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Hamiltonian Analysis of the Proposal by Chen et al., *Phys. Rev. Lett.* 100 (2008) 232002

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Abstract. A decomposition of the gluon field as a sum of "physical" and "pure gauge" fields was recently proposed in a series of articles [1–3]. The proposal is purely classical. Here we consider the quatization of the corresponding $U(1)$ model. It is shown that the quantization leads to a model which is non-equivalent to the initial one. The generalization to other gauge groups is easy and straightforward.

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1 Introduction

The aim of the present article is to investigate the quantization of the model suggested by Chen et al. [1–3]. These papers are focused on the definitions of physical quantities such as momentum and spin in Quantum Chromodynamics. The heart of the considerations therein is a decomposition of the gauge field into separate "physical" and "pure gauge" *fields*. The procedure leads to a change in the nucleon angular momentum distribution among quarks and gluons and eventually solves the nucleon spin puzzle, but also rises a lot of questions [4–6].

A characteristic feature of any gauge model is that we use auxiliary (unphysical) variables for its description. For example, the gauge fields (e.q. photons, gluons) have both physical and unphysical *components*. The important point is that the separation of the unphysical components can be achieved only by the usage of non-local operators. As a consequence, the quantization of the physical degrees of freedom alone is a very hard task which can be solved only in a few simple cases. The Dirac quatization in the whole phase space is used instead for any realistic model. Therefore the quantization of the model proposed by Chen et al. seems to be a very interesting task. Here we consider the proposed model as a Hamiltonian system with constraints [7]. For our purposes it is enough to investigate the simplest case of a pure $U(1)$ gauge field (free Electrodynamics), because the specificity of the treatment is the same both for Abelian and non-Abelian gauge fields. The interaction with matter does not change the considerations as well.

2 Decomposition of the *U*(1) Gauge Field

The basic idea in Refs. [1–3] is to represent the gauge field A_μ as a sum of two fields — "physical" field $\hat{A}_\mu$ and "pure gauge" field $\bar{A}_\mu$

$$A_\mu = \hat{A}_\mu + \bar{A}_\mu. \tag{1}$$

The conditions, which are proposed to distinguish the nature of these fields in the case of $U(1)$ local symmetry, are

$$\partial_i \hat{A}_i = 0, \tag{2}$$

$$\bar{F}_{\mu\nu} \equiv \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu = 0. \tag{3}$$

It is argued in Ref. [2] that the physical component $\hat{A}_\mu$ and the pure-gauge component $\bar{A}_\mu$ are derived quantities which are uniquely expressed in terms of the field variable A_μ as solutions of a system of differential equations (2,3) subjected to proper boundary conditions. This is a purely classical argument and cannot be used in the quantum case. Indeed, it does not follow automatically from the fact that an equation can be solved when the fields in it are real functions, it can be solved as well when the fields are operators. For example, it is well known [8] that the much simpler Lorenz condition $\partial_\mu A^\mu = 0$ cannot be imposed as an operator equation. Therefore, a quantization procedure in which one solves eqs. (2,3) as operator equations is an open question. Moreover, it is even possible that such procedure is not self consistent. On the other hand there are several standard methods for quantization of fields subjected to additional constraints: Dirac quantization, BRST quantization, Master Equation. In all these methods the field components are quantized independently and the constraints are satisfied only between physical states. Hereafter we shall follow the same approach. Our Hamiltonian analysis is written having in mind the Dirac quantization with functional integral, but all listed quantization methods can be applied to the considered problem and will give the same result.

Equation (1) effectively doubles the gauge potential, while eqs. (2,3) are used to decrease back the number of the degrees of freedom. In this sense, among eqs. (1–3) the really important one is eq. (1). This is the reason to start our analysis considering the consequences of eq. (1) itself, i.e., for a moment we treat $\hat{A}$ and $\bar{A}$ as independent fields.

The pure electromagnetic Lagrangian is

$$L = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu}, \tag{4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Substituting eq. (1) into eq. (4), we obtain

$$\mathcal{L} = -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - \frac{1}{2}\hat{F}_{\mu\nu}\bar{F}_{\mu\nu} \tag{5}$$

($\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$). The Lagrangian (5) possesses an enlarged gauge symmetry. It is not only $U(1)$ invariant but it is also invariant under the so called Stuckelberg gauge symmetry:

$$\begin{aligned} \delta\hat{A}_\mu &= C_\mu \\ \delta\bar{A}_\mu &= -C_\mu \end{aligned} \tag{6}$$

where C is an arbitrary real $4-$vector field. The existence of the gauge transformation (6) allows us to prove that the models with Lagrangians (5) and (4) are physically equivalent. The easiest way to show this equivalence is to perform the following change of variables:

$$A_\mu = \hat{A}_\mu + \bar{A}_\mu \tag{7}$$

$$B_\mu = \frac{1}{2}(\hat{A}_\mu - \bar{A}_\mu). \tag{8}$$

The Jacobian of this transformation is 1 and the Lagrangian (5) does not depend on the field B. Therefore, the continual integration over B field is trivial and the transition amplitude for the model with Lagrangian (5) coincides with the transition amplitude for the pure quantum electromagnetic field. A more canonical approach to the proof of the equivalence involves the usage of the following gauge fixing:

$$B_\mu = 0. \tag{9}$$

The Faddeev–Popov determinant which corresponds to constraints (6) and gauge conditions (9) is 1. Therefore, there are no ghosts, and the gauge conditions allow trivially to integrate over the field B.

The above considerations demonstrate that the Lagrangian (5) is just an uneconomic way to describe a well known model. Its potential advantage compare to the Lagrangian (4) is if, eventually, we find a gauge in which the fields $\hat{A}$ and $\bar{A}$ are the physical and pure gauge part of the electromagnetic field. In other words, the question is whether eqs. (2,3) can be used as gauge conditions for the Stuckelberg gauge transformation (6).

3 Hamiltonian Analysis

Let $\hat{p}_\mu$ and $\bar{p}_\mu$ are the momenta conjugate to the fields $\hat{A}_\mu$ and $\bar{A}_\mu$. From the Lagrangian (5) we obtain the following primary constraints:

$$\hat{p}_0 = 0 \tag{10}$$

$$\bar{p}_0 = 0 \tag{11}$$

$$\hat{p}_i - \bar{p}_i = 0 \tag{12}$$

and the following secondary constraints:

$$\partial_i \hat{p}_i = 0 \tag{13}$$

$$\partial_i \bar{p}_i = 0. \tag{14}$$

Constraints (10,11) indicate that the corresponding dynamically conjugated variables $\hat{A}_0$ and $\bar{A}_0$ are Lagrange multipliers. The really important constraints which generate the gauge transformations of the physical degrees of freedom are (12–14). However not all of these constraints are independent and so, we have to choose a proper subset of them. We use the freedom in this choice to make $\hat{A}_\mu$ the physical part of the electromagnetic field A_μ, i.e., to make this field $U(1)$ gauge invariant. This can be done by dropping out from constraints (12–14) the constraint (13). Altogether the independent constraints we choose are

$$\partial_i \bar{p}_i = 0 \tag{15}$$

$$\hat{p}_i - \bar{p}_i = 0 \tag{16}$$

Using eqs. (15,16), we can write a first order Lagrangian equivalent to the Lagrangian (5).

$$\mathcal{L}' = \partial_0 \hat{A}_i \hat{p}_i + \partial_0 \bar{A}_i \bar{p}_i - \frac{1}{4}\hat{p}_i^2 - \frac{1}{4}\bar{p}_i^2 - \frac{1}{2}H_i^2 + A_0 \partial_i \bar{p}_i + \Lambda_i(\hat{p}_i - \bar{p}_i). \tag{17}$$

Here, $\hat{A}_0$, $\bar{A}_0$, and Λ_i are Lagrange multipliers and, as usual, $H_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$. Certainly, the canonical Hamiltonian of the model is not uniquely determined: $\mathcal{H}' = \frac{1}{2}\hat{p}_i^2 - H_i^2$, $\mathcal{H}'' = \frac{1}{2}\bar{p}_i^2 - H_i^2$, and so on are equally good and are a matter of redefinition of Λ_i. Note that $\hat{A}_0$ and $\bar{A}_0$ participate in (17) only through their combination $A_0 = \hat{A}_0 + \bar{A}_0$. This is a consequence of the already discussed fact that not all constraints are independent and reflects the Stuckelberg symmetry of the Lagrange multipliers. Thus the combination of Lagrange multipliers $\frac{1}{2}(\hat{A}_0 + \bar{A}_0)$ decouple from all other fields and the functional integration over it is trivial.

In our system of independent constraints on the physical degrees of freedom eq. (15) is the generator of the $U(1)$ gauge transformation of the field $\bar{A}_i$, and eqs. (16) are the generators of the Stuckelberg symmetry. We want to preserve the $U(1)$ symmetry but rid off from the Stuckelberg one. Therefore, we have to apply a particular gauge fixing which gives us the opportunity to interpret the conditions (2,3) proposed in Refs. [2, 3] as gauge fixing conditions for the Stuckelberg symmetry. Consider first eqs. (3) ($\bar{F} = 0$). In general an antisymmetric rank 2 tensor (which transforms in $(0,1) \oplus (1,0)$ representation of the Lorentz group) has six independent components and same is the number of independent conditions in eqs. (3). But we have only three gauge generators and so, the closest to the conditions in eqs. (3) we can use is

$$\bar{F}_{jk} = 0 \ \ \forall i, j \tag{18}$$

or, equivalently

$$\bar{H}_i = 0. \tag{19}$$

(It is worth to be mentioned that eqs. (18) are the conditions used in the earliest work [1] but thrown away in later works in favor of eqs. (3).)

The great advantage of the gauge (19) is that it does not fix the $U(1)$ freedom of the $\bar{A}$ field because the corresponding Poisson brackets with the $U(1)$ constraint (15) are 0. However, eqs. (19) do not fix the Stuckelberg gauge freedom either — the determinant of the matrix of Poisson brackets between constraints (16) and gauge fixing conditions (19) is 0. A way out is to add a new gauge fixing condition and we have a candidate ready for it — eq. (2). It is easy to check that all third order minors of the rectangular matrix of Poisson brackets between constraints (16) and gauge fixing conditions (2, 19) are with non-zero (proportional to Δ) determinant and thus we indeed have a gauge fixing.

The problem is that the gauge conditions are more than the constraints and therefore one of them have to be dropped. It is not possible this to be eq. (2), so it must be one of the eqs. (19). But all of these conditions are independent which leads us to the conclusion that the model in which we simultaneously fulfill eqs. (2,18) is not equivalent to the free Quantum Electrodynamics. The situation is even worse when we try to use eqs. (3) (which involve both physical coordinates and Lagrange multipliers) as gauge fixing. In this case we have seven independent gauge condition and only four (three for the physical degrees of freedom and one for the Lagrange multipliers) gauge generators.

4 The Electromagnetic Field Momentum

Finally, some remarks on the definition of the conserved quantity momentum in QED which is in the heart of the discussion in Refs. [1, 2].

The bare kinetic operator corresponding to the Lagrangian (4) (after Fourier transformation) is

$$K = k^2\eta_{\mu\nu} - k_\mu k_\nu. \tag{20}$$

It is non-invertible, and therefore it is not possible to make any perturbative calculations in this theory. The cure of the problem is to add into the Lagrangian a term proportional to $\partial_\mu A^\mu$, e.g.

$$\delta L = -\frac{\alpha}{2}(\partial_\mu A^\mu)^2. \tag{21}$$

This is the so called α gauge term[1]. The kinetic operator now is $K = k^2\eta_{\mu\nu} - (1-\alpha)k_\mu k_\nu$, its inverse is well defined, and so is the perturbation theory. So, we have to use not the Lagrangian L from eq. (4) but $L + \delta L$ when we define

[1] The usage of other Lorentz invariant gauge fixing terms requires generalization of the Noether theorem to Lagrangians with higher derivatives.

the Noether currents in the quantized theory. The choice $\alpha = 1$ corresponds to the diagonal (Feynman - 't Hooht) gauge. In this gauge the components of the gauge potential and the corresponding asymptotic states are solutions of the massless Klein–Gordon equation. This feature together with the requirement the operator $\partial_\mu A^\mu$ between physical states to give 0 solve the energy positiveness problem for the second quantized electromagnetic field [8]. When $\alpha \neq 1$ the kinetic operator is non-diagonal but all its eigenvalues are proportional to k^2 — three of them are k^2 and one is αk^2 (the last eigenvalue corresponds to pure gauge degree of freedom). The important point here is that for any $\alpha \neq 0$ the asymptotic states admit free particle interpretation. However, the situation is completely different if there is no gauge fixing ($\alpha = 0$) in which case the energy positiveness cannot be proved.

The proposed in Refs. [1–3] representation of the electromagnetic potential as a sum of physical and pure gauge parts does not solve the problem with the kinetic operator kernel listed above. The kinetic operators both for $\hat{A}$ and $\bar{A}$ fields have zero modes and some well chosen terms have to be added into the Lagrangian to improve the situation. The only possible term involving $\hat{A}$ field is $\delta L' = f(\partial_i \hat{A}_i)$ where f is some function. Unfortunately, this term is not Lorentz invariant and plugging it into the Lagrangian results in a non-Lorentz invariant theory. This means no plane wave external states and break down of the explicit Lorentz invariance in all orders of the perturbation theory. In other words — a lot of problems. Everything must be done from the scratch. Probably, it is possible to be handled, as it is possible to deal with non-invariant regularizations, but the clever idea is to keep the manifest Lorentz invariance throughout the calculations and fix the coordinate system at the final (the Coulomb gauge is equivalent to the Lorenz one plus fixation of the coordinate frame). See Refs. [5, 6] for more arguments on this point.

5 Conclusions

The performed Hamiltonian analysis shows that the decomposition of the gauge field into physical and pure gauge parts proposed in Refs. [1–3] leads to huge enlargement of the gauge symmetry in the model. An important feature of the emerged Stuckelberg gauge symmetry is that it is an Abelian symmetry and it acts only on the gauge field components. Its generators are given by eqs. (16) and their form do not depend neither on the type of the gauge group of the model in consideration nor on the interaction with additional matter fields. Therefore, our analysis can be applied *en bloc* to any gauge system (with Abelian or non-Abelian gauge group, with or without matter) and the conclusion will be the same — eqs. (2,3) cannot be used as gauge fixing conditions. As a result, the model proposed in Refs. [1–3] cannot be quantized consistently.

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References

[1] X.S. Chen, X.F. Lü, W.M. Sun, F. Wang, and T. Goldman (2008) *Phys. Rev. Lett*. **100** 232002; arXiv:0806.3166.

[2] X.S. Chen, W.M. Sun, X.F. Lü, F. Wang, and T. Goldman (2009) *Phys. Rev. Lett*. **103** 062001; arXiv:0904.0321.

[3] X.-S. Chen, B.-C. Zhu, Physical decomposition of the gauge and gravitational fields, arXiv:1006.3926

[4] S. C. Tiwari, Comment on "Spin and Orbital Angular Momentum in Gauge Theories: Nucleon Spin Structure and Multipole Radiation Revisited", arXiv:0807.0699

[5] X. Ji, Comment on "Spin and Orbital Angular Momentum in Gauge Theories" by X. S. Chen et. al. (PRL100, 232002 (2008)) arXiv:0810.4913

[6] X. Ji, Comment on "Does Gluons Carry Half of the Nucleon Momentum?" by X. S. Chen et. al. (PRL103, 062001 (2009)) arXiv:0910.5022

[7] see, e.g., M. Henneaux and C. Teitelboim (1992) "Quantization of Gauge Systems", Princeton Univ. Press.

[8] N.N. Bogoliubov and D.V. Shirkov (1980) "Introduction to the Theory of Quantized Fields", John Wiley & Sons Inc; 3rd edition