Bulg. J. Phys. **41** (2014) 172–179

Higher-Order Singletons and Partially Massless Fields*

X. Bekaert[1], **M. Grigoriev**[2]

[1]Laboratoire de Mathématiques et Physique Théorique, Unité Mixte de Recherche 7350 du CNRS, Fédération de Recherche 2964 Denis Poisson Université François Rabelais, Parc de Grandmont, 37200 Tours, France
[2]Tamm Theory Department, Lebedev Physics Institute Leninsky prospect 53, 119991 Moscow, Russia

Received *27 August 2014*

Abstract. We review the following higher-spin holographic duality conjecture: the $O(N)$ model at an isotropic Lifshitz point of criticality order $\ell + 1$ should be dual to a higher-spin gravity theory whose spectrum contains a tower of partially massless symmetric tensor fields of all even spins and all odd depths between 1 and $2\ell - 1$. More precisely, the Gaussian fixed point corresponding to free higher-order singletons on the d-dimensional boundary should be described in the bulk by the Vasiliev equations based on the symmetry algebra of the polywave equation of order 2ℓ. Moreover, an elementary renormalization group analysis suggests that for $\ell = 2$ and $6 \leqslant d \leqslant 10$ both the free and the interacting isotropic Lifshitz point should be described by the same bulk theory but with distinct boundary conditions.

PACS codes: 11.25.Hf, 11.25.Tq, 64.60.ae

1 Introduction

The duality conjectured in 1998 by Maldacena, called "AdS/CFT correspondence" or "gauge/gravity duality", relates IIB superstring theory around the background geometry $AdS_5 \times S^5$ and maximally supersymmetric super Yang-Mills (SYM) theory on the conformal boundary of the five-dimensional anti de Sitter (AdS) spacetime but it was soon generalised to other superstring vacua [1]. Nowadays, this body of ideas has been considerably extended so that, by now, the "holographic duality" or "bulk/boundary correspondence" generally refers to a duality relating a gravity theory, not necessarily a string theory, around a *weakly-curved* background, to a *strongly-coupled* conformal field theory (CFT)

*Based on the talk given by X.B. at the international conference "Mathematics Days in Sofia" held in July 2014 at Sofia (Bulgaria) on the boundary, not necessarily a gauge theory, nor supersymmetric, nor even relativistic (the only constraint is that the CFT should possess an expansion in the inverse of the number N of fields, such that its large-N limit correspond to a tractable semiclassical limit on the gravity side) [2]. However, if one believes in the strong form of the duality then a natural question is: What is the bulk dual of a *free* CFT?

Free (or integrable) CFTs have an infinite number of hidden symmetries, thus by Noether theorem their spectrum contains an infinite tower of traceless conserved currents with unbounded spin (including the spin-two stress-energy tensor). Therefore the AdS/CFT dictionary suggests that free (or integrable) CFTs should be dual to higher-spin (HS) gravity theories [3, 4] whose spectrum contains an infinite tower of gauge fields with unbounded spin (including the spin-two graviton). One should stress that even if the boundary CFT is free, the n-point correlators of the bilinear boundary currents do not vanish, thus the n-point vertices of the bulk theory are also not vanishing. In other words, the bulk dual of a *free* CFT should be an *interacting* HS gravity theory.

Such ideas emerged progressively in a series of papers more than a decade ago [5–10]: the idea was born in the context of the Maldacena conjecture for AdS_5 [5, 6], and then pursued in any dimension, first at the level of kinematics [7] and later at a dynamical level. This lead to the holographic duality conjecture between bosonic HS gravity around a strongly-curved AdS_{d+1} background and the singlet sector of a d-dimensional theory of (a large number of) free conformal scalars in the vector representation of an internal symmetry group [8–10]. A further important step was the observation by Klebanov and Polyakov of the holographic degeneracy of four-dimensional bosonic HS gravity in the sense that both the Gaussian (free) and the Wilson-Fisher (strongly-coupled) fixed points of the three-dimensional $O(N)$-model should be dual to the same four-dimensional bulk theory, but with distinct boundary conditions [10]. This provided the first example of an exotic AdS/CFT correspondence between a *strongly-coupled* condensed-matter CFT and a *strongly-curved* non-stringy AdS theory. Since bosonic HS gravity exists in any dimension, the holographic degeneracy argument also applies to the pair of fixed points of the five-dimensional $O(N)$-model that should be dual to the same six-dimensional bulk theory [11–13].

The agreement between Vasiliev's four-dimensional HS gravity and the sector of bilinear operators formed out of free conformal scalars in three dimensions has been verified very early at the level of scalar cubic couplings in [14, 15] but, more recently, generic checks of the conjecture for AdS_4/CFT_3 at the cubic level [16] (see also [17, 18]) prompted a revived interest in the correspondence which has since been generalised in various directions (see [19] for a review).

The goal of the paper is to briefly review the generalisation proposed in [20] of the HS holographic duality, by spelling out the CFT side in details in Section 2 and by summarizing the main evidences in favor of the conjecture in Section 3.

2 CFT: the Isotropic Lifshitz Point

The isotropic Lifshitz[1] point is a special type of multicritical renormalization-group (RG) fixed point that was originally introduced in the tricritical case in 1975 [22] and has been experimentally observed in a ternary mixture of homopolymer and diblock copolymer for the first time in 1995 [23] (see also [24] for more details).

The fundamental fields of the $O(N)$ vector model are N real scalar fields that form an N-vector multiplet $\vec{\phi} = (\phi^1, \dots, \phi^N)$ of the internal symmetry group $O(N)$. Actually, complex scalar fields can also be considered, in which case the symmetry is enhanced to $U(N)$. The vector model in d dimensions at an isotropic Lifshitz point[2] of criticality order $\ell + 1$ is described by a Wilsonian action whose quadratic part starts at order 2ℓ in the derivative expansion. In practice, this means that the couplings in front of the ℓ first terms appearing in the derivative expansion of the quadratic piece (*i.e.* the mass term $\int \vec{\phi}^* \cdot \vec{\phi}\, d^dx$, the standard kinetic term $\int \vec{\phi}^* \cdot \Box\vec{\phi}\, d^dx$, *etc*, till the term $\int \vec{\phi}^* \cdot \Box^{\ell-1}\vec{\phi}\, d^dx$) are set to zero. Moreover, the couplings of the interacting piece also have to be fine-tuned in order to be at a fixed point. In practice, this means that $\ell + 1$ experimental parameters have to be fixed, so that one speaks of a (multi)critical fixed point of order $\ell + 1$.

The actual kinetic term is (at lowest order in the derivative expansion)

$$S_{\text{free}}[\vec{\phi}] = \frac{1}{2}\, c_4 \int \vec{\phi}^* \cdot \Box^\ell \vec{\phi}\; d^dx \tag{1}$$

and it contains the polywave operator $\Box^\ell$ of order 2ℓ. The standard critical fixed point corresponds to $\ell = 1$ and is of order 2. In the physical applications, the isotropic Lifshitz point is mostly considered in the tricritical case $\ell = 2$ because it is already very hard to approach it experimentally [23,24]. This case corresponds to the kinetic operator $\Box^2$ of fourth order.[3] In the general case, the engineering dimension of the scalar fields is $\Delta_{\vec{\phi}} = (d - 2\ell)/2$ and the quadratic action S_{free} is conformally invariant. An on-shell scalar field $\vec{\phi}$ with scaling dimension $(d - 2\ell)/2$ and solution of the polywave equation $\Box^\ell \phi = 0$ has been called (scalar) "singleton of order ℓ" in [20] (and "ℓ-lineton" in [30]) since for $\ell = 1$ it is the usual scalar singleton (also called "Rac" in [31]) subject to the d'Alembert equation. A singleton of order $\ell \leqslant d/2$ spans an irreducible (in general, indecomposable) representation (irrep) of the conformal algebra $\mathfrak{o}(d, 2)$ denoted by $\mathcal{D}(\frac{d-2}{2}, 0)$. The usual singleton ($\ell = 1$) saturates the unitarity bound

[1] Although the union of the adjectives "isotropic" and "Lifshitz" may sound like an oxymoron (since, usually, the celebrated "Lifshitz point" rather correspond to some anisotropic scale symmetry), we followed the standard RG terminology for these multicritical fixed points (see *e.g.* [21]).

[2] A seminal study of the large-N limit of Lifshitz points was provided in the paper [25] (see also [26] and references therein to recent works).

[3] Notice that this CFT was also considered in the context of the Weyl anomaly [27,28].

but higher-order singletons are not unitary (the polywave equation is higher-derivative). One should stress that unitarity is not a fatal issue for the physical applications in condensed matter (in fact one works in the Wick-rotated signature for statistical physics).

The $O(N)$-singlet bilinears are the analogues of the "single-trace operators" of adjoint models (like $\mathcal{N} = 4$ SYM) and are also referred as "single-trace" by a conventional abuse of terminology. The single-trace sector of vector models is much more simple than the one of adjoint models because the former is spanned only by the bilinears $(\partial \cdots \partial \vec{\phi}^*) \cdot (\partial \cdots \partial \vec{\phi})$. From the point of view of the conformal algebra $\mathfrak{o}(d,2)$, the bilinear operators belong to the tensor product $\mathcal{D}(\frac{d-2\ell}{2}, 0)^{\otimes 2}$ of two singleton representation. Due to the $O(N)$ symmetry, the allowed deformations of the Gaussian action (1) are $O(N)$-singlets, which are called "multi-trace operators" since they are products of single-trace ones. Double-trace deformations are the simplest ones and they can be seen as mediated by an auxilliary scalar field via the Hubbard-Stratonovich trick. A corollary of [32] is that the scaling dimensions (Δ_{free} and Δ_{int}) of the Hubbard-Stratonovich field (respectively at the undeformed fixed point and at the interacting fixed point driven by the double-trace deformation) are conjugate to each other ($\Delta_{\text{free}} + \Delta_{\text{int}} = d + \mathcal{O}(1/N)$) in the large-N limit.

The deformation with the lowest scaling dimension is the quartic interaction $|\vec{\phi}|^4$ which is precisely a double-trace deformation corresponding to a scalar Hubbard-Stratonovich field $\sigma \sim |\vec{\phi}|^2$. At the Gaussian fixed point, the scaling dimension of operators is equal to their engineering dimension. Therefore, the auxilliary field σ has scaling dimension equal to $\Delta_{\text{free}} = 2\Delta_{\vec{\phi}} = d - 2\ell$ at the Gaussian fixed point and can be unitary there only if $\Delta_{\text{free}} > (d-2)/2 \Leftrightarrow d > 4\ell - 2$. At the interacting fixed point corresponding to the sole deformation by $|\vec{\phi}|^4$, the field σ has scaling $\Delta_{\text{int}} = d - \Delta_{\text{free}} = 2\ell$ in the large-N limit, thus it can be (i) unitary only if $\Delta_{\text{int}} > (d-2)/2 \Leftrightarrow d < 4\ell + 2$ and (ii) relevant only if $d < 2\Delta_{\text{int}} = 4\ell$. In this sense, the "critical dimension" is $d = 4\ell$ (since mean-field theory is only efficient for $d > 4\ell$). The scalar Hubbard-Stratonovich field can be unitary at both fixed points (holographic degeneracy) only in the range $4\ell - 2 < d < 4\ell + 2$.

Let us now pursue this elementary RG analysis but restrict to the two physically relevant cases (ℓ equal to 1 or 2). The potentially relevant deformations take the form (at lowest order in the derivative expansion)

$$S_{\text{int}}[\vec{\phi}] = \frac{1}{2} \int \Big(u_4 \, |\vec{\phi}|^4 \, + \, u_6 \, |\vec{\phi}|^6 \Big) \, d^d x \tag{2}$$

For the critical fixed point ($\ell = 1$), the double-trace deformation $|\vec{\phi}|^4$ is the only relevant interaction for $d \geqslant 3$. More precisely, the triple-trace deformation $|\vec{\phi}|^6$ is marginal for $d = 3$ (see *e.g.* [29] for a related discussion). The variation of the RG behaviour with the dimension d is summarized for the critical case in the table 2 of [11], so one can now consider the tricritical isotropic Lifshitz point ($\ell = 2$). In that case, the double-trace deformation $|\vec{\phi}|^4$ is the only relevant deformation in the range $6 < d < 8$ but the triple-trace deformation $|\vec{\phi}|^6$ becomes relevant for $d < 6$. In the range $8 < d < 10$ the interacting fixed point is expected to become ultraviolet attractive (an example of asymptotic safety).

3 AdS: Partially Massless Fields

At kinematical level (*i.e.* at the level of two-point functions and in the large-N limit), the AdS_{d+1}/CFT_d correspondence can be seen, from a group-theoretical perspective, as a mere intertwiner of $\mathfrak{o}(d,2)$-representations [33, 34] where the latter algebra is either realised as isometries in the bulk or as conformal transformations on the boundary.

The maximal compact subalgebra $\mathfrak{o}(d,2)$ is $\mathfrak{o}(2) \oplus \mathfrak{o}(d)$ and corresponds to the time translations generated by the (conformal) Hamiltonian and to the $\mathfrak{o}(d)$-rotations, both acting in a natural way on the boundary $\cong S^1 \times S^{d-1}$. Accordingly, the simplest irreps of $\mathfrak{o}(d,2)$ denoted by $\mathcal{D}(\Delta, s)$ are caracterized by the energy (scaling dimension) $\Delta > 0$ and by the spin $s \in \frac{1}{2}\mathbb{N}$ of the ground state (conformal primary operator). In particular, a partially conserved traceless symmetric tensor field of rank $s \geqslant t \geqslant 1$ on the boundary generates the irrep $\mathcal{D}(d+s-t-1, s)$. By "partially conserved" (of depth t) one means that the conservation law is weaker (only the t-th divergence vanishes on-shell) [35]. The AdS counterpart of $\mathcal{D}(d+s-t-1, s)$ is a "partially massless" field, *i.e.* a symmetric tensor field on AdS_{d+1} with a fine-tuned mass such that it possesses exotic gauge symmetries containing t derivatives of the gauge parameter of rank $s-t$ [36]. The case $t = 1$ reproduces the standard AdS/CFT dictionary between boundary conserved currents and bulk gauge fields. The generalisation of the dictionary to higher-depths was proposed in [35]. For $s > 0$ and $t > 1$, the irreps $\mathcal{D}(d+s-t-1, s)$ of $\mathfrak{o}(d,2)$ are not unitary (the bound is saturated for $t = 1$) but their $\mathfrak{o}(d+1,1)$ counterparts are unitary. In other words, bosonic partially massless fields are unitary on de Sitter (dS) spacetimes but not on anti de Sitter spacetimes.

The following theorem ensures the validity, at kinematical level, of our conjecture [20]. The tensor product of two higher-order scalar singletons decomposes as the sum of $\mathfrak{o}(d,2)$-irreps describing the partially conserved-currents/massless-fields of all ranks $s \in \mathbb{N}$ and all *odd* depths t (= $2k-1$) ranging from 1 to $2\ell-1$:

$$\mathcal{D}\Big(\frac{d-2\ell}{2}, 0\Big) \otimes \mathcal{D}\Big(\frac{d-2\ell}{2}, 0\Big) = \bigoplus_{s=0}^{\infty} \bigoplus_{k=1}^{\ell} \mathcal{D}(d+s-2k, s)\,. \qquad (3)$$

It is a generalisation of the Flato-Fronsdal theorem [31, 37] to higher order $\ell > 1$ and depth $t > 1$. From the holographic perspective, the left-hand-side corresponds to the singlet sector of the $U(N)$ vector model (for the $O(N)$ it is restricted to the symmetric product [11]) while the right-hand-side can be interpreted either as the CFT spectrum of (composite) primary operators, or as the AdS spectrum of elementary fields. The latter coincides with the spectrum of fields arising from the linearised Vasiliev's equations based on the symmetry algebra of the free higher-order singleton [20]. In fact, the nonlinear equations proposed by Vasiliev for bosonic higher-spin gravity around $(A)dS_{d+1}$ [4] remain consistent when based on the symmetry algebra of the free singleton of any order ℓ (because it is a suitable quotient of the "off-shell" higher-spin algebra). This provides the first example of a full interacting theory involving partially massless fields.

4 AdS/CFT: Conjecture

Putting all these facts together support the following particular case of the conjecture in [20]: the large-N limit of the $O(N)$ vector model at a tricritical isotropic Lifshitz point should be dual to the bosonic higher-spin gravity with an infinite tower of partially massless fields of all even spins 0, 2, 4, ... and of depths 1 and 3. The RG analysis shows that in the range $6 < d < 10$, the situation is very similar to the usual case $\ell = 1$ in that the theory is holographically degenerate. More precisely, in the large-N limit the two fixed points are related by a Legendre transformation on the CFT side and by a change of boundary condition for the bulk scalar field. However, in the more physical range $2 < d < 6$ the situation is more complicated because triple-trace deformations become relevant and require more investigations. The discussion of multitrace deformations of vector theories and their bulk duals in [29] should help to adress this issue.

The dS/CFT version of this proposal is also of interest. Remember that, though partially massless fields are not unitary on AdS, their dS analogues are. Following the conjecture of [38], one might speculate that the Euclidean $Sp(N)$ vector model with anticommuting scalars at isotropic Lifshitz points might be dual to unitary bosonic HS theories of partially massless tensor fields around de Sitter spacetime.

Acknowledgments

X.B. is grateful to the organisers of the "Mathematics Days in Sofia" for their invitation to this enjoyable meeting and the opportunity to present his talk. The work of M.G. was supported by RFBR grant 13-01-00386.

References

[1] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz (2000) *Phys. Rept.* **323** 183; [hep-th/9905111].

[2] J. McGreevy (2010) *Adv. High Energy Phys*. **2010** 723105; [arXiv:0909.0518 [hep-th]].
[3] M.A. Vasiliev (1990) *Phys. Lett*. **B243** 378.
[4] M.A. Vasiliev (2003) *Phys. Lett*. **B567** 139;[hep-th/0304049].
[5] B. Sundborg (2001) *Nucl. Phys. Proc. Suppl*. **102** 113; [hep-th/0103247].
[6] E. Sezgin, P. Sundell (2001) *JHEP* **0109** 036; [hep-th/0105001].
[7] S.E. Konstein, M.A. Vasiliev, V.N. Zaikin (2000) *JHEP* **0012** 018; [hep-th/0010239].
[8] A. Mikhailov (2002) "Notes on higher spin symmetries," NSF-ITP-01-181, ITEP-TH-66-01 [hep-th/0201019].
[9] E. Sezgin, P. Sundell (2002) *Nucl. Phys. B* **644** (2002) 303; [*Erratum-ibid. B* **660** (2003) 403] [hep-th/0205131].
[10] I.R. Klebanov, A.M. Polyakov (2002) *Phys. Lett. B* **550** 213; [hep-th/0210114].
[11] X. Bekaert, E. Joung, J. Mourad (2012) *Fortsch. Phys.* **60** 882; [arXiv:1202.0543 [hep-th]].
[12] J. Maldacena, A. Zhiboedov (2013) *Class. Quant. Grav.* **30** 104003; [arXiv:1204.3882 [hep-th]].
[13] S. Giombi, I.R. Klebanov, B.R. Safdi (2014) *Phys. Rev. D* **89** 084004; [arXiv:1401.0825 [hep-th]].
[14] A.C. Petkou (2003) *JHEP* **0303** 049; [hep-th/0302063].
[15] E. Sezgin, P. Sundell (2005) *JHEP* **0507** 044; [hep-th/0305040].
[16] S. Giombi, X. Yin (2011) *JHEP* **1104** 086; [arXiv:1004.3736 [hep-th]].
[17] N. Colombo, P. Sundell, "Higher Spin Gravity Amplitudes From Zero-form Charges," arXiv:1208.3880 [hep-th].
[18] V. Didenko, E. Skvortsov (2013) *JHEP* **1304** 158; [arXiv:1210.7963 [hep-th]].
[19] S. Giombi, X. Yin (2013) *J. Phys. A* **46** 214003; [arXiv:1208.4036 [hep-th]].
[20] X. Bekaert, M. Grigoriev (2013) *Nucl. Phys. B* **876** 667.
[21] C. Bervillier (2004) *Phys. Lett. A* **331** 110; [hep-th/0405027].
[22] R.M. Hornreich, M. Luban, S. Shtrikman (1975) *Phys. Rev. Lett.* **35** 1678.
[23] F.S. Bates, W. Maurer, T.P. Lodge, M.F. Schulz, M.W. Matsen, K. Almdal, K. Mortensen (1995) *Phys. Rev. Lett.* **75** 4429.
[24] D. Schwahn, K. Mortensen, H. Frielinghaus, K. Almdal (1999) *Phys. Rev. Lett.* **82** 5056.
[25] R.M. Hornreich, M. Luban, S. Shtrikman (1975) *Phys. Lett. A* **55** 269.
[26] M. Shpot, Y. Pis'mak (2012) *Nucl. Phys. B* **862** 75; [arXiv:1202.2464 [hep-th]].
[27] R.J. Riegert (1984) *Phys. Lett. B* **134** 56.
[28] E.S. Fradkin, A.A. Tseytlin (1984) *Phys. Lett. B* **134** 187.
[29] S. Elitzur, A. Giveon, M. Porrati, E. Rabinovici (2007) *Nucl. Phys. Proc. Suppl.* **171** 231.
[30] C. Iazeolla, P. Sundell (2008) *JHEP* **0810** 022; [arXiv:0806.1942 [hep-th]].
[31] M. Flato, C. Fronsdal (1978) *Lett. Math. Phys.* **2** 421.
[32] S.S. Gubser, I.R. Klebanov (2003) *Nucl. Phys. B* **656** 23; [hep-th/0212138].
[33] V.K. Dobrev (1999) *Nucl. Phys. B* **553** 559; [arXiv:hep-th/9812194].

[34] N. Aizawa. V.K. Dobrev, “Intertwining Operator Realization of anti de Sitter Holography,” arXiv:1406.2129 [hep-th].

[35] L. Dolan, C.R. Nappi, E. Witten (2001) *JHEP* **0110** 016; [hep-th/0109096].

[36] S. Deser, A. Waldron (2001) *Nucl. Phys. B* **607** 577; [hep-th/0103198].

[37] M.A. Vasiliev (2004) *JHEP* **0412** 046; [hep-th/0404124].

[38] D. Anninos, T. Hartman, A. Strominger, “Higher spin realization of the dS/CFT correspondence,” arXiv:1108.5735 [hep-th].