplifying method to what we have previously used in [1, 4].

3 A Reconstruction Theorem for Sections of a Vector Bundle

Notations and conventions. Here we take M to denote an oriented, connected and paracompact smooth manifold of general dimension d, whose unital $\mathbb{R}$-algebra of smooth real-valued functions we denote by $\mathcal{C}^\infty(M, \mathbb{R})$. All vector bundles over M are assumed to be smooth.

Let S be a smooth vector bundle of rank r over M, endowed with a scalar product $\mathscr{B}$. We are interested in describing collections of smooth global sections $\xi_i \in \Gamma(M, S)$ (where $i = 1 \ldots s$) through certain smooth global sections $E_{ij} \stackrel{\mathrm{def.}}{=} E_{\xi_i, \xi_j} \in \Gamma(M, \mathrm{End}(S))$ of the bundle of endomorphisms of S, which are built as bilinear combinations of the sections ξ_i. More precisely, we consider the global endomorphisms of S whose action on smooth sections is given by:

$$E_{ij}(\xi) \stackrel{\text{def.}}{=} \mathscr{B}(\xi, \xi_j)\xi_i \ , \ \forall \xi \in \Gamma(M, S) \ , \ \forall i, j = 1 \dots s \tag{4}$$

and seek a set of conditions satisfied by these objects which allow us to reconstruct the sections ξ_i from knowledge of E_{ij}. Setting $\mathscr{B}_{ij} \stackrel{\text{def.}}{=} \mathscr{B}(\xi_i, \xi_j) \in \mathcal{C}^\infty(M, \mathbb{R})$, an easy computation shows that E_{ij} satisfy:

$$\mathrm{tr} E_{ij} = \mathscr{B}_{ij} \tag{5}$$

and that $\mathcal{O}_{ij} = E_{ij}$ is a particular solution of the system of equations:

$$\begin{aligned} &\mathcal{O}_{ij} \circ \mathcal{O}_{kl} = \mathrm{tr}(\mathcal{O}_{kj})\mathcal{O}_{il} \ , \ \forall i, j, k, l = 1 \dots s \\ &\mathcal{O}_{ij}^t = \mathcal{O}_{ji} \ , \ \forall i, j = 1 \dots s \ , \end{aligned} \tag{6}$$

where $(\ )^t$ denotes the transpose taken with respect to $\mathscr{B}$. The equations obtained from the first relations of (6) by setting $j = k = l = i$ give s decoupled systems of the same type satisfied by $\mathcal{O}_i \stackrel{\text{def.}}{=} \mathcal{O}_{ii}$:

$$\boxed{\begin{aligned} &\mathcal{O}_i^2 = \mathrm{tr}(\mathcal{O}_i)\mathcal{O}_i \\ &\mathcal{O}_i^t = \mathcal{O}_i \ . \end{aligned}} \tag{7}$$

A particular case of interest in the applications of this paper is when ξ_i are mutually orthonormal at every point of M, which amounts to setting $\mathscr{B}_{ij} = \delta_{ij} 1_M$ in (5). This leads us to consider the further conditions:

$$\mathrm{tr}(\mathcal{O}_i) = 1_M \ ,$$

the diagonal part of which has to be considered together with (7). The one can prove the following:

Proposition. Giving an everywhere orthonormal system of global smooth sections $\xi_i \in \Gamma(M, S)$ is equivalent to giving a system of global endomorphisms $\mathcal{O}_i \in \Gamma(M, \mathrm{End}(S))$ which satisfy the following conditions:

$$\boxed{\begin{aligned} &\mathcal{O}_i^2 = \mathcal{O}_i \ , \\ &\mathcal{O}_i^t = \mathcal{O}_i \ , \\ &\mathrm{tr}(\mathcal{O}_i) = 1_M \ , \\ &\mathrm{tr}(\mathcal{O}_i \circ \mathcal{O}_j) = 0_M \ \text{ for } \ i < j \ . \end{aligned}} \tag{8}$$

Furthermore, a solution $(\mathcal{O}_i)_{i=1\dots s}$ of this system determines the corresponding everywhere $\mathscr{B}$-orthonormal system of sections of S via the relations:

$$\mathcal{O}_i = E_{\xi_i, \xi_i} \ , \ \text{ where } \ E_{\xi_i, \xi_i}(\xi) \stackrel{\text{def.}}{=} \mathscr{B}(\xi, \xi_i)\xi_i \ ,$$

up to *independent* ambiguities of the form $\xi_i \to -\xi_i$.

In this approach, the third row equations in (8) impose the unit norm conditions $\mathscr{B}(\xi_i, \xi_i) = 1_M$ while the last row of equations impose orthogonality of these sections at all points of M.

Constrained flat sections. Let $Q \in \Gamma(M, \mathrm{End}(S))$ be a smooth global endomorphism of S and $D : \Gamma(M, S) \to \Omega^1(M) \otimes_{\mathcal{C}^\infty(M,\mathbb{R})} \Gamma(M, \mathrm{End}(S))$ be a connection on S which is compatible with $\mathscr{B}$ in the sense that $\mathscr{B}$ is D-flat.

By definition, a *Q-constrained and D-flat section* of S is a smooth global section $\xi \in \Gamma(M, S)$ which satisfies the CGK conditions (2). Equation (3) for D-flatness implies that the $\mathscr{B}$-pairing of any two D_m-flat spinors is constant on M and in particular that any two solutions ξ_1, ξ_2 of (2) have constant $\mathscr{B}$-pairing.

Since the parallel transport of D_m preserves $\mathscr{B}$ by virtue of (3), it immediately follows (see [1]) that any two solutions of (2) which are linearly independent at a point are linearly independent everywhere and can be replaced by two solutions of (2) which are $\mathscr{B}$-orthogonal everywhere and whose local values at any point span the same subspace of the fiber of S at that point as the two original solutions. This implies that, when the $\mathbb{R}$-vector space of solutions of (2) is non-vanishing, we can always find a basis of solutions which consists of sections of S that are everywhere $\mathscr{B}$-orthogonal. In [1], we showed that the global endomorphisms $E_{\xi,\xi'}$ defined through:

$$E_{\xi,\xi'}(\xi'') \stackrel{\text{def.}}{=} \mathscr{B}(\xi'', \xi')\xi \ \ , \ \ \forall \xi'' \in \Gamma(M, S)$$

satisfy the identities:

$$D^{\mathrm{ad}}_m(E_{\xi,\xi'}) = E_{D_m\xi,\xi'} + E_{\xi,D_m\xi'} \ \ , \ \ \forall \xi, \xi' \in \Gamma(M, S)$$

and:

$$Q \circ E_{\xi,\xi'} = E_{Q\xi,\xi'} \ \ , \ \ E_{\xi,\xi'} \circ Q^t = E_{\xi,Q\xi'} \ \ , \ \ \forall \xi, \xi' \in \Gamma(M, S) \ \ ,$$

where D^{ad} is the connection induced by D on the bundle $\mathrm{End}(S)$.

Proposition. Let $\xi \in \Gamma(M, S)$. Then the following statements are equivalent:

- $Q\xi = 0$
- $Q \circ E_{\xi,\xi} = 0$
- $E_{\xi,\xi} \circ Q^t = 0$

Proposition. Let $\xi \in \Gamma(M, S)$ such that ξ is nowhere vanishing. Then the following statements are equivalent:

- $D_m\xi = 0$
- $D^{\mathrm{ad}}_m(E_{\xi,\xi}) = 0$

Corollary. Let $\xi \in \Gamma(M, S)$ such that ξ is nowhere vanishing. Then ξ satisfies (2) iff. $E_{\xi,\xi}$ satisfies:

$$D_m^{\mathrm{ad}}(E_{\xi,\xi}) = Q \circ E_{\xi,\xi} = 0 \ \ .$$

Using previous results, this implies:

Theorem. Giving s solutions $\xi_1, \ldots, \xi_s$ of (2) which are $\mathscr{B}$-orthonormal everywhere is equivalent to giving s globally-defined endomorphisms $\mathcal{O}_1, \ldots, \mathcal{O}_s \in \Gamma(M, \mathrm{End}(S))$ which satisfy (8) as well as the conditions:

$$\boxed{D_m^{\mathrm{ad}}(\mathcal{O}_i) = Q \circ \mathcal{O}_i = 0 \ , \ \forall i = 1 \ldots s \ .} \tag{9}$$

Furthermore, a solution $(\mathcal{O}_i)_{i=1\ldots s}$ of (8) determines the corresponding everywhere $\mathscr{B}$-orthonormal system of sections of S via the conditions:

$$\mathcal{O}_i = E_{\xi_i,\xi_i} \ \ ,$$

up to *independent* ambiguities of the form:

$$\xi_i \to -\xi_i \ \ .$$

Since by the argument recalled above any system of independent solutions can be replaced (upon making linear combinations with coefficients from $\mathcal{C}^\infty(M, \mathbb{R})$) with a system of solutions which are $\mathscr{B}$-orthonormal everywhere, this result gives a complete characterization of nontrivial solutions to the problem (2).

4 Application of the Reconstruction Theorem to Spinors and the Translation to Forms

We shall present in what follows the implications of the reconstruction theorem to our approach through geometric algebra developed in [1, 4, 6], which uses an isomorphic realization of the Clifford bundle $\mathrm{Cl}(T^*M)$ of T^*M as the Kahler-Atiyah bundle $(\wedge T^*M, \diamond)$, where the so-called geometric product $\diamond : \wedge T^*M \times \wedge T^*M \to \wedge T^*M$ is an associative (but non-commutative) fiberwise composition which makes the exterior bundle into a bundle of unital associative algebras.

Let (M, g) be a pseudo-Riemannian manifold and S be a pin bundle on M, with underlying pin representation $\gamma : (\wedge T^*M, \diamond) \to (\mathrm{End}(S), \circ)$. Let $\mathscr{B}$ be an admissible bilinear form on S (see [6] for a detailed discussion), which we assume to be a fiberwise scalar product. We also assume that the signature (p, q) of g is such that we are in the normal case, i.e. the Schur algebra of γ equals the base field $\mathbb{R}$.

With the sub-bundle $\wedge^\gamma T^*M$ of $\wedge T^*M$ defined as in [1], the restriction of γ gives an isomorphism of bundles of algebras from $\wedge^\gamma T^*M$ to $\mathrm{End}(S)$, whose inverse we denote by $\gamma^{-1}:\mathrm{End}(S)\to\wedge^\gamma T^*M$ (we shall use the same notation for the induced map on sections). As in [1], we let $\check{T}\stackrel{\mathrm{def.}}{=}\gamma^{-1}(T)\in\Omega^\gamma(M)\stackrel{\mathrm{def.}}{=}\Gamma(M,\wedge^\gamma T^*M)$ denote the 'vertical dequantization' of any globally-defined endomorphism $T\in\Gamma(M,\mathrm{End}(S))$.

Consider a system of global endomorphisms $\mathcal{O}_{ij}\in\Gamma(M,\mathrm{End}(S))$ as before, where $i,j=1\ldots s$. Applying γ^{-1} and using the results of [1] we find that (6) is *equivalent* with the following system of equations for the inhomogeneous forms $\check{\mathcal{O}}_{ij}\stackrel{\mathrm{def.}}{=}\gamma^{-1}(\mathcal{O}_{ij})\in\Omega^\gamma(M)$:

$$\begin{aligned}&\check{\mathcal{O}}_{ij}\diamond\check{\mathcal{O}}_{kl}=\mathcal{S}(\check{\mathcal{O}}_{kj})\check{\mathcal{O}}_{il}~~,~~\forall i,j,k,l=1\ldots s\\&\tau_{\mathscr{B}}(\check{\mathcal{O}}_{ij})=\check{\mathcal{O}}_{ji}~~,~~\forall i,j=1\ldots s~~,\end{aligned}\tag{10}$$

while (7) are *equivalent* with the following decoupled systems for the 'diagonal' components $\check{\mathcal{O}}_i\stackrel{\mathrm{def.}}{=}\check{\mathcal{O}}_{ii}$:

$$\boxed{\begin{aligned}&\check{\mathcal{O}}_i\diamond\check{\mathcal{O}}_i=\mathcal{S}(\check{\mathcal{O}}_i)\check{\mathcal{O}}_i\\&\tau_{\mathscr{B}}(\check{\mathcal{O}}_i)=\check{\mathcal{O}}_i~~.\end{aligned}}~~,\tag{11}$$

Here, $\tau_{\mathscr{B}}$ is the *modified reversion* defined through:

$$\tau_{\mathscr{B}}\stackrel{\mathrm{def.}}{=}\tau\circ\pi^{\frac{1-\epsilon_{\mathscr{B}}}{2}}=\begin{cases}\tau~, & \text{if }\epsilon_{\mathscr{B}}=+1\\ \tau\circ\pi~, & \text{if }\epsilon_{\mathscr{B}}=-1\end{cases}~~,\tag{12}$$

with $\epsilon_{\mathscr{B}}$ discussed in loc. cit., while $\mathcal{S}:\Omega^\gamma(M)\to\mathcal{C}^\infty(M,\mathbb{R})$ is the trace on the reduced Kähler-Atiyah algebra $(\Omega^\gamma(M),\diamond)$ defined in [1],

$$\mathcal{S}(\omega)\stackrel{\mathrm{def.}}{=}\omega^{(0)}N_{p,q}\mathrm{rk}S~~,\tag{13}$$

where $\omega^{(0)}\in\mathcal{C}^\infty(M,\mathbb{R})$ denotes the rank zero component of ω and $N_{p,q}$ equals 1 or 2 according to whether the fiberwise representation γ is faithful or not. One has [1]:

$$\mathcal{S}(\check{T})=\mathrm{tr}(T)~~,~~\forall T\in\Gamma(M,\mathrm{End}(S))$$

The geometric product $\diamond$ defined in [1] can be viewed as a deformation of the wedge product parameterized by the metric g, which reduces to the latter in the limit $g\to\infty$.

For systems of everywhere orthonormal spinors, the previous results imply:

Proposition. Giving s smooth global sections $\xi_1,\ldots,\xi_s\in\Gamma(M,S)$ which are everywhere $\mathscr{B}$-orthonormal amounts to giving s globally-defined smooth forms

$\check{\mathcal{O}}_1, \ldots, \check{\mathcal{O}}_s \in \Omega^\gamma(M)$ specified up to independent signs (i.e. up to independent ambiguities of the form $\check{\mathcal{O}}_i \to -\check{\mathcal{O}}_i$) such that the following system of relations is satisfied, where $i, j = 1 \ldots s$:

$$\boxed{\begin{array}{l} \check{\mathcal{O}}_i \diamond \check{\mathcal{O}}_i = \check{\mathcal{O}}_i \ , \\ \tau_{\mathscr{B}}(\check{\mathcal{O}}_i) = \check{\mathcal{O}}_i \ , \\ \mathcal{S}(\check{\mathcal{O}}_i) = 1_M \ , \\ \mathcal{S}(\check{\mathcal{O}}_i \diamond \check{\mathcal{O}}_j) = 0_M \ \text{ for } \ i < j \ . \end{array}} \tag{14}$$

Furthermore, a solution of this system determines the corresponding sections ξ_i through the relations:

$$\check{\mathcal{O}}_i = \check{E}_{\xi_i,\xi_i}$$

up to *independent* ambiguities of the type:

$$\xi_i \to -\xi_i \ ,$$

where the inhomogeneous forms $\check{E}^{(k)}_{\xi_i,\xi_j}$ were introduced and discussed in [1, 4, 6]. To gain some understanding of the content of these conditions, consider the rank expansions:

$$\check{\mathcal{O}}_j = \sum_{k=0}^{d} \check{\mathcal{O}}_j^{(k)} \ \text{ with } \ \check{\mathcal{O}}_j^{(k)} \in \Omega^k(M) \ .$$

Using definition (12) of $\tau_{\mathscr{B}}$ shows that the second equations in (14) amount to the condition that only those ranks $k \in \{0, \ldots, d\}$ which satisfy the following condition need to be considered in these rank expansions (i.e., we have $E_j^{(k)} = 0$ for any k which does not satisfy this condition):

$$k(k - \epsilon_{\mathscr{B}}) \equiv_4 0 \tag{15}$$

i.e. $k \equiv_4 0, \epsilon_{\mathscr{B}}$. Using (13) shows that the third equations in (14) amount to:

$$\check{\mathcal{O}}_j^{(0)} = \frac{1}{N_{p,q}\mathrm{rk}S} \ . \tag{16}$$

Relation (16) suggests the rescaling $\check{\boldsymbol{\mathcal{O}}}^{(k)} \stackrel{\text{def.}}{=} (N_{p,q}\mathrm{rk}S)\check{\mathcal{O}}^{(k)} \in \Omega^k(M)$, so that:

$$\check{\mathcal{O}}_i = \frac{1}{N_{p,q}\mathrm{rk}S} \sum_{k=0}^{d} \check{\boldsymbol{\mathcal{O}}}_i^{(k)} \ . \tag{17}$$

Then (16) amounts to the condition $\check{\boldsymbol{\mathcal{O}}}_i^{(0)} = 1$ and hence the second and third rows of (14) are equivalent with the statement that $\check{E}_i$ have the following rank expansions:

$$\check{\mathcal{O}}_i = \frac{1}{N_{p,q}\mathrm{rk}S}\left[1_M + \sum_{\substack{k=1 \\ k(k-\epsilon_{\mathscr{B}})\equiv_4 0}}^{d} \check{\boldsymbol{\mathcal{O}}}^{(k)}\right] \quad . \tag{18}$$

Constrained generalized Killing spinors. Consider the situation in which S is endowed with a $\mathscr{B}$-compatible connection D_m and with a global endomorphism $Q \in \Gamma(M, \mathrm{End}(S))$. Then solutions of (2) will be called *constrained generalized Killing spinors* (CGKS). As shown in [1], the connection D_m induces a derivation $\check{D}_m$ of the reduced Kähler-Atiyah algebra $\Omega^\gamma(M)$ while Q induces an element $\check{Q} \stackrel{\text{def.}}{=} \gamma^{-1}(Q)$ of this algebra. Furthermore, we have $\check{D_m(T)} = \check{D}_m(\check{T})$ and $Q \check{\circ} T = \check{Q} \diamond \check{T}$ for all $T \in \Gamma(M, \mathrm{End}(S))$. Hence the results of previous subsection give the following theorem, which gives a precise mathematical encoding of the "method of bilinears" [7] in the situation at hand:

Theorem. Giving s globally-defined smooth spinors $\xi_1, \ldots, \xi_s$ which satisfy (2) and which are $\mathscr{B}$-orthonormal everywhere is equivalent to giving s globally-defined forms $\check{\mathcal{O}}_1, \ldots, \check{\mathcal{O}}_s \in \Omega^\gamma(M)$ which satisfy (14) as well as the conditions:

$$\boxed{\check{D}_m^{\mathrm{ad}}(\check{\mathcal{O}}_i) = \check{Q} \diamond \check{\mathcal{O}}_i = 0_M} \;\; , \;\; \forall i = 1 \ldots s \;\; . \tag{19}$$

Furthermore, a solution $(\check{\mathcal{O}}_i)_{i=1\ldots s}$ of (14) determines the corresponding everywhere $\mathscr{B}$-orthonormal system of sections of S via the conditions:

$$\check{\mathcal{O}}_i = \check{E}_{\xi_i, \xi_i} \;\; ,$$

up to *independent* ambiguities of the form:

$$\xi_i \to -\xi_i \;\; .$$

5 Application to Compactifications of M-Theory on 8-Manifolds

Since for these backgrounds $p - q \equiv_8 0$, we are in the normal simple case, so $\Omega^\gamma(M) = \Omega(M)$. For these values of p and q (namely $p = 8$ and $q = 0$), one has (up to rescalings by smooth nowhere vanishing real-valued functions defined on M) two admissible pairings $\mathscr{B}_\pm$ on S (see [6]), both of which are symmetric and have the types $\epsilon_{\mathscr{B}_\pm} = \pm 1$. Since any choice of admissible pairing leads to the same result, we choose to work with $\mathscr{B} \stackrel{\text{def.}}{=} \mathscr{B}_+$ without loss of generality. Then $\tau_{\mathscr{B}}$ coincides with the canonical reversion τ of the Kähler-Atiyah algebra of (M, g), which is defined through:

$$\tau(\omega) \stackrel{\text{def.}}{=} (-1)^{\frac{k(k-1)}{2}}\omega \;\; , \;\; \forall \omega \in \Omega^k(M) \;\; . \tag{20}$$

Upon rescaling by a smooth function, we can in fact take $\mathscr{B}$ to be a fiberwise scalar product on S and denote the corresponding norm through $|| \; ||$.

5.1 The case $s = 1$

This was studied in [2, 3] and more completely in [1] using geometric algebra techniques. Below we recall the main results as an illustration of our reconstruction theorem and as a preparation for the case $\mathcal{N} = 2$.

As stated in Section 2, the condition that S admits a nowhere-zero section which is everywhere of rank one is *equivalent* with the condition that (14) admits at least one solution $\check{\mathcal{O}} \in \Omega(M) = \Omega^\gamma(M)$. Since $N_{8,0} = 1$ and $\mathrm{rk}S = 16$, the general definition (13) becomes:

$$\mathcal{S}(\omega) = 16\omega^{(0)} \ , \ \forall \omega \in \Omega(M) \ .$$

The second relation of (14) means that the rank expansion of $\check{\mathcal{O}}$ contains only forms of ranks $k = 0, 1, 4, 5$ and 8. This case was studied in [2, 3] and more completely in our work [1] using Kähler-Atiyah algebra techniques. The non-quadratic relations in the reconstruction theorem applied to this case amount to the statement that $\check{\mathcal{O}} \in \Omega(M)$ expands as

$$\check{\mathcal{O}} = \frac{1}{16}(1 + V + \Phi + Z + b\nu) \ , \tag{21}$$

where $b \in \mathcal{C}^\infty(M, \mathbb{R})$, $V \in \Omega^1(M), \Phi \in \Omega^4(M)$, $Z \in \Omega^5(M)$ and ν is the canonical volume form of (M, g). In [1], we used the normalization $||\xi|| = \sqrt{2}$ rather than $||\xi|| = 1$ as we do in this paper. As a consequence, the quantities $a, b, \bar{K}, Y$ and $\bar{Z}$ used in that paper correspond to what we call a, b, V, Φ and Z in this paper up to a factor of 2. Defining $\varphi \stackrel{\text{def.}}{=} *Z \in \Omega^3(M)$, one finds the following relations which hold globally on M:

$$(1 \mp b)\Phi^\pm = \pm(V \wedge \varphi)^\pm \ . \tag{22}$$

as well as:

$$||V||^2 = 1 - b^2 \ , \ ||\varphi|| = \sqrt{7}(1 - b^2) \ , \ \iota_V \varphi = 0 \ . \tag{23}$$

One also finds the globally valid relation:

$$||V||^2 ||\iota_{x\wedge y}\varphi||^2 = ||x \wedge y \wedge V||^2 \ , \ \forall x, y \in \Omega^1(M) \ . \tag{24}$$

The reconstruction theorem tells us that imposing

$$\check{\mathcal{O}} \diamond \check{\mathcal{O}} = \check{\mathcal{O}} \tag{25}$$

guarantees that $\check{\mathcal{O}} = \check{E}_{\xi,\xi}$ for some globally-defined normalized spinor $\xi \in \Gamma(M, S)$ which is determined up to sign by this condition. This recovers the Fierz identities of Martelli-Sparks purely from Kähler-Atiyah algebra relations. The CGKS equations in Kähler-Atiyah form

$$\check{D}^{\mathrm{ad}}_m(\check{\mathcal{O}}) = \check{Q} \diamond \check{\mathcal{O}} = 0_M$$

recover the remaining results in [2, 3].

The geometry can be summarized by saying that M admits a singular (Stefan-Sussmann) foliation whose leaves of codimension one carry a longitudinal G_2 structure and whose leaves of codimension zero carry (positively or negatively oriented) $\mathrm{Spin}(7)$ structures. The codimension zero leaves are subsets on M along which one of the two Weyl components of ξ vanishes (which one of them vanishes determines the orientation of the corresponding $\mathrm{Spin}(7)$ structure). In general, the structure group of M does not globally reduce, which is why one needs the theory of Stefan-Sussmann foliations rather than regular foliation theory. The torsion coefficients of these longitudinal G-structures can be computed explicitly using a combination of geometric algebra methods with various properties of G_2 and $\mathrm{Spin}(7)$ structures given in [8, 9]. We can also extract the complete expressions for the background fluxes, which have not yet been given in the literature.

Let $\mathcal{D}_V \stackrel{\text{def.}}{=} \ker V = \{X \in \mathcal{X}(M) | V(X) = 0\} = \{X \in \mathcal{X}(M) | g(V^\sharp, X) = 0\} \subset \mathcal{X}(M)$ be the singular (Stefan-Sussmann) distribution defined by the orthogonal complement of the vector field $V^\sharp$. Its tangent fiber $\mathcal{D}_V(p)$ at a point $p \in M$ has rank seven when $V(p) \neq 0$ and rank eight when $V(p) = 0$, the last condition defining a closed subset of M which in turn is a disjoint union of subsets M_V^+ and M_V^- consisting of those points of M where one of the two Weyl components of ξ vanishes.

Conditions (23) and (24) mean that, on the G_2 locus $U_V \stackrel{\text{def.}}{=} M \setminus (M_V^+ \sqcup M_V^-)$, the 3-form $\frac{1}{||V||}\varphi$ is longitudinal to $\mathcal{D}_V$ and its restriction to $\mathcal{D}_V$ is the canonically normalized 3-form of a G_2 structure on $\mathcal{D}_V$, which is compatible with the metric given by the restriction of g to $\mathcal{D}_V$. On the locus $M_\pm$, the singular distribution $\mathcal{D}_V$ coincides with the restriction of the tangent space and one finds that the retstriction of the four-forms $\Phi^\pm$ satisfy the defining conditions of a $\mathrm{Spin}_\pm(7)$ structure. The Majorana spinor ξ is determined (up to sign) by the smooth function $b \in \mathcal{C}^\infty(M, [-1, 1])$ and by the following data:

(a) the Frobenius distribution $D_V|_{U_V}$ together with its G_2 structure

(b) the $\mathrm{Spin}(7)$ structures on the two topological bundles $TM|_{M_V^{(\pm)}}$ (whose orientations relate to that of TM as explained above).

Remark. A very particular case in which b is everywhere degenerate is the case $b = +1$ everywhere (the case $b = -1$ everywhere is very similar) i.e. $\xi \in \Gamma(M, S_+)$ (then $\mathrm{d}_p b = 0$ for all $p \in M$). In this case, V vanishes identically and $D_V = TM$ has constant rank equal to eight, thus all points of M are regular for D_V. Relations (23) give $\varphi = 0$ while (24) and (22) are trivially satisfied. Relations

$$||\Phi||^2 = 14 \ , \quad * \Phi = \Phi \ , \quad \Phi \bigtriangleup_2 \Phi = -14\Phi$$

reflect the fact that ξ (equivalently, Φ) defines a canonically-normalized $\mathrm{Spin}(7)$ structure on M which is compatible with the metric.

5.2 The case $s = 2$

As previously stated, here and in our previous work [4], the internal parts of our supersymmetry generators are not assumed to have definite chirality*, which is a surprisingly nontrivial generalization of what was previously studied in the literature [10], complicating computations quite drastically.

In this case we have two spinors in eight dimensions, and the results of Sections 2 and 3 show that giving two globally-defined spinors on M which are everywhere $\mathscr{B}$-orthonormal is *equivalent* to giving a solution $(\check{\mathcal{O}}_1, \check{\mathcal{O}}_2)$ of the following system of equations:

$$\check{\mathcal{O}}_1 \diamond \check{\mathcal{O}}_1 = \check{\mathcal{O}}_1 \ , \ \ \check{\mathcal{O}}_2 \diamond \check{\mathcal{O}}_2 = \check{\mathcal{O}}_2 \tag{26}$$
$$\tau(\check{\mathcal{O}}_1) = \check{\mathcal{O}}_1 \ , \ \ \tau(\check{\mathcal{O}}_2) = \check{\mathcal{O}}_2 \tag{27}$$
$$\mathcal{S}(\check{\mathcal{O}}_1) = \mathcal{S}(\check{\mathcal{O}}_2) = 1_M \tag{28}$$
$$\mathcal{S}(\check{\mathcal{O}}_1 \diamond \check{\mathcal{O}}_2) = 0_M \tag{29}$$

The first three rows form independent systems for $\check{\mathcal{O}}_1$ and $\check{\mathcal{O}}_2$ of the type studied before (they characterize the existence of everywhere $\mathscr{B}$-normalized global sections ξ_1, ξ_2 of S) while the last relation enforces $\mathscr{B}$-orthogonality everywhere of ξ_1 and ξ_2 (and, in particular, linear independence of these spinors everywhere). Using the results for the case $s = 1$, we therefore know that the first three rows of (26) are solved by:

$$\check{\mathcal{O}}_i = \frac{1}{16}(1 + V_i + \Phi_i + Z_i + b_i \nu) \ , \tag{30}$$

where $V_i \in \Omega^1(M), \Phi_i \in \Omega^4(M), Z_i \in \Omega^5(M)$ and $b_i \in \mathcal{C}^\infty(M, \mathbb{R})$, for $i = 1$ and $i = 2$.

Applying the reconstruction theorem to this case shows that the full system of Fierz relations (26) plus the GKS constraint (19) (differential and algebraic)

$$\check{D}_m^{\mathrm{ad}}(\check{\mathcal{O}}_i) = \check{Q} \diamond \check{\mathcal{O}}_i = 0_M \ , \ \ i = 1, 2$$

is equivalent with two copies of the system found in the $s = 1$ case plus a single quadratic constraint which couples these two systems (the 4th relation in (26)). This gives a drastic simplification of the system of relations that we extracted in previous work and provides a way to describe the geometry in terms of certain foliations. Namely, we find that, on the 'generic locus', M carries a *codimension three* foliation endowed with a five dimensional $\mathrm{SU}(2)$ structure in

*This was also attempted in [5] for the same compactifications.

the sense of Conti and Salamon [11]. Once again, one needs the theory of Stefan-Sussmann foliations, which also produces other loci beyond the generic locus, carrying lower codimensional foliations endowed with different longitudinal G-structures.

The compactification space M becomes a Calabi-Yau fourfold when $s = 2$ and $K(D,Q) \subset \Gamma(M, S_+)$ or $K(D,Q) \subset \Gamma(M, S_-)$, in which case $\kappa = 0$ and the CGK equations amount to the conditions that $f = d\Delta^{-3/2}$ and F a primitive $(2,2)$ form [10]. However, there is no reason to require that $K(D,Q)$ consists of Majorana-Weyl spinors! Therefore, our general solutions can be called "generalized Calabi-Yau fourfolds".

They are of interest for generalizing (and thus solving the problem of G-flux) in F-theory by considering an F-theory limit which should make sense when the leaves of our foliations admit T^2 fibrations.

6 Generalization and Conclusions

Using the reconstruction theorem, one can similarly characterize supersymmetric backgrounds of this type which preserve more than two supersymmetries, since the reconstruction theorem allows one to reduce the case of s CGK spinors to s copies of a single CGK spinor plus a set of $\frac{s(s-1)}{2}$ quadratic algebraic constraints, thus allowing a systematic study for $s > 1$. In general, one finds a description though Stefan-Sussmann foliations carrying various longitudinal G-structures.

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