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Multiplication of Generalized Functions: Introduction*

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This Introduction was written in 1989 to the book by Ch. Ya. Christov and B. P. Damianov titled "Multiplication of Generalized Functions" in the series "Lecture notes for young scientists" (in Bulgarian) published by the Bulgarian Academy of Sciences Publishing House, Sofia (1989). This is the last publication of academician Christov.

With the development of science all concepts have been practically subject to generalizations. So it was with the concept of a function - perhaps the most important concept of one of the basic sciences – Mathematics. After the polynomials, fraction-rational and algebraic functions, the elementary transcendental functions, such as $\exp(x)$, $\cos x$, $\ln x$ and others, were next introduced as well as the so-called special functions like the functions of Bessel, Euler, Laplace. They have all been introduced as solutions of various differential equations, used in natural sciences and engineering. Typical for these solutions is that they cannot be expressed in terms of the known elementary functions.

The functions described above are actually not just a generalization of the notion of function: they are only introduction of various specific (interesting and useful) examples of functions. A generalization should require replacement of the very definition of function with another – a more general one. According to the classical definition, a function is a rule by which to each value of the real variable x (independent variable or argument) running some set M juxtaposes a value of another variable y running some set N; this is written as $y = f(x)$.

We obtain a possible generalization replacing the variable x (or y) with a set of variables $(x, y, z, \ldots)$ (respectively with set $(u, v, w, \ldots)$). But this generalization is in fact a classical one since it does not modify the spirit of the definition. Other generalizations – but also classical ones – are obtained when we allow x (or y) to take complex values or accept that several values of y correspond to a single value of x (for example, in the complex domain $w = \sqrt[n]{z}$ is a multi-valued function of the latter kind). On the contrary, in case of generalized functions or *distributions*, as well as in some variants of theirs, x or y are not numbers in the classical sense and not to each x one or several values of y would correspond.

*This text was translated and submitted by B. P. Damyanov

There are two approaches to the definition of distributions: sequential and functional. The first approach is similar to the transition from rational to real numbers. Following Cantor, each number in the new sense (real number) is defined as a fundamental sequence of numbers, convergent according to the criterion of Cauchy in the old sense (rational numbers). Such a sequence can have its limit in the set of the old numbers, with no generalization in this case, or it can have no such limit and then it defines a new, *irrational* number. The manifold of rational and irrational numbers presents the set of real numbers. (The algebraic operations with the sequences of numbers are performed with their terms: if $a = \{a_i\}$, $b = \{b_i\}$ $(i = 1, 2 \ldots)$, then $a + b = \{a_i + b_i\}$. Two different sequences of rational numbers do not always present two different real numbers: a criterion for equivalence of the sequences is introduced according to which any real number is defined by a class of equivalent sequences (class of equivalence).

The distributions are defined in a similar way as classes of sequences of classical functions convergent in some (*weak*) sense. They may consist of real or complex functions, one- or multi-valued ones, depend of one or more arguments, and run various sets of arguments. In this way we obtain different spaces of generalized functions. Below, if nothing else is specified, we will operate for simplicity with complex one-valued generalized functions with one real argument x.

Further, the construction of a system of generalized functions based on the sequential approach is reduced to answering the following questions:

1. Which class of ordinary functions the elements defining the sequences belong to? (Not all classical functions are to be involved, although this hides the danger that the obtained generalized functions will cover only partly the set of ordinary functions.)
2. When will a sequence of ordinary functions be convergent? (Not always a sequence of functional values has to be convergent for each x. If we accept such a requirement, the above procedure will not go beyond ordinary functions.)
3. When will two convergent sequences, from the class of admitted elements, define the same generalized function? In other words, we need to determine classes of equivalent sequences. (These classes may have joint elements, but then the definitions of the algebraic and analytic operations will be more complicated.)
4. How can the algebraic and analytic operations be generalized in the transition from ordinary to generalized functions? Generally, this operation is applied to the corresponding elements of the sequences that belong to the classes of each of the arguments. However, this generalization is not automatically guaranteed since this rule does not ensure existence or uniqueness of the result. Another question also appears: will the sequences of ordinary functions that result from this operation, cover the equivalence class of another generalized function. In this respect another generaliza-

tion is also possible. (Also, there might exist operations that hold for the generalized functions but do not hold for their representatives.)

These questions and the correspondence between the generalized and ordinary functions, as well as a study of the properties of the various generalizations present the object of the *sequential* theory.

The *functional approach* is similar, for example, to the way of introducing of infinitely remote points on the plane. We know that every two (different) lines in the plane have a point of intersection. However, this rule has an exception: the parallel lines have no intersection point. This exception however can be removed if "ideal elements" are included, i.e. if we admit that two parallel lines have a joint point at infinity. Then, series of theorems, e.g. those of the projective geometry, acquire a simpler and unique formulation.

A similar approach of completion with ideal elements is applied in the functional approach at the definition of distributions. Let Φ be a linear topological space of infinitely-differentiable functions $\varphi(x)$, where x is real, that are integrable together with their derivatives. If $f(x)$ is a partly discontinuous function of x, then

$$a[\varphi(x)] = \int_{-\infty}^{\infty} f(x)\varphi(x)\mathrm{d}x \tag{1}$$

is a linear functional of $\varphi(x)$ with values in $\mathbb{C}$ (or in $\mathbb{R}$). This however does not cover the set of all linear functional over Φ.Indeed, let us define

$$a[\varphi(x)] = \varphi^{(k)}(\xi). \tag{2}$$

Here $\xi$is an arbitrary chosen value of x, and $\varphi^{(k)}(\xi)$ is the derivative of order k of $\varphi(x)$ at the point ξ. It is not difficult to show that for each ξ and k we obtain a functional that cannot be presented as in (1). To any linear combination of these elements (with arbitrary complex coefficients) there again corresponds such a functional.

Nevertheless, we can assume that any functional (1) over the set Φ is determined by a function if the set F of functions can be generalized by a completion with 'ideal elements' (similar to the infinitely remote points in the geometry of lines). But it might be argued that if we accept such 'functions', as well as the infinitely remote points, this will contradict for example to the Bolzano–Weierstrass theorem for existence of a compression point. The latter theorem states that such a point exists only if the points are in a finite interval on a given line that does not contain its infinite point. It is however enough to consider as 'remote point' not the distance but the angle between the lines connecting the two points in consideration with a fixed point outside the line. In this way we arrive at a new topology leading to compactness of the space so that the mentioned theorem will be always valid.

In general, the question whether the ideal elements will be outside the system under consideration or will be included in it depends on the topology accepted.

The same is valid about generalized functions. One can prove that, although there exist according to (1), no classical functions $f(x)$ corresponding to the functionals (2) or to some of their linear combinations, there are still sequences that converge in the appropriate topologies of the spaces Φ and F and satisfy (1) provided its left part is given by (2). Their boundary elements can be considered as generalized functions.

Thus the following three general consequences are obtained:

1. Many assertions depend on the topologies of the spaces Φ and F.
2. The two approaches – sequential and functional – appropriately formulated may lead to the same result, i.e., to the same set of generalized functions.
3. Two terminologies can be introduced:
   - A) The generalized functions are 'ideal' elements, their set being complement to a class of ordinary functions.
   - B) They are functionals defined on the set Φ.

In mathematical literature the terminology B) is more popular but we think that for the needs of Physics terminology A) is also natural. Also it suits well to the very term "generalized functions". Of course, the two terminologies do not contradict one another.

The necessity for generalization of the classical functions has been realized long ago in many fields both of Mathematics and Physics. In the early 20s, Paul Dirac has proposed, for the needs of quantum mechanics, the well known δ-function, defined as a "function" by means of the property

$$\int_{-\infty}^{\infty} \delta(x-y)\varphi(y)\mathrm{d}y = \varphi(x)\,.$$

His aim was to introduce a 'continuous' analogue of the Kroneker symbol δ_{ij} (characterized by $\sum_i \delta_{ij} A_i = A_j$) aimed to describe physical quantities – masses or charges concentrated at a point. It is clear that there is no classical function with such properties (especially - differentiable), still as an exceptionally useful mathematical idealization of physical quantities, the Dirac function received a broad application.

In 1950 Laurent Schwartz published his mathematically-strict theory of generalized functions that included both the δ-function and its derivatives, defined as linear functionals over appropriate spaces of classical so-called 'test-functions'. This theory received well-deserved popularity and was applied to various fields of Mathematics and Physics, such as the theory of differential equations, quantum mechanics and relativistic theory of quantized fields. Moreover, the name *distributions*, given by Schwartz to a given class of generalized functions (D') obtained universal acceptance in mathematical literature for all generalized functions, introduced as linear functionals over spaces of functions. Let us note that the term 'distributions' originates from the description of densities of physical quantities such as masses, charges, etc.

Of course, there exist also other theories of generalized functions, enlarging the classical notions – the hyper-functions of Sato, the generalized functions of Colombeau, the non-standard analysis and others. The Schwartz distributions however remain the most popular among these theories and they are often identified with that name.

But despite of the indisputable role of the Schwartz distributions in various fields of science, there still exists the difficult problem of their application – the *problem of multiplication* of the distributions. It consists of the impossibility to define for them this operation so as to be associative, with a unique result, coordinated to the product of ordinary functions.

This problem appears as a basic one in many applications of the distributions. Let us give a specific example. It is well known that the wave function of a particle at a given moment (both in impulse and coordinate presentation) can be only a function from the class $L_2(R^n)$ of quadratic-integrable functions. On the other hand, it is well known that most convenient for work is the so-called plane wave $\psi_0(x) = \mathrm{e}^{ikx}$. However, neither the plane wave, nor its Fourier image $\varphi_0(p)$ belong to class $L_2(R^n)$. The latter is necessary in connection with their physical interpretation that uses their squares or products. The solution of this problem is known – we have to use the so-called *wave packages*: plane waves with amplitudes that converge to zero at large distances. Just the wave packages in impulsive representation are examples of generalized functions that have to be multiplied.

Other examples are given by classical diffraction of singular potentials, certain approaches in the theory of general relativity, description of shock waves in mathematical physics. Also related to the problems of multiplication of distributions are renormalizations in quantum field theory.

Despite of the lack of a correct general definition, particular results were obtained concerning the existence of some distributional products. Due to the importance of this problem, series of publications have appeared in mathematical and physical literature lately – aiming a correct definition of the operation of multiplication either in the whole set or in some subsets of the distributions. Given the impossibility of a complete solution of this problem, each of the proposed approaches has its specific aims and achievements.

A detailed study of the problem of multiplication of distributions and a presentation of the different approaches to its solution represent the subject of the present book (see Addendum). Also considered in short are some recent theories related to distributions, such as the 'non-linear' theory of generalized functions of Colombeau and the non-standard analysis of distributions.

ADDENDUM from the editor

The cited book "*Multiplication of Generalized Functions*" by Ch. Ya. Christov and B. P. Damyanov is based on ideas and results of the following papers of Ch. Ya. Christov obtained in collaboration with B. P. Damyanov (and D. Danov) published mainly in *the Bulgarian Journal of Physics*.

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