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Graphical Method for Computing an Infinite Family of Feynman Periods

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Abstract. The goal of the present study is to demonstrate the possibility to analyze and evaluate certain multiple integrals that give residues of Feynman amplitudes by means of graph-theoretical methods. The method is applied to calculate the "wheel with N spokes" family of "Feynman periods" in a massless scalar quantum field theory. All calculations are performed in position space. We show that this specific integration problem can be mapped to a problem of enumerating weighted paths on an associated graph with weighted edges and vertices. The problem can be further reduced to the known problem of enumerating 2-Motzkin paths with $N-2$ steps, which gives the Catalan number C_{N-1}. Relevant decompositions of Catalan numbers are also derived.

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1 Introduction

Ultraviolet (UV) renormalization procedure of Feynman amplitudes in position space can be formulated as a problem of extension of distributions. [1] The renormalization procedure is simplified and made explicit in a dilation invariant massless quantum field theory – see the works of Ussyukina, [2] Nikolov *et al.*, [3] and references therein. (A massive quantum field theory, on the other hand, is known to have the same leading singularities as its massless limit.)

A homogeneous function $G(\vec{x})$, $\vec{x} = (x_1, x_2, \dots x_n)$ with a possible singularity at the origin has a unique extension as a (Scwartz) distribution on $\mathbb{R}^n$ if the density $G(\vec{x})\mathrm{d}x_1\mathrm{d}x_2 \dots \mathrm{d}x_n$ has a positive degree of homogeneity. (The latter guarantees that it is a locally integrable around the origin.) Otherwise it is called (superficially) divergent. If the density is homogeneous of degree zero, the case we shall encounter in this paper, then $G(\vec{x})$ is logarithmically divergent. As Feynman amplitudes are in one to one correspondence with graphs, we shall also speak of divergent graphs. We say that a graph is primitively divergent if it (is divergent but) has no divergent subgraph. Given a norm $\rho(\vec{x})$ in $\mathbb{R}^n$, a primitively divergent amplitude $G(\vec{x})$ admits an analytic regularization of the type

$$G^\epsilon(\vec{x}) = [\rho(\vec{x})]^\epsilon \, G(\vec{x}), \ \epsilon > 0. \tag{1}$$

One proves [2, 3] that $G^\epsilon(\vec{x})$ has a simple pole at $\epsilon = 0$ and, moreover, there exists a number (a linear functional of $G(\vec{x})$) res G called the residue of $G(\vec{x})$ such that for a logarithmically divergent amplitude the limit

$$G^r(\vec{x}) = \lim_{\epsilon \to 0} \left[G^\epsilon(\vec{x}) - \frac{1}{\epsilon} \operatorname{res} G \; \delta(\vec{x}) \right] \tag{2}$$

is a well defined (associate homogeneous) distribution on $\mathbb{R}$, called renormalized amplitude. The residue res G is verified to be a period according to the definition of Kontsevich and Zagier [4].

"The wheel with N spokes" is an extension of the wheels with three and four spokes in massless scalar theories - see the paper of Schnetz [5] and references to earlier work cited there. The wheel family of Feynman periods has been first calculated by Broadhurst, [6] and the more general case has been treated as an application of Schnetz's theory of graphical functions. [5] In the present paper we also consider the wheel with N spokes family. The main contribution of this work is that it provides a direct derivation of the wheel formula using combinatorial graphical methods related to the so called Motzkin paths. This direct derivation could possibly be applied to the calculation of other families of periods.

The paper is organized as follows. In Section 2 we define the problem. The calculations employed in Section 3 allow us to make the correspondence between the problem of interest and the associated graph, which we define and analyze in Section 4. We summarize the results in Section 5.

2 Wheel with n spokes

We consider the wheel with N spokes (see Figure 1 for an example), which is a 4-dimensional $N\,(N \geq 3)$-loop primitively divergent Feynman graph. By

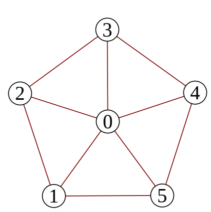


Figure 1. Graph of the wheel with five spokes.

choosing the origin of the coordinate system at the center of the wheel (the 0^{th} vertex), and labeling the remaining vertices by $1, 2, ...N$ we can write the propagator

$$G_N = \prod_{l=1}^{N} x_l^{-2} x_{l,l+1}^{-2},\ x_{N,N+1} = x_{N,1}. \tag{3}$$

For simplicity of presentation we have omitted the arguments of the functions in the latter and in the text bellow. Next, we parameterize G_N by the spherical coordinates of the N independent 4–vectors x_l:

$$x_l = r_l \omega_l,\ r_l \geq 0,\ \omega_l^2 = 1,\ l = 1, 2, \ldots N. \tag{4}$$

(Note that, we recover the complete 4–point graph of the φ^4 theory for $N = 3$.)

Following the works of Nikolov *et al.* [3, 7] and Todorov, [8] we shall sketch a calculation which uses a norm for the analytic regularization

$$G_N^{\varepsilon} = \left(\frac{R}{\ell}\right)^{\varepsilon} G_N,\ R = \max(r_1, r_2, \ldots r_N), \tag{5}$$

where ℓ is a scale parameter.

We compute the residue $\operatorname{res} G_N$ by first noticing that the symmetry of the integrand allows us to choose $R = r_N$ and multiply the result by N:

$$\operatorname{res} G_N = N \int \mathrm{d}^4 x_N \delta(r_N - 1) \int_{r_L \leq 1} \mathrm{d}^4 x_L \ldots \int_{r_1 \leq 1} \mathrm{d}^4 x_1 G_N. \tag{6}$$

Further, we rewrite $\operatorname{res} G_N$ as a sum of integrals over finite simplexes, obtained from the standard one

$$r_N (= 1) \geq r_L \geq \ldots r_1 (\geq 0),\ N = L + 1, \tag{7}$$

by a permutation of the subscripts $\sigma : \ (L, L-1, \ldots 1) \to (\sigma_L, \sigma_{L-1}, \ldots \sigma_1)$. We have

$$\operatorname{res} G_N = N\, \pi^{2N} \sum_{\sigma \in S_L} \mathcal{I}_\sigma, \tag{8}$$

where with S_L we denote the symmetric group and $\mathcal{I}_\sigma$ is the integral of

$$\tilde{G}_{\sigma;N} = \frac{1}{\pi^{2N}} \int_{\mathbb{S}^3} \mathrm{d}\omega_1 \ldots \int_{\mathbb{S}^3} \mathrm{d}\omega_N G_N \tag{9}$$

over the simplex $1 \geq r_{\sigma_L} \geq \cdots \geq r_{\sigma_1} \geq 0$.

It turns out that the angular integral in Eq. (9) is a polylogarithmic function (Li) whose argument depends on the sector of radial variables (see Appendix 5), and hence, Eq. (8) simplifies to

$$\mathrm{res}\, G_N = N(2\pi^2)^N \sum_{\sigma\in S_L} \int_0^1 \mathrm{d}r_{\sigma_L} \int_0^{r_{\sigma_L}} \mathrm{d}r_{\sigma_{L-1}} \cdots \int_0^{r_{\sigma_2}} \mathrm{d}r_{\sigma_1} P Li_{L-1}(Q_\sigma). \tag{10}$$

Here, with P and Q_σ we denote specific cases of the more generally defined functions

$$P_{\sigma;l} = \prod_{j=l}^{L} r_{\sigma_j}^{-1},\ P \equiv P_{\sigma;1} \tag{11}$$

$$\begin{aligned} Q_{\sigma;l,k_l} &= \left(\frac{r_{\sigma_l}}{r_{\sigma_{l-1}}}\right)^{2k_{l-1}} Q_{\sigma;l-1,k_{l-1}},\ 2 \le l \le L, \\ 1 \le &k_l \le L/2,\ \text{even}\, L\ [(L+1)/2\, \text{odd}\, L]\,, \\ Q_\sigma &\equiv Q_{\sigma;1,1} = \prod_{l=1}^{L+1} \frac{r_{l,l+1}}{R_{l,l+1}}, \\ R_{l,l+1} &= \max(r_l, r_{l+1}),\ r_{l,l+1} = \min(r_l, r_{l+1}), \\ R_{N,N+1} &= R_{N,1},\ r_{N,N+1} = r_{N,1}. \end{aligned} \tag{12}$$

The motivation for the introduction of $P_{\sigma;l}$ and $Q_{\sigma;l,k_l}$ becomes clear in the next section where we use them.

Following these preliminaries, we can now formulate the problem addressed in the rest part of the paper: Evaluate the integral

$$\mathcal{I} = \frac{\mathrm{res}\, G_N}{N\pi^{2N}} = \sum_{\sigma\in S_L} \mathcal{I}_\sigma. \tag{13}$$

As we shall see, this problem is related to the classification of the integrals $\mathcal{I}_\sigma$ in sets with respect to their value; More precisely, with respect to the coefficients $\mathcal{I}_\sigma/\mathcal{I}_I$, where I stands for trivial r–ordering. The classification depends on the permutation σ of the corresponding r–ordering as described in the next section. The latter problem is an interesting result on its own although it is complementary to our study.

3 Evaluation of $\mathcal{I}_\sigma$

Let us consider a permutation of L numbers and its corresponding r – ordering

$$(\sigma) = (\sigma_L, \sigma_{L-1}, \dots \sigma_1), \tag{14}$$

$$r_{\sigma_L} > r_{\sigma_{L-1}} > \cdots > r_{\sigma_1}. \tag{15}$$

A single, say the l^{th}, integration step from the evaluation of $\mathcal{I}_\sigma$ is given by

$$\mathcal{I}_{\sigma_l}[P_{\sigma;l}\, Li_{L+l-2}\left(Q_{\sigma;l,k_l}\right)] = \left(\frac{1}{2k_l}\right) P_{\sigma;l+1}\, Li_{L+l-1}\left(Q_{\sigma;l+1,k_{l+1}}\right), \quad (16)$$
$$1 \le l \le L.$$

This expression is obtained by changing the variable, followed by the application of formula $Li_{l+1}(z) = \int_0^z t^{-1} Li_l(t)\,\mathrm{d}t$.

By considering the iterated integral $\mathcal{I}_\sigma$ as a sequence of L successive applications of $\mathcal{I}_{\sigma_{(\cdot)}}$ on the initial integrand we can utilize the induction from Eq. (16) to find

$$\mathcal{I}_\sigma = 2^{L+1}\mathcal{I}_{\sigma_L}\mathcal{I}_{\sigma_{L-1}}\ldots\mathcal{I}_{\sigma_1}\left[PLi_{L-1}(Q_\sigma)\right] = \prod_{l=1}^{L} k_l^{-1}\, 2\zeta(2L-1), \quad (17)$$

where we have used $Li_{2L-1}(1) = \zeta(2L-1)$.

Equation (17) shows that for any given ordering (14) the integral $\mathcal{I}_\sigma$ is evaluated in terms of ζ–value. However, different orderings may lead to different coefficients that multiply 2ζ. The possible coefficients are determined by the products of the members of the allowed sets of k_l's. From the structure of the initial integrand $PLi_{L-1}(Q_\sigma)$ and the induction from Eq. (16) we conclude that the allowed sets of k_l's consist of all chains of positive integers that obey the conditions

$$k_1 = k_L = 1,\ |k_{l+1} - k_l| \le 1,\ 1 \le l \le L-1. \quad (18)$$

Specifically, $k_{l+1} - k_l > 1$ ($k_{l+1} - k_l < 1$) when $r_{\sigma_{l+1}}$ appears in the nominator (denominator) of $Q_{\sigma;l,k_l}$, and $k_{l+1} - k_l = 1$ when $Q_{\sigma;l,k_l}$ does not depend on $r_{\sigma_{l+1}}$ for the given ordering.

Let s $(\cup s = S_L)$ stands for the subset of permutations $\sigma \in S_L$ for which we have the same value of the coefficient in Eq. (17). With the use of Eq. (18) it is straightforward to work out the denominators of the possible coefficients

$$\begin{aligned} n_s &= \prod_{l=1}^{L} k_l = 1, 2, 4, 8, 12, 16, \ldots S, \\ S &= \begin{cases} l!^2, & L = 2l \quad (\text{even } L), \\ l!(l+1)!, & L = 2l+1 \quad (\text{odd } L). \end{cases} \end{aligned} \quad (19)$$

The set of k_l's contributing to the product in Eq. (19) can be any set corresponding to some $\sigma \in s$.

In particular, for the case of trivial ordering

$$\mathcal{I}_I = 2\zeta(2L-1),\ (I) = (1, 2, \ldots L),\ n_s = 1,\ |s| = 2^{L-1}, \quad (20)$$

where $|s|$ is the cardinality of s.

On invoking Eq. (17), Eq. (19) and Eq. (20) we can split the sum in Eq. (13) into classes s

$$\mathcal{I} = \sum_s \frac{1}{n_s}\,|s|\,\mathcal{I}_I. \tag{21}$$

Thus, the problem for evaluation of $\mathcal{I}$ reduces to finding the factor $\sum_s(|s|/n_s)$.

Consider an argument of the polylogarithm

$$Q_\sigma = \frac{\dots r_l^2 \dots r_{l_1}^2 \dots r_{l_{k_l-1}}^2 \dots}{\dots}, \\ \dots > r_l > \dots > r_{l_1} > \dots > r_{l_{k_l-1}} > \cdots, \quad 1 \le l \le L. \tag{22}$$

and let the argument of the polylogarithm under the integral sign at the l^{th} integration step is of the form

$$Q_{\sigma;l,k_l} = \frac{\dots r_l^{2k_l} \dots}{\dots}, \quad 1 \le l \le L. \tag{23}$$

If we assume that the symmetries up to the $(l-1)^{\text{th}}$ integration are already taken into account, there are k_l possibilities for r_l (l can accept k_l values) to have the same result after taking the l^{th} integral. Noting that the number of permutations connected with the fore-mentioned symmetries for a given integral $\mathcal{I}_\sigma$ in Eq. (17) equals n_s in (19), we conclude that the number of permutations $|s|$ producing the coefficient $1/n_s$ in Eq. (17) is divisible by n_s, i.e. that

$$N_L(n_s) = \frac{|s|}{n_s},\ N_L(n_s) \in \mathbb{N}. \tag{24}$$

Furthermore, the integral $\mathcal{I}_\sigma$ is also symmetric with respect to the change $r_{k_l} \leftrightarrow r_{k_{l+1}}$ when there are integration steps for which $k_{l+1} - k_l = 0$. Both symmetries imply to group the permutations producing the same sequences of k_l's in subclasses $\tilde{s}$ ($\cup\tilde{s} = s$). Therefore, the numbers $N_L(n_s)$, for a given s, can be thought to account for the sum of the numbers of permutations from the classes $\tilde{s}$, or

$$N_L(n_s) = \sum_{\sigma\in\tilde{s}} 2^{\sum_{l=2}^{L}(1-|k_l-k_{l-1}|)}, \tag{25}$$

where σ is representative of the class $\tilde{s}$.

It follows then, that if one is not interested in the values of the coefficients $1/n_s$ and their multiplicities $|s|$ but only on their ratio Eq. (25), the formula (21) may be simplified to

$$\mathcal{I} = \sum_s N_L(n_s)\,\mathcal{I}_I. \tag{26}$$

In particular, one may check and prove that

$$N_L(1) = 2^{L-1},\ N_L(2) = (L-2)2^{L-2},\ N_L(n_S) = \begin{cases} 2, & (\text{even}\, L), \\ 1, & (\text{odd}\, L). \end{cases} \quad (27)$$

With the aim of illustration, we shall prove only the formulas for $N_L(1)$ and $N_L(n_S)$. For each of these specific cases there is only one subclass $\tilde{s}$, since there is only one possible set of k_l's which product (19) gives $n_s = 1, n_S$ according to the rules (18).

We calculate $N_L(1)$ by noticing that in order to have $n_s = 1$, we must count only the permutations for which $k_l = 1\,\forall l$. (For such permutations the argument of the polylogarithm at the l^{th} integration step is $Q_{\sigma;l,1} = r_l^2$.) Then, $k_{l+1} - k_l = 0$ that justifies (see Eq. (25)) the respective formula in Eq. (27).

For even L, the value of $N_L(n_S)$ reflects the fact that there are $|s| = 2\,n_S$ permutations producing coefficient $1/n_S$. Indeed, let us inspect the argument of the polylogarithm

$$Q_\sigma = \left(\frac{r_{j_1} r_{j_3} \ldots r_{j_{L-1}}}{r_{j_2} r_{j_4} \ldots r_{j_{L-2}}} \right)^2 \quad (28)$$

for such a permutation σ. Taking into account the symmetry of the integral $\mathcal{I}_\sigma$ with respect to the permutations of indexes in the nominator and the permutations of indexes in the denominator of Q_σ, the $L/2$ possible permutations producing an argument of the type of Q_σ, and the single integration step for which $k_{L/2+1} - k_{L/2} = 0$ implying doubling of every permutation, we find that $|s| = 2\,(L/2)!^2 = 2\,n_S$. Similar but simpler argumentation holds for odd L.

4 Associated Graph

By applying the results of the analysis made in Section 3 we shall develop a simple system of rules for pictorial representation of the iterated integral $\mathcal{I}$. More precisely, we shall show how to draw a graph corresponding to the factor $\sum_s (|s|/n_s)$, and how to calculate this factor and the summands in it. Hereafter we will refer to such a graph as "associated".

4.1 Graph construction

We formulate the rules for drawing an associated graph for a given L as follows:

1. For even (odd) L draw an isosceles trapezoid (triangle) of vertices, so that there are L vertices placed on the first floor (the base of the figure-graph), there are $L-2$ vertices placed on the second floor $L-2$, etc. until floor $L/2$ [$(L+1)/2$]], where there are only two (one) vertices left. Attach a number k to all vertices of the k'th floor.

2. If k_l is the label of the vertex on the k_l^{th} row in the l^{th} column (counting from right), connect only the vertices satisfying the condition $|k_l - k_{l-1}| \leq 1$ with directed from right to left edges. (Note that there may be edges only between the vertices from neighboring columns.) Attach numbers to the edges – attach "1" to the diagonal edges obeying $|k_l - k_{l-1}| = 1$, and "2" to the horizontal edges for which $|k_l - k_{l-1}| = 0$.

We clarify the reasoning for such a design. The graph is constructed by putting floors of vertices on top of each other in the form of a trapezoid or a triangle and by attaching the number of the floor to all vertices from this floor. Every vertex in the l^{th} column (counting from right) and the k_l^{th} row corresponds to the l^{th} integration step and is labeled by the power k_l of r_l^2 in $Q_{\sigma;l,k_l}$, respectively. There are L consecutive integrations in $\mathcal{I}_\sigma$, hence L columns of vertices. The number of rows in a given column is determined by the fact, that the power k_{l-1} of r_{l-1}^2 may be kept, increased or decreased (not less than 0) by one when added to the power k_l of r_l^2 in $Q_{\sigma;l,k_l}$ as a result of the $(l-1)^{\text{th}}$ integration, see Eqs. (16), (18). Therefore, for even (odd) L the number of rows in the columns increases by one until the $(L/2)^{\text{th}}$ [$((L+1)/2)^{\text{th}}$] row and then decreases to one again with steps of one. This justifies the trapezoidal/trianglar positioning of the vertices.

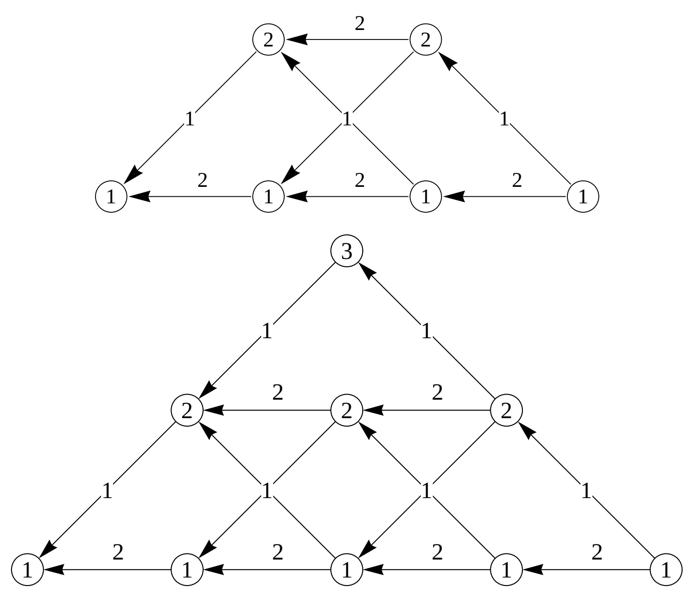


Figure 2. Associated graphs for wheel with five ($L = 4$) spokes (up), and wheel with six ($L = 5$) spokes (down).

Furthermore, we connect by arrows only the vertices from the neighboring columns, so the difference between their labels is in accord with the rules (18). We give the horizontal edges label "2" in order to account for the symmetry connected with the case of $k_{l+1} - k_l = 0$ (see the previous Section for details). Hence, by design, the sequence of integrations, which contributes to $N_L(n_s)$ by $|\tilde{s}|$ (see Eq. (25)), is modeled by a single path starting from bottom-right vertex, which follows the directed edges in a certain sequence of internal $L-2$ vertices (depending on the representative permutation $\sigma \in \tilde{s}$), and ending in the bottom left vertex. It should be noted, that the orientation is not necessary for the calculations, and it is introduced only to conveniently guide the eye when one tracks the paths. Summing over all paths one gets the factor $\sum_s(|s|/n_s)$, as explained in the next subsection.

Illustrative examples of associated graphs for even and odd value of L are shown in Figure 2.

4.2 Calculation on the Graph

Let us summarize the rules of calculation on an associated graph introduced in the previous subsection.

1. Split the paths into classes s distinguished by the number

$$n_s = \prod_{l=1}^{L} k_l. \tag{29}$$

   A path falls into class s if the product of the set of labels of the vertices through which the path goes $\{k_l\}$ equals n_s.

2. For every path in a given class s, find the product of the labels attached to all edges of this path. Then sum the contributions of these products from all paths $\mathcal{A}_s$ belonging to the class s, multiply by n_s, and find the multiplicity

$$|s| = n_s \sum_{\mathcal{A}_s} 2^{\sum_{l=2}^{L}(1-|k_l-k_{l-1}|)}. \tag{30}$$

3. Sum the ratios $|s|/n_s$ for all classes s to find

$$\frac{\mathcal{I}}{\mathcal{I}_I} = \sum_{s,\mathcal{A}_s} 2^{\sum_{l=2}^{L}(1-|k_l-k_{l-1}|)}. \tag{31}$$

In steps 1. and 2. we find the numbers n_s from Eq. (19) and the respective cardinalities $|s|$ of s. In step 3. we merely construct formula (21).

As we have already shown in Section 3 (and also have by construction in Eq. (30)) the multiplicity $|s|$ is proportional to n_s, and hence, n_s cancels in Eq. (31).

Therefore, we set the labels of all vertices to "1" (consequently $n_s = 1, \forall s$), without changing the result. Further, we notice that the sum in Eq. (31) can be considered as a weighted sum over the Motzkin paths [9] with $L-1$ steps $\mathcal{M}_{1,L-1}$. (Motzkin paths with $L-1$ steps are counted by the Motzkin numbers $M_L = 1, 2, 4, 9, 21, 51, 127, \ldots$, starting from $L = 2$. [10,11]) Since for a given Motzkin path, every horizontal edge contributes to the weight with which the path enters the sum with a factor of two, the said weight is equal to the number of the 2-Motzkin paths that arise if we assume that the horizontal edges of the given path can be two types. Thence, the problem reduces to the enumeration of 2-Motzkin paths $\mathcal{M}_{2,L}$ with $L-1$ steps, $\mu_{2,L-1} = |\mathcal{M}_{2,L}|$, which is given by the respective Catalan number $C_L = 2, 5, 14, 42, 132, 429, 1430\ldots$, starting from $L = 2$. [10] Accordingly, we have

$$\frac{\mathcal{I}}{\mathcal{I}_I} = \sum_{\mathcal{M}_{1,L}} 2^{\sum_{l=2}^{L}(1-|k_l-k_{l-1}|)} = \mu_{2,L-1} = C_L,\; C_L = \frac{1}{L+1}\binom{2L}{L}. \quad (32)$$

By combining Eq. (13) and Eq. (32) we eventually obtain the residue/period of the wheel with N spokes

$$\operatorname{res} G_N = 2\pi^{2N}\binom{2N-2}{N-1}\zeta(2N-3). \quad (33)$$

Moreover, in view of (32), we find an additional result. Namely, if we do not cancel n_s in the sum $\sum_s |s|/n_s$ this sum can be considered as a decomposition of the respective Catalan number related to different r-orderings of the multiple integrals $\mathcal{I}_\sigma$ and their symmetries.

Example. *Let us consider the case of the wheel with five ($L = 4$) spokes in Figure 1. The corresponding associated graph is plotted in Figure 2 (left frame). There are four paths, that we collect in three classes s with respect to n_s [Step 1., Eq. (29)] and find their multiplicities $|s|$ [Step 2., Eq. (30)], see Table 1.*

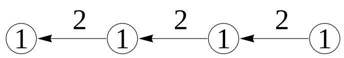
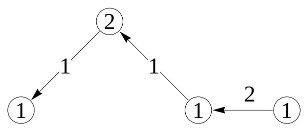
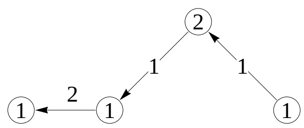
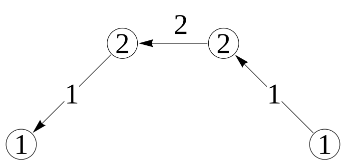
Summing over the three classes (Step 3.), we find

$$\frac{\mathcal{I}}{\mathcal{I}_I} = 1\times 8 + \frac{1}{2}\times 8 + \frac{1}{4}\times 8 = C_4. \quad (34)$$

Obviously, if we cancel n_s in Eq. (34), or equivalently set the labels of vertices to "1" in Table 1, we find $N_4(1) = 8,\ N_4(2) = 4,\ N_4(4) = 2$.

Calculation on the associated graph allows us to obtain recursive relations for decomposition of Catalan numbers, see Appendix 5. In Appendix 5 we present decompositions of Catalan numbers C_L for several values of L.

Table 1. Calculation on the associated graph for $L = 4$

Classes of Paths	Numbers n_s	Multiplicities $\|s\|$
	1	8
		
	2	8
	4	8

5 Summary

We calculated the "wheel with N spokes" family of Feynman periods. The integral defining the period is decomposed into a sum of integrals over all simplexes of radial variables labeled by the symmetric group. By classifying the integrals over simplexes in equivalence classes with respect to their value and using some symmetry considerations, we show that the integration problem can be mapped to a problem of enumerating weighted paths on an associated graph. Every path of the associated graph represents a symmetry class (with respect to changes of radial variables) of multiple integrals. The symmetry properties are taken into account by the labels (weights) of edges and vertices. The result of the calculation on the associated graph for the wheel with N spokes is the Catalan number C_{n-2}, which is obtained by further reducing the problem to the known problem of enumerating 2-Motzkin paths with $N - 2$ steps when the classification of integrals over simplexes is of no importance. Counting on the associated graph one finds decompositions of Catalan numbers related to the different orderings of r_l.

Further extension of this work would involve the calculation of more complex families of Feynman periods by generalizing the combinatorial approach of this paper.

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Appendix A: Integration over the Angular Variables

According to Eq. (8) and Eq. (9) the residue of the wheel graph with $N(= L+1)$ spokes is given by

$$\operatorname{res} G_N = N\pi^{2N} \sum_{\sigma \in S_L} \mathcal{I}_\sigma, \tag{A1}$$

where

$$\mathcal{I}_\sigma = \int_0^1 \mathrm{d}r_{\sigma_L} \int_0^{r_{\sigma_L}} \mathrm{d}r_{\sigma_{L-1}} \dots \int_0^{r_{\sigma_2}} \mathrm{d}r_{\sigma_1}\, \tilde{G}_{\sigma;N}, \tag{A2}$$

$$\tilde{G}_{\sigma;N} = \frac{1}{\pi^{2N}} \int_{\mathbb{S}^3} \mathrm{d}\omega_0 \dots \int_{\mathbb{S}^3} \mathrm{d}\omega_L G_N, \tag{A3}$$

$$G_N = \prod_{l=1}^{N} x_l^{-2} x_{l,l+1}^{-2},\ x_{N+1} = x_1. \tag{A4}$$

The integral over the spherical angles ω_i from Eq. (A3) can be computed in 4-dimensional space-time by substituting the expansion of $x_{l,l+1}^{-2}$ in Gegenbauer polynomials $C_k^1(\cdot)$

$$\begin{aligned} x_{l,l+1}^{-2} &= (r_l^2 + r_{l+1}^2 - 2r_l r_{l+1}\omega_l\omega_{l+1})^{-1} \\ &= \frac{1}{R_{l,l+1}^2} \sum_{k=0}^{\infty} \left(\frac{r_{l,l+1}}{R_{l,l+1}}\right)^k C_k^1(\omega_l\omega_{l+1}), \end{aligned} \tag{A5}$$

$$\begin{aligned} R_{l,l+1} &= \max(r_l, r_l+1),\ r_{l,l+1} = \min(r_l, r_{l+1}), \\ R_{N,N+1} &= R_{N,1},\ r_{N,N+1} = r_{N,1} \end{aligned} \tag{A6}$$

in the propagator G_N, and applying the integral formula

$$\int_{\mathbb{S}^3} \mathrm{d}\omega C_l^1(\omega_1\omega_2) C_k^1(\omega_2\omega) = \frac{|\mathbb{S}^3|}{l+1} \delta_{lk} C_l^1(\omega_1\omega_2), \tag{A7}$$

where $|\mathbb{S}^3| = 2\pi^2$ is the volume of the unit hypersphere in 4 dimensions. After some elementary calculations one ends up with a product of an inverse monomial of r_l's (P in Eq. (A10)), and a series which is summed up to the polylogarithmic function defined by

$$Li_k(\eta) = \sum_{n=1}^{\infty} \frac{\eta^n}{n^k}. \tag{A8}$$

Finally, putting $\tilde{G}_N$ from Eq. (A3) in the expression for $\mathcal{I}_\sigma$ in Eq. (A2), and substituting the latter in Eq. (A1), one obtains the following expression for the residue

$$\operatorname{res} G_N = N\left(2\pi^2\right)^N \sum_{\sigma\in S_L} \int_0^1 \mathrm{d}r_{\sigma_L} \int_0^{r_{\sigma_L}} \mathrm{d}r_{\sigma_{L-1}} \dots \int_0^{r_{\sigma_2}} \mathrm{d}r_{\sigma_1} P Li_{L-1}(Q_\sigma), \tag{A9}$$

where

$$P = \prod_{l=1}^{L} r_l^{-1}, Q_\sigma = \prod_{l=1}^{N} \frac{r_{l,l+1}}{R_{l,l+1}}. \tag{A10}$$

Appendix B: Recursive Relations for Decomposition of Catalan Numbers

Calculation on the associated graph allows us to obtain the following recursive relations:

$$\begin{aligned} C(L,L) =& C(L-1,L) + \left[\frac{1}{l!^2}\right] 2l!^2 \\ &+ \left[\frac{1}{l(l-1)}\right] l(l-1)\left(C(L-2,L-2) - C(L-3,L-2)\right), \\ &\text{even } L,\ L \geq 6,\ l = L/2, \end{aligned} \tag{B1}$$

$$\begin{aligned} C(L,L) =& C(L-1,L) + \left[\frac{1}{l!(l+1)!}\right] l!(l+1)! \\ &+ \left[\frac{1}{l}\right] 2l\left(C(L-1,L-1) - C(L-2,L-1)\right) \\ &\text{odd } L,\ L \geq 5,\ l = (L-1)/2, \end{aligned} \tag{B2}$$

$$C(3,3) = 2C(2,2) + \left[\frac{1}{2}\right] 2, \tag{B3}$$

$$C(4,4) = 2C(3,3) + \left[\frac{1}{2}\right] 2C(2,2) + \left[\frac{1}{4}\right] 8. \tag{B4}$$

Here $C(L,L)$ denotes the Catalan number C_L. We assume at each step of the calculation that we have found the preceding Catalan numbers written in the form

$$C(k-1,k-1) = f_1 + \sum_{l=2}^{k-1} f_l C(l,l), \quad 4 \le k \le L, \tag{B5}$$

(as in Eq. (B3) and Eq. (B4)) where f_l $(l = 2, 3, \dots L-1)$ are numerical coefficients. Then, we define the respective numbers $C(k, k-1)$ as follows:

$$C(k-1,k) = \sum_{l=3}^{k} f_{l-1} C(l,l), \quad 4 \le k \le L. \tag{B6}$$

The square brackets in Eqs. (B1), (B2), (B3), and Eq. (B4) stand to remind us not to cancel the fractions. (One can, of course, multiply them.)

The desired decomposition for C_L is obtained by substituting the decompositions for the preceding Catalan numbers $C(k,k)\,(k < L)$ from Eq. (B5) and $C(k-1,k)\,(k \le L)$ from Eq. (B6) in the respective expression Eq. (B1) (or (B2)), and rewriting the latter in terms of the fractions.

For example, consider the case $L = 5$. By the application of the definition from Eq. (B6), one finds

$$C(3,4) = 2C(3,3), \tag{B7}$$

$$C(4,5) = 2C(4,4) + \left[\frac{1}{2}\right] 2C(3,3). \tag{B8}$$

Next, one applies the expressions from Eq. (B7) and Eq. (B8) in Eq. (B2) for the case $L = 5$ and obtaines

$$C(5,5) = 2C(4,4) + \left[\frac{1}{2}\right] 2\left(2C(4,4) - 3C(3,3)\right) + \left[\frac{1}{12}\right] 12. \tag{B9}$$

Note that the above expression has to be used in the calculation for $C[5,6]$.

On substituting Eq. (B3) and Eq. (B4) in Eq. (B9), and taking into account that $C(2,2) = 2$, we obtain the decomposition of C_5:

$$C(5,5) = 1 \times 16 + \frac{1}{2} \times 24 + \frac{1}{4} \times 36 + \frac{1}{8} \times 32 + \frac{1}{12} \times 12. \tag{B10}$$

Appendix C: Decomposition of Catalan Numbers C_L for $L = 2, 3, 4, 5, 6$

Here we present decompositions of Catalan numbers C_L for several values of L:

$$C_2 = 1 \times 2,\ L = 2, \tag{C1}$$

$$C_3 = 1 \times 4 + \frac{1}{2} \times 2,\ L = 3, \tag{C2}$$

$$C_4 = 1 \times 8 + \frac{1}{2} \times 8 + \frac{1}{4} \times 8,\ L = 4, \tag{C3}$$

$$C_5 = 1 \times 16 + \frac{1}{2} \times 24 + \frac{1}{4} \times 36 + \frac{1}{8} \times 32 + \frac{1}{12} \times 12,\ L = 5, \tag{C4}$$

$$C_6 = 1 \times 32 + \frac{1}{2} \times 64 + \frac{1}{4} \times 120 + \frac{1}{8} \times 160 + \frac{1}{12} \times 48 + \frac{1}{16} \times 128 + \frac{1}{24} \times 96 + \frac{1}{36} \times 72,\ L = 6. \tag{C5}$$

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