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Many-Body Composite Bosons from the Viewpoint of Functional Renormalization *

Yuya Tanizaki[1,2]

[1]Department of Physics, The University of Tokyo
[2]Theoretical Research Division, Nishina Center, RIKEN

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Abstract. Fermionic functional renormalization group is applied to describe Bose–Einstein condensation of composite dimers in a two-component fermionic system with a short-range interaction. Since fermions are bounded in dimers, they do not affect long-range nature of the system. This requires us to invent a renormalization group procedure, which controls the dispersion relation of composite bosons without affecting that of fermionic excitations. In order to realize it without introducing auxiliary bosonic fields, we introduce a vertex infrared regulator so as to compute the critical temperature.

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1 Introduction

In this paper, we report our recent studies on applications of fermionic functional renormalization group (fermionic FRG, or f-FRG) to many-body dimers of two-component fermions [1, 2].

Wilson's renormalization group [3] now becomes an important method of field theories in order to connect physical phenomena at different length scales. FRG [4–6] significantly improves this technology by relating Green functions at different energy scales instead of effective couplings. This generalization enables us to take into account nonlocal behaviors of Green functions, which often appears in field theories at finite densities and finite temperatures.

Fermionic FRG provides an unbiased and systematic study of many-body fermionic systems, and it is complementary to the auxiliary field method [7–11]. In previous studies, one controls low-energy fermionic excitations by adding an infrared (IR) regulator as a fermion bilinear operator to the Hamiltonian, and

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computes fermionic Green functions using f-FRG. In this study, we consider a new type of regulator, called the vertex IR regulator, and show its usefulness to study low-energy bosonic excitations [1, 2]. We apply it to the two-component fermionic system with a contact attractive interaction, described by

$$S[\overline{\psi}, \psi] = \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x} \left[\overline{\psi} \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi + g \overline{\psi}_\uparrow \overline{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right], \tag{1}$$

with T the temperature, μ the chemical potential, m the fermion mass, and g the coupling constant. This system shows the BCS-BEC crossover [12, 13], and we consider the region with a small positive scattering length, i.e., a deep BEC regime.

2 FRG with a vertex IR regulator

We briefly describe the general formalism of FRG with the vertex IR regulator [1, 2]. We consider a field theory with a classical action $S[\phi]$ of a field ϕ, which contains a bare propagator and a four-point vertex:

$$S[\phi] = \frac{1}{2} \phi_{\alpha_1} G^{-1, \alpha_1 \alpha_2} \phi_{\alpha_2} + \frac{1}{4!} g^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \phi_{\alpha_1} \phi_{\alpha_2} \phi_{\alpha_3} \phi_{\alpha_4}, \tag{2}$$

where α_i denotes the label to be summed up, such as space-time coordinates, particle species, and internal degrees of freedom. We add a vertex IR regulating term δS_k, which depends on a parameter k smoothly, to this action:

$$\delta S_k[\phi] = \frac{1}{4!} g_k^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \phi_{\alpha_1} \phi_{\alpha_2} \phi_{\alpha_3} \phi_{\alpha_4}, \tag{3}$$

with the boundary conditions $g_{k=\infty} = -g$ and $g_{k=0} = 0$. Let us consider the system described by the action $S + \delta S_k$, and define the k-dependent one-particle-irreducible (1PI) effective action $\Gamma_k[\varphi]$ of this theory. The flow equation of $\Gamma_k[\varphi]$ is determined as [1, 2]

$$\begin{aligned} \partial_k \Gamma_k[\varphi] = \; & \frac{1}{4!} \partial_k g_k^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \Big(\varphi_{\alpha_1} \varphi_{\alpha_2} \varphi_{\alpha_3} \varphi_{\alpha_4} + 6 \varphi_{\alpha_1} \varphi_{\alpha_2} G_{k, \alpha_3 \alpha_4} \\ & + 3 G_{k, \alpha_1 \alpha_2} G_{k, \alpha_3 \alpha_4} + 4 \varphi_{\alpha_1} G_{k, \alpha_2 \beta_2} G_{k, \alpha_3 \beta_3} G_{k, \alpha_4 \beta_4} \Gamma_k^{(3), \beta_2 \beta_3 \beta_4} \\ & + G_{k, \alpha_1 \beta_1} G_{k, \alpha_2 \beta_2} G_{k, \alpha_3 \beta_3} G_{k, \alpha_4 \beta_4} \Gamma_k^{(4), \beta_1 \beta_2 \beta_3 \beta_4} \\ & + 3 G_{k, \alpha_1 \beta_1} G_{k, \alpha_2 \beta_2} G_{k, \alpha_3 \beta_3} G_{k, \alpha_4 \beta_4} G_{k, \gamma_1 \gamma_2} \Gamma_k^{(3), \beta_1 \beta_2 \gamma_1} \Gamma_k^{(3), \gamma_2 \beta_3 \beta_4} \Big), \end{aligned} \tag{4}$$

with $G_k[\varphi] = \left(\frac{\delta_L}{\delta \varphi} \frac{\delta_R}{\delta \varphi} \Gamma_k[\varphi] \right)^{-1}$ the field dependent propagator, and $\Gamma_k^{(n)}[\varphi]$ the n-th functional derivative of $\Gamma_k[\varphi]$. By taking the vertex expansion of the 1PI effective action $\Gamma_k[\varphi]$ in terms of fields φ in (4), we can obtain the flow equation

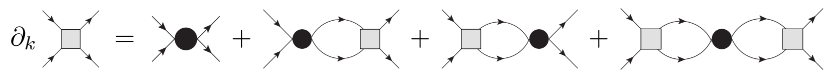

Figure 1. Flow equation of the four-point vertex function $\Gamma_k^{(4)}$ in the vacuum with the vertex IR regulator [1,2]. Blobs and boxes denote $\partial_k g_k$ and $\Gamma_k^{(4)}$, respectively.

for each 1PI vertex function. Since $\delta S_{k=0}[\phi] = 0$, $\Gamma_k[\varphi]$ converges to the usual 1PI effective action at $k = 0$. On the other hand, since $\delta S_{k=\infty}[\phi]$ is the negative of the interaction term in $S[\phi]$, the flow of FRG with the vertex IR regulator starts from the free theory.

3 Fermionic FRG for many-body composite bosons

3.1 Flow of the four-point vertex in the vacuum

Before applying the f-FRG formalism with a vertex IR regulator to BEC of composite bosons, we first consider its flow in the vacuum. In this limit, all the diagram with a fermion closed loop vanishes automatically, since there are no antiparticles in non-relativistic physics. Therefore, the fermionic self-energy correction $\Sigma_k(p)$ vanishes: $\Sigma_k \equiv 0$.

Let us consider the f-FRG flow of four-point vertex function. Its diagrammatic expression is greatly simplified in this limit and given in Fig.1. Since there are no explicit relative momentum dependence in the contact interaction, the solution depends only on the center-of-mass momenta p, which is denoted as $\Gamma_k^{(4)}(p)$. We define a p-dependent function $M(p)$ by

$$M(p) = \int^{\Lambda} \frac{\mathrm{d}^3 \boldsymbol{l}}{(2\pi)^3} \int \frac{\mathrm{d}l^0}{2\pi} \frac{1}{G^{-1}(\frac{p}{2}+l)G^{-1}(\frac{p}{2}-l)}, \tag{5}$$

where Λ is the UV cutoff of the spatial momentum. The flow equation in Fig. 1 takes the form

$$\partial_k \Gamma_k^{(4)}(p) = \partial_k g_k(p) \left[M(p)\Gamma_k^{(4)}(p) - 1 \right]^2 . \tag{6}$$

We can integrate the differential equation (6) with the initial conditions $\Gamma_{k=\infty}^{(4)}(p) = 0$ and $g_{k=\infty}(p) = -g$ so as to find that

$$\frac{1}{\Gamma_k^{(4)}(p)} = \frac{1}{g + g_k(p)} + M(p). \tag{7}$$

Now, we can choose the explicit form of the vertex IR regulator $g_k(p)$ so as to realize the dispersion relation given in Fig. 2. By setting $g_k = g^2 R_k/(1 - gR_k)$,

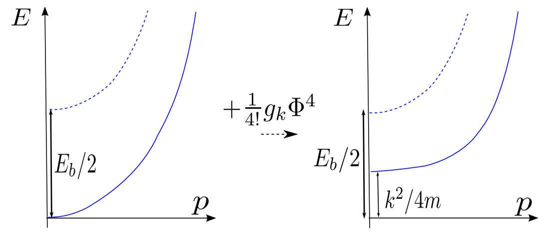

Figure 2. Dispersion relations of single-atom excitations and composite-boson excitations before and after introduction of the vertex IR regulator. The dashed line in each figure shows the dispersion relation of fermionic excitations, and they are gapped by half of the binding energy $E_b/2(= -\mu)$. The solid line shows the dispersion relation of composite-boson excitations. The vertex IR regulator only affects the composite-particle excitation significantly.

we get [1, 2]

$$\Gamma^{(4)}_k(p) = -\frac{(4\pi a_\mathrm{s}/m)}{\left(\sqrt{1+ma_\mathrm{s}^2\left(ip^0+\frac{\boldsymbol{p}^2}{4m}\right)}-1\right)+(4\pi a_\mathrm{s}/m)R_k(\boldsymbol{p})}. \tag{8}$$

In order to obtain this expression, we perform the coupling renormalization $1/g = m/4\pi a_\mathrm{s} - \Lambda/2\pi$ with the positive scattering length $a_\mathrm{s} > 0$, and take the continuum limit $\Lambda \to \infty$. In order to realize the vacuum condition, we put the chemical potential equal to the half of the binding energy: $\mu = -1/2ma_\mathrm{s}^2$.

Therefore, the fermionic four-point vertex function $\Gamma^{(4)}_k(p)$ represents the inverse propagator of composite bosons, and the IR regulating function $R_k(\boldsymbol{p})$ inside $g_k(p)$ can control low-energy bosons as we expected in Fig. 2.

3.2 Fermionic self-energy in the BEC limit through f-FRG

Let us consider the many-body effect of the fermionic self-energy to find the number density of particles in the BEC limit. In this section, we simply approximate $\Gamma^{(4)}_k$ by the one discussed in Sec.3.1.

Since we are interested in the low-density system, the flow equation for the self-energy can be simplified only by taking into account a single closed loop, and we get Fig.3. By substituting the solution $\Gamma^{(4)}_k(p)$ of the flow equation given in Fig.1 into the flow of self-energy in Fig.3, we obtain

$$\partial_k\Sigma_k(p) = \int_l^{(T)} \frac{\partial_k\Gamma^{(4)}_k(p+l)}{G^{-1}(l)-\Sigma_k(l)}. \tag{9}$$

As an approximation, let us neglect Σ_k on the r.h.s. of (9), then the solution of

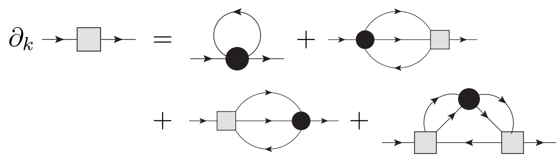

Figure 3. Approximate flow equation of the self-energy Σ_k for a low-density system with the vertex IR regulator [1, 2].

the flow equation (9) is given by

$$\begin{aligned}\Sigma_k(p) &= T\sum_{l^0}\int\frac{\mathrm{d}^3\boldsymbol{l}}{(2\pi)^3}\frac{\Gamma_k^{(4)}(p+l)}{G^{-1}(l)}\\ &\simeq \int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{(8\pi/m^2a_s)n_B\left(\boldsymbol{q}^2/4m+\widetilde{R}_k(\boldsymbol{q})\right)}{ip^0+\boldsymbol{q}^2/4m+\widetilde{R}_k(\boldsymbol{q})-(\boldsymbol{q}+\boldsymbol{p})^2/2m-1/2ma_s^2},\end{aligned} \tag{10}$$

with n_B the Bose-Einstein distribution function, $\mu = -1/2ma_s^2$, and $\widetilde{R}_k = (8\pi/m^2a_S)R_k$. In order to justify this approximation, we evaluate the magnitude of the self-energy:

$$|\Sigma_k(p)| \lesssim \frac{1}{2ma_s^2}\times(\sqrt{2mT}a_s)^3\times n_B(k^2/4m), \tag{11}$$

which is much smaller than that of the chemical potential $|\mu| = 1/(2ma_s^2)$ when $a_s \to 0^+$ as long as $n_B(k^2/4m) \sim 1$. Therefore, Σ_k on the r.h.s. of (9) is negligible for the most part of the flow in the deep BEC limit [1, 2].

In order to determine the ratio between the critical temperature T_c and the Fermi energy $\varepsilon_F(:= (3\pi^2n)^{2/3}/2m)$, we must calculate the number density n of fermions. In this formulation, it is determined as

$$n = T\sum_{p^0}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{-2e^{-ip^00^+}}{G^{-1}(p)-\Sigma_0(p)}. \tag{12}$$

When we expand the integrand in terms of $\Sigma_0(p)$, the main contribution of this integration comes from the term $G(p)\Sigma_0(p)G(p)$, which gives

$$n \simeq 2\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}n_B(\boldsymbol{q}^2/4m) = \frac{(2mT_c)^{3/2}}{\pi^2}\sqrt{\frac{\pi}{2}}\zeta(3/2). \tag{13}$$

We get the critical temperature $T_c/\varepsilon_F = 0.218$, which is nothing but the BEC transition temperature of free Bose gas, without introducing auxiliary bosonic fields [1, 2].

4 Summary

A new formalism of f-FRG is proposed by introducing a vertex IR regulator in the four-fermion interaction, and its exact evolution equation of the 1PI effective action is considered. This enables us to describe BEC of composite particles in a systematic way without using auxiliary fields [1, 2].

As a concluding remark, let us mentions that establishing the universal description of f-FRG for the whole region of the BCS-BEC crossover is very important [2] (see also Refs. [14, 15] for the study on the BCS region in this context). To deepen our understandings on the BCS-BEC crossover from FRG viewpoints, it is also important to relate these studies of fermionic FRG with that of FRG with partial bosonization techniques [16–21].

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