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Plane Symmetric Cosmological Model with Quadratic Equation of State

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Abstract. A plane symmetric cosmological model with perfect fluid has been studied in general theory of relativity. The general solutions of the corresponding Einstein's field equations have been obtained with quadratic equation of state (EoS) $p = \alpha\rho^2 - \rho$, where α is the constant and strictly $\alpha \neq 0$. The physical and geometrical properties of the model are studied.

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1 Introduction

The accelerating universe has been evidenced by observational data collected from different sources like supernova type Ia, large scale structures and cosmic microwave background radiation [1-4]. The unknown form of energy component usually named as dark energy (DE) having negative pressure has been considered as the prime source in the cosmic history. However, the search for the source with negative pressure or best fit Dark Energy candidate have considered cosmological constant, dynamical models, modified and higher dimensional theories [5-7].

On large scales, the homogeneous and isotropic nature of the universe has been confirmed by the recent observations of large scale structure (LSS) [8] and cosmic microwave background radiation (CMBR) [9,10]. In general, it has been observed that anisotropic models do not isotropize sufficiently as they evolve into future. This isotropy problem can be solved by inflation.

Ananda & Bruni [11,12] studied the general relativistic dynamics of Robertson-Walker models and proved that the behavior of the anisotropy at the singularity found in the brane scenario can be recreated in the general relativistic context by considering a quadratic term in equation of state [EoS] given by

$$P = P_0 + \alpha\rho + \beta\rho^2,$$

where P_0, α and β are parameters. Above equation represents the first terms of the Taylor expansion of any equation of state of the form $P = P(\rho)$ about $\rho = 0$.

Dark energy universe with different equations of state has been discussed by Nojiri and Odintsov [13], and Capozziello [14] and proved that the quadratic equation of state may describe dark energy or unified dark matter. Rahman et al. [15], Feroze and Siddiqui [16] have modelled electron as a spherically symmetric charged perfect fluid distribution of matter characterized by quadratic equation of state in general relativity. Maharaj and Takisa [17] have obtained new exact solutions of the Einstein-Maxwell field equations with charged anisotropic matter distribution and a quadratic equation of state. Chavanis [18,19] has studied cosmological models based on a quadratic equation of state unifying vacuum energy, radiation and dark energy and also the cosmological models describing the early inflation, the intermediate decelerating expansion and the late accelerating expansion. Sharma and Ratanpal [20] have obtained a class of solutions describing the interior of a static spherically symmetric compact anisotropic star and shown that the model admits an equation of state which is quadratic in nature. Malaver [21] has studied the behavior of compact relativistic objects with anisotropic matter distribution considering quadratic equation of state and new solutions to the Einstein-Maxwell system of equations are found in terms of elementary functions. Recently, Takisa et al. [22] have modelled a charged anisotropic relativistic star with quadratic equation of state.

Our motivation to consider a quadratic equation of state includes its importance in the brane world models and the study of dark energy and general relativistic dynamics for different models. Also, it is not unnatural to choose quadratic form of equation of state to study anisotropy problems. Hence, the plane symmetric cosmological model containing a perfect fluid with quadratic equation of state has been studied. The general solutions of the Einstein's field equations for plane symmetric space-time have been obtained. The physical and geometrical aspects of the model are discussed.

2 Metric and Field Equations

In view of the importance of the plane symmetry which has been exploited to study the cosmological models by [23-29], we have considered the line element in plane symmetric form as

$$ds^2 = dt^2 - A^2(dx^2 + dy^2) - B^2 dz^2 \,, \tag{2.1}$$

where$A$ and B are scale factors and are functions of time t only.

The Einstein field equations, in natural limits ($8\pi\, G = 1$ and $c = 1$) are

$$R_{ij} - \frac{1}{2} R g_{ij} = -T_{ij} \,, \tag{2.2}$$

where R_{ij} is the Ricci tensor, R is the Ricci scalar, and T_{ij} is the energy-momentum tensor.

The energy-momentum tensor T_{ij} for the perfect fluid is given by

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}\,, \tag{2.3}$$

where ρ is the energy density, p is the pressure, and u^i is the four velocity vector satisfying $g_{ij}u^i u_j = 1$.

Here, we have assumed an equation of state (EoS) in the general form $p = p(\rho)$ for the matter distribution.

We have considered it in quadratic form as

$$p = \alpha\rho^2 - \rho\,, \tag{2.4}$$

where α is the constant and strictly $\alpha \neq 0$.

However, we can take $\gamma = 0$ to avoid complexities in calculations. This will not affect the quadratic nature of equation of state.

The Einstein field equations (2.2) for the plane symmetric metric (2.1) with the help of equations (2.3) reduce to equations:

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}^2}{A^2} = \rho, \tag{2.5}$$

$$2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} = -p, \tag{2.6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -p, \tag{2.7}$$

where overhead dot $(\cdot)$ denote differentiation with respect to time t.

The energy conservation equation $T^{ij}_{;j} = 0$ leads to the following expression:

$$\dot{\rho} + \left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)(\rho + p) = 0\,. \tag{2.8}$$

We define the spatial volume V and average scale factor a for the plane symmetric universe as

$$V = a^3 = A^2 B. \tag{2.9}$$

The mean Hubble parameter H for the plane symmetric universe is defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(H_x + H_y + H_z\right), \tag{2.10}$$

where $H_x = \dfrac{\dot{A}}{A}$, $H_y = \dfrac{\dot{A}}{A}$, $H_z = \dfrac{\dot{B}}{B}$ are the directional Hubble parameters in the directions of x, y and z axes respectively.

3 Solutions of the Field Equations

Subtracting equation (2.7) from equation (2.6) and using equation (2.9), we get

$$\frac{d}{dt}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{\dot{V}}{V}=0. \tag{3.1}$$

After solving equation (3.1), we can write the metric functions A and B explicitly as

$$A(t)=D_1 V^{\frac{1}{3}}\exp\left(X_1\int\frac{dt}{V}\right), \tag{3.2}$$

$$B(t)=D_2\, V^{\frac{1}{3}}\exp\left(X_2\int\frac{dt}{V}\right), \tag{3.3}$$

where D_1, D_2 and X_1, X_2 are the constants of integration.

The isotropic subcase, $A=B$ is not possible here as $X_1+X_2\neq 0$ in equations (3.2) to (3.3).

Using equation (2.9) in equation (2.8), we get

$$\dot{\rho}+\frac{\dot{V}}{V}(\rho+p)=0. \tag{3.4}$$

From equations (2.5)-(2.7), we obtain

$$\frac{\ddot{V}}{V}=\frac{3}{2}(\rho-p)\,. \tag{3.5}$$

After solving equation (3.5) and integrating, we obtain

$$\dot{V}=\sqrt{3\rho V^2+c_1}\ , \tag{3.6}$$

where c_1 is an integration constant.

Again integrating above equation (3.6), we get

$$\int\frac{dV}{\sqrt{3\rho V^2+c_1}}=t+t_0, \tag{3.7}$$

where the integration constant t_0 can be taken to be zero, since it only gives a shift in time.

Now, using equation (2.4) and equation (3.4), we obtain energy density in terms of volume as

$$\rho=(\alpha\log V)^{-1}\,. \tag{3.8}$$

It can be seen here that energy density is a positive quantity.

Using equations (3.8) in equation (3.7) and choosing $c_1 = t_0 = 0$, we get

$$\int \frac{dV}{\sqrt{3(\alpha \log V)^{-1} V^2}} = t\,. \tag{3.9}$$

Integrating above equation (3.9), we obtain

$$V = e^{3\left(\frac{t^2}{4\alpha}\right)^{1/3}} \quad . \tag{3.10}$$

Using equation (3.10) in equations (3.2) and (3.3), we obtain the scale factors A and B as

$$A(t) = D_1 \, e^{\left(\frac{t^2}{4\alpha}\right)^{1/3}} \exp[X_1 F(t)] \quad , \tag{3.11}$$

$$B(t) = D_2 \, e^{\left(\frac{t^2}{4\alpha}\right)^{1/3}} \exp[X_2 F(t)] \quad , \tag{3.12}$$

where

$$F(t) = \frac{\alpha^{1/3}}{6} \left\{ \sqrt{3\alpha}\alpha^{1/3} \mathrm{erf}\left[\sqrt{3/\alpha}\,(t/2\alpha)^{1/3}\right] - 3(4t)^{1/3} e^{-3\left(\frac{t^2}{4\alpha}\right)^{1/3}} \right\}$$

and erf is the "error function".

Using equation (3.11) and (3.12) in equation (2.1), the model takes the form as

$$ds^2 = dt^2 - D_1^2 e^{\left(\frac{t^2}{4\alpha}\right)^{2/3}} \left\{\exp[X_1 F(t)]\right\}^2 (dx^2 + dy^2) - D_2^2 e^{\left(\frac{t^2}{4\alpha}\right)^{2/3}} \left\{\exp[X_2 F(t)]\right\}^2 dz^2. \tag{3.13}$$

Using equation (3.10) in equation (3.8), we obtain energy density as

$$\rho = \frac{1}{3}\left(\frac{2}{\alpha t}\right)^{2/3} \quad . \tag{3.14}$$

From equations (2.4) and (3.14), we obtain pressure as

$$p = \frac{\alpha}{9}\left(\frac{2}{\alpha t}\right)^{4/3} - \frac{1}{3}\left(\frac{2}{\alpha t}\right)^{2/3} \quad . \tag{3.15}$$

The mean Hubble parameter H is found to be

$$H = \frac{1}{3}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = \frac{2\,t^{-1/3}}{3\,(4\alpha)^{1/3}} \tag{3.16}$$

The expansion scalar θ is found to be

$$\theta = 3H = \frac{2\, t^{-1/3}}{(4\alpha)^{1/3}}\,. \tag{3.17}$$

The mean anisotropy parameter Δ is found to be

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{3X}{4}(4\,\alpha\, t)^{2/3}\; e^{-6\left(\frac{t^2}{4\alpha}\right)^{1/3}}\,. \tag{3.18}$$

The shear scalar σ^2 is found to be

$$\sigma^2 = \frac{3}{2}\,\Delta\; H^2 = \frac{X}{2\,(4\alpha)^{2/3}} e^{-6\left(\frac{t^2}{4\alpha}\right)^{1/3}} \tag{3.19}$$

The deceleration parameter q is found to be

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = \frac{(4\alpha)^{1/3}}{2\, t^{2/3}} - 1 \quad . \tag{3.20}$$

4 Discussion

(I) At initial time $t = 0$, we get the finite spatial volume V. It expands exponentially as time t increases and becomes infinitely large as $t \to \infty$ as shown in Figure 1.

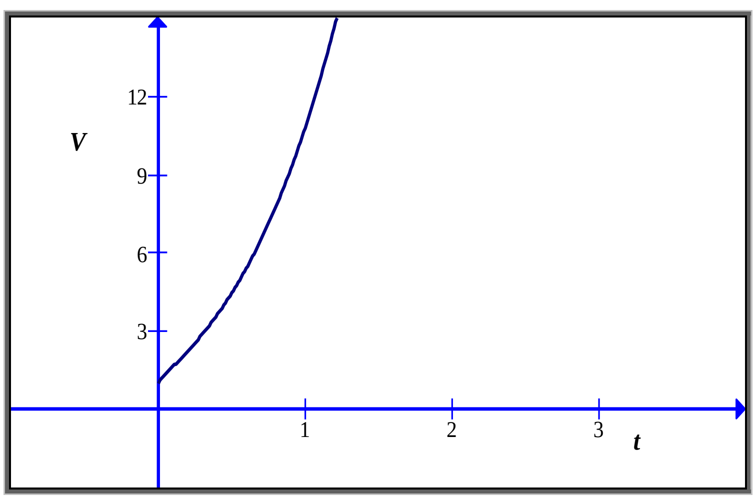

Figure 1. The variation of volume V vs. t for $\alpha = 0.5$.

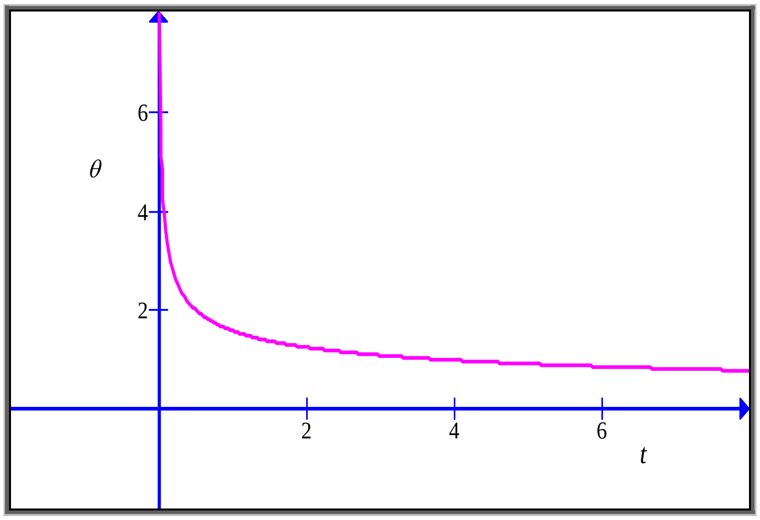

Figure 2. Expansion scalar θ vs. t for $\alpha = 0.5$.

(II) The evolution of expansion scalar θ for $\alpha = 0.5$ has been shown in Figure 2. It is observed that the expansion scalar θ starts with infinite value at $t = 0$ but as cosmic time t increases it decreases and becomes constant after some finite time.

(III) From Figure 3, it is observed that the evolution of the energy density ρ is infinite at $t = 0$ and as cosmic time t increases it decreases and becomes

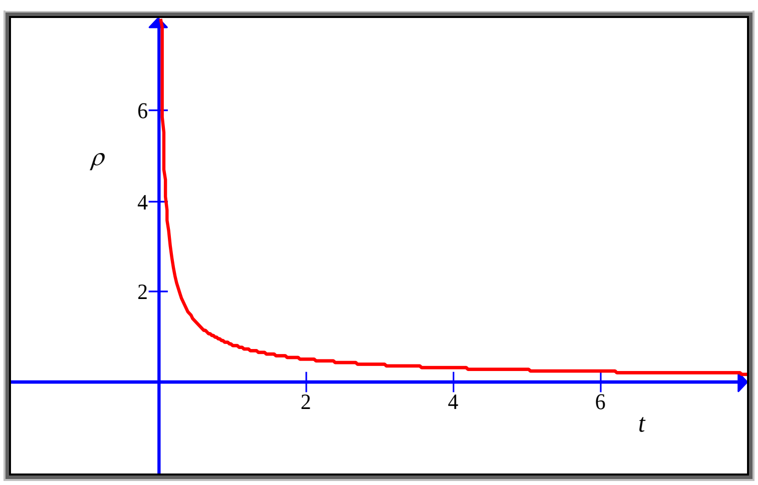

Figure 3. Evolution of energy density ρ vs. t for $\alpha = 0.5$.

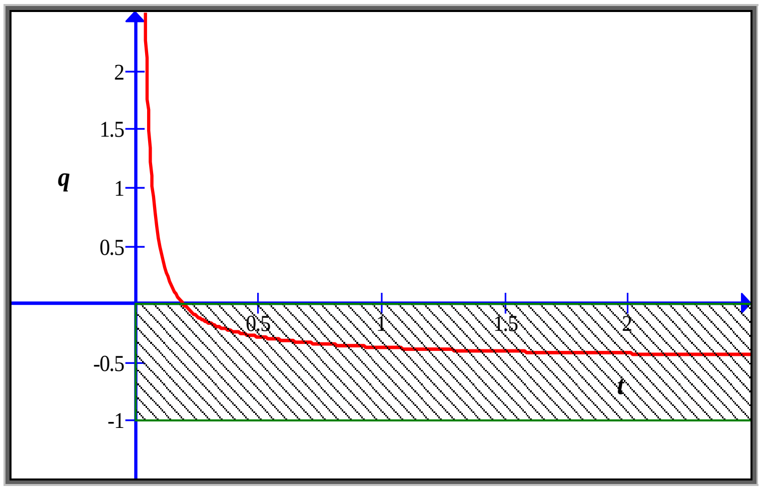

Figure 4. Deceleration parameter q vs. t for $\alpha = 0.5$.

constant after some finite time for $\alpha = 0.5$.

(IV)The deceleration parameter q is decelerating at $t = 0$. If time t increases then it decreases and becomes negative after some finite time for $\alpha = 0.5$ as shown in Figure 4. It implies that the universe accelerates after an epoch of deceleration.

5 Conclusion

A plane symmetric cosmological model has been studied with quadratic equation of state. It has been observed that the deceleration parameter q for the plane symmetric cosmological model is in the range $-1 \leq q \leq 0$ (shaded region in Figure 4) which matches with the observations made by Riess et al. [30] and Perlmutter et al. [1].

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