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Dedicated to the memory of
Professor Christo Christov (1915-1990)

Quantum Entanglement

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Abstract. Expository paper providing a historical survey of the gradual transformation of the "philosophical discussions" between Bohr, Einstein and Schrödinger on foundational issues in quantum mechanics into a quantitative prediction of a new quantum effect, its experimental verification and its proposed (and loudly advertised) applications. The basic idea of the 1935 paper of Einstein-Podolsky-Rosen (EPR) was reformulated by David Bohm for a finite dimensional spin system. This allowed John Bell to derive his inequalities that separate the prediction of quantum entanglement from its possible classical interpretation. We reproduce here their later (1971) version, reviewing on the way the generalization (and mathematical derivation) of Heisenberg's uncertainty relations (due to Weyl and Schrödinger) needed for the passage from EPR to Bell. We also provide an improved derivation of the quantum theoretic violation of Bell's inequalities. Soon after the experimental confirmation of the quantum entanglement (culminating with the work of Alain Aspect) it was Feynman who made public the idea of a quantum computer based on the observed effect.

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1 Introduction

One thing that troubled Einstein most with the Copenhagen Interpretation was the "instantaneous reduction of the wave function" – and hence of the probability distribution – when a measurement is performed. After Bohr's talk at the fifth (the famous!) Solvay Congress in October 1927, he made a comment concerning the double slit experiment. Bohr's probability wave is spread over the detector screen, but as soon as the electron is detected at one point, the probability becomes zero everywhere else – instantly (see, e.g., [21] Ch. 8, Sect. *Berlin and Brussels*). If the reduction of the probability wave of a single particle may not have seemed so paradoxical, the consequences of the thought experiment that Einstein, Podolsky and Rosen [17] proposed in 1935 appear really drastic. It involves two correlated particles such that measuring the coordinate or the momentum of one of them fixes the corresponding quantity of the other, possibly *distant particle*. Only two senior physicists reacted to the EPR paper at the time: Schrödinger [32] sympathized with the authors and, after a correspondence with Einstein ([21], Ch. 9, Sect. *The cat in the box*; [19], Sect. 1.3), introduced the term *entanglement* (as well as the notorious cat – [33]); Bohr [11] challenged Einstein's notion of physical reality and rejected the idea that its quantum mechanical description is incomplete. The younger "working particle physicists" ignored the discussion (probably dismissing it as "metaphysical"). A gradual change of attitude only started in the 1950's with the work of David Bohm, an American physicist that had to leave the US after loosing his job as an early victim of the McCarthy era [20]. He reformulated the EPR paradox (first in his textbook on Quantum Theory [9], then, more thoroughly, in an article with his student Aharonov [10]) in terms of the electron spin variables. The reduction to a finite dimensional quantum mechanical problem soon allowed a neat formulation – in the hands of John Bell [4, 7] – and opened the way to its experimental verification. In a few more decades it gave rise to a still hopeful and fashionable outburst of activity under the catchy names of "quantum computers" or "quantum information".

The present paper aims to highlight the key early steps of what is being advertised as "a new quantum revolution" [3]. We begin in Sect. 2 by reviewing the post Heisenberg development and understanding of the *uncertainty relations* using the algebraic formulation of quantum theory. Sect. 3 is devoted to the early history of the subject – from EPR, Schrödinger and Bohr through Bohm to Bell, Clauser, Shimony et al. [14] who proposed to use polarized photons to test Bell's inequalities to the ultimate realization of this proposal in the work of Alain Aspect (see his later reviews [2, 3] and references cited there). Sect. 4 deals with the actual derivation of Bell-CHSH inequalities for classical "local hidden variables", following [5]; in describing their violation in quantum theory we introduce a maximally entangled U(2) invariant state. Sect. 5 overviews the work of Feynman [18] and of Manin and Shore (see [25] and references therein) and ends with a general outlook. We briefly discuss the (partly philosophical – as reviewed in [19]) issue of *nonlocality* siding with the dissenting view of a mathematical physicist [16].

2 Weyl-Schrödinger's Uncertainty Relations

Much of the early discussions on the meaning of quantum mechanics, turned around the uncertainty relations which restrict the set of legitimate questions one can ask about the microworld. Heisenberg justified in 1927 his uncertainty principle for the measurement of the position x and the momentum p by analyzing the unavoidable disturbance of the microsystem by any experiment designed to determine these variables. In Weyl's book [38] of the following year (1928) one finds a derivation of a more precise relation for *the product of dispersions (mean square deviations)* $\sigma_x^2 \sigma_p^2$ from the properties of the wave function describing the quantum state. Soon after, Schrödinger [31] and others (for a review and more references – see [37]) extend Weyl's analysis to general pairs of noncommuting operators – a necessary step towards a realistic test of entanglement. Let us point out that, while the validity of Heisenberg's popular arguments for the limitations concerning individual measurements have been questioned in recent experiments [29], the mathematical results about the dispersions of incompatible observables which we proceed to review are impermeable.

In order to display the generality and simplicity of the uncertainty relations we shall adopt (and begin by reminding) the algebraic formulation of quantum theory, which, having the aura of abstract nonsense, is seldom taught to physicists.

We start with a (noncommutative) *unital star algebra* $\mathcal{A}$ – a complex vector space equipped with an associative multiplication (with a unit element 1) and an *antilinear antiinvoltion* $*$ such that

$$(AB)^* = B^*A^* \,, \;\; (\lambda A)^* = \bar{\lambda} A^* \,, \;\; (A^*)^* = A \;\text{ for }\; A, B \in \mathcal{A} \,, \;\; \lambda \in \mathbb{C} \quad (2.1)$$

where the bar over a complex number stands for complex conjugation. The *hermitean* elements A of $\mathcal{A}$ (such that $A^* = A$) are called *observables*. A *state* is a (complex valued) *linear functional* $\langle A \rangle$ on $\mathcal{A}$ satisfying *positivity*: $\langle A^* A \rangle \geq 0$ and *normalization*: $\langle 1 \rangle = 1$.

Proposition 2.1 If A is an observable, $A = A^*$, then its *expectation value* $\langle A \rangle$ is real; moreover,

$$A^* = A \,, \; B^* = B \quad \Rightarrow \quad \langle BA \rangle = \overline{\langle AB \rangle}. \qquad (2.2)$$

Proof. The implication (2.2) is a consequence of the positivity (and hence the reality) of both $\langle (A+B)^2 \rangle$ and $\langle (A+iB)(A-iB) \rangle$. The reality of $\langle A \rangle$ for $A = A^*$ follows from (2.2) for $B = 1$.

Remark 2.1 The more common definition of a (*pure*) quantum state as a vector $|\Psi\rangle$ of norm 1 in a Hilbert space (or rather a 1-dimensional projection $|\Psi\rangle\langle\Psi|$) is recovered as a special case for $\langle A \rangle = \langle \Psi | \, A \, | \Psi \rangle \equiv \mathrm{tr} \, (A \, |\Psi\rangle\langle\Psi| \,)$. The reality of the expectation value in this case appears as a corollary of the spectral decomposition theorem for hermitean operators while in the above formulation it is an elementary consequence of the algebraic positivity condition. Furthermore, our definition applies as well to a *mixed* state (or a *density matrix*). The set of all (admissible) states form a convex manifold $\mathcal{S}$. The pure states appear as *extreme points* (or indecomposable elements) of $\mathcal{S}$.

We now proceed to the formulation and the *elementary proof* of the Schrödinger uncertainty relation which is both stronger and more general than Weyl's precise mathematical statement of Heisenberg's principle. Let A, B be two (noncommuting) observables with expectation value zero. (The general case is reduced to this by just replacing A, B by $A - \langle A \rangle$, $B - \langle B \rangle$.)

Proposition 2.2 (*Schrödinger's uncertainty relation*) The product of the dispersions of the above observables exceeds the sum of squares of the expectation values of the hermitean and the antihermitean parts of their product:

$$\sigma_A^2 \sigma_B^2 (\equiv \langle A^2 \rangle \langle B^2 \rangle\,) \geq \langle AB \rangle \langle BA \rangle = \\ = (\frac{1}{2}\langle AB + BA \rangle)^2 + (\frac{1}{2i}\langle AB - BA \rangle)^2 \geq |\frac{1}{2}\langle [A,B] \rangle|^2. \quad (2.3)$$

The *proof* consists of a direct application of the elementary Schwarz inequality for a positive quadratic form, taking (2.2) into account. The Heisenberg-Weyl uncertainty relation is obtained as a special case for $A = p$, $B = q$ using $[q,p] = i\hbar$. We shall apply the inequality (2.3) to the case of two orthogonal polarization vectors in Sect. 4 below.

Remark 2.2 We note that there are further strengthenings of the uncertainty relations (see e.g. [24]) but Proposition 2.2 will be sufficient for our purposes.

3 From EPR to Bell's Inequalities

In 1935, already at Princeton, (the 56-year-old) Einstein with two younger collaborators, Boris Podolsky (Taganrog, 1896 – Cincinnati, 1966) and Nathan Rosen (Brooklyn, 1909 – Haifa, 1995) proposed something new. They consider a state of two particles travelling with opposite momenta (in opposite directions) along the x-axis with a (non-normalizable) wave function

$$\Psi(x_1, x_2) = \int u_p(x_1)\, u_{-p}(x_2)\, e^{i\frac{p\,d}{\hbar}} \frac{dp}{2\pi\hbar} = \delta(x_1 - x_2 + d)\,, \quad (3.1) \\ u_p(x) = e^{i\frac{px}{\hbar}}\,.$$

If one measures the position of the first particle x_1 the position of the second one would be fully determined ($x_2 = x_1 + d$). If one measures instead its momentum and finds the value $p_1 = p$ then the momentum of the second particle will be $p_2 = -p$. None of the operations on the first particle should disturb the second one (as the distance d between the two can be made arbitrarily large). It then appears that the second particle should have both definite position and definite momentum. As quantum mechanics cannot accommodate both the authors conclude that it has to be considered incomplete. The precise wording of the abstract of [17] is more nuanced: "... A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality...". (The paper was actually written by Podolsky and Einstein was not happy: he found the wording obscuring the simple message... – see [19], Sect. 1.3.)

Two quantum theorists responded to the EPR paper (both born in the 1880's): Schrödinger [31], a sympathizer, who, after a correspondence with Einstein, introduced the term *entanglement* [33], and Bohr, the father of the Copenhagen Interpretation, who was upset. Bohr's reaction was recorded by his faithful collaborator (since 1930), the Belgian physicist Léon Rosenfeld (1904-1974 – characterized in [22] as a Marxist defender of complementarity) and later told by Bohr's grandson Tomas. In Rosenfeld's words:

"This onslaught came down upon us as a bolt from the blue. ... As soon as Bohr had heard my report of Einstein's argument, everything else was abandoned: we had to clear up such a misunderstanding at once. We should reply by taking up the same example and showing the right way to speak about it. In great excitement, Bohr immediately started dictating to me the outline of such a reply. Very soon, however, he became hesitant: 'No, this won't do, we must try all over again ... we must make it quite clear ...' ... Eventually, he broke off with the familiar remark that he 'must sleep on it'. The next morning he at once took up the dictation again, and I was struck by a change of the tone: there was no trace of the previous day's sharp expressions of dissent. As I pointed out to him that he seemed to take a milder view of the case, he smiled: 'That's a sign', he said, 'that we are beginning to understand the problem.' Bohr's reply was that yes, nature is actually so strange. The quantum predictions are beautifully consistent, but we have to be very careful with what we call 'physical reality'."

Bohr's reply [11] to EPR carried the same title *Can quantum-mechanical description of physical reality be considered complete?* and also appeared in the Physical Review four months later. Compared with the clear message of EPR it appears tortured: " Indeed the finite interaction between object and measuring agencies conditioned by the very existence of the quantum of action entails – because of the impossibility of controlling the reaction of the object on the measuring instruments if these are to serve their purpose – the necessity of a final renunciation of the classical ideal of causality and a radical revision of our attitude towards the problem of physical reality." No wonder that that the younger generation was repelled by such a metaphysical twist in the discussion.

The next step was only made some 15 years later by David Bohm (1917-1992) who had the opportunity to discuss the matter with Einstein. In his 1951 book on Quantum Theory and later in [10] he reduced the problem in the case of electron spins for observables that are *dichotomic* – a crucial advance for the subsequent development (see [3], Sect. 3). It was for such type of observables that Bell could derive in 1964 [4] his inequalities. Bell's paper, which now counts over 9000 citations, remained essentially unnoticed until the 1969 work of Clauser et al. [14] (see Chapter 7 of [20], in particular, Picture 7.4).

Remark 3.1 Bell was originally motivated by the work of Bohm on hidden variables and by the observation that it provided a counterexample to von Neumann's "no hidden variable" theorem (of 1932 – translated into English, as if just to counter Bohm's theory, in 1955 [28]). Remarkably, he ended up by establishing a better theorem of this type – both more general and concrete – stimulating new experiments. (For interesting later comments on von Neumann's theorem and Bell's critique – see [13, 30].)

Freire is right to call all participants in these developments *quantum dissidents* – physicists trying to upset what the historian of physics Max Jammer (1974) had termed the "almost unchallenged monocracy of the Copenhagen school" (see Sect. 2.1 of [20]). Having a doctorate in philosophy (under Carnap in 1953) before acquiring a second doctorate in physics (under Eugene Wigner in 1962) was not viewed as an asset for Abner Shimony in the physics community. The authoritative intervention of Wigner (Nobel Prize, 1963) was needed to defend him from strong and unfair criticism after his first paper on the foundations of quantum mechanics (see Sect. 7.3: "Philosophy enters the labs: the first experiments" of [20]). When John Clauser still a graduate student wanted to prepare an experiment to check Bell's inequalities (independently of Shimony) and asked the advice of Feynman about his project the answer was: "You will be wasting your time" (see [27]). Happily, when Shimony learned about Clauser's proposal to do the same (Bell-) type experiment that he has assigned to his graduate student Horne, he followed Wigner's advice and called Clauser; so rather than engaging in a fierce competition they started a fruitful collaboration. Even after their landmark 1969 paper [14] and the first experimental confirmation of the quantum mechanical violation of the Bell-CHSH inequalities (by Clauser and Freedman, 1972) debates on the foundations of quantum mechanics were formally forbidden by the editor (Samuel Goudsmit, 1902-1978) of Physical Review. Clauser and Shimony then started publishing *Epistemological Letters* – a hand-typed, mimeographed, "underground" physics newsletter about quantum physics distributed by a Swiss foundation (1973-1984). According to Clauser, much of the early work on Bell's theorem was published only there (including Bell's paper [6] and the responses to it by Shimony, Clauser and Horne; for more on this story – see [23]).

4 Inequalities Separating Hidden Variable and Quantum Predictions for a 2-State System

The EPR paradox rephrased in terms of dichotomic observables says that measuring e.g. a photon polarization we shall instantly determine the polarization of its distant entangled partner without disturbing it. On the other hand, as we shall recall shortly, polarizations in two directions differing by an angle θ cannot be determined simultaneously unless $\sin 2\theta = 0$. The paradox would be resolved if photon polarization in all directions was fully determined by some additional statistical parameters termed *hidden variables* (HV). This is a natural assumption. To cite [3] "when biologists observe strong correlations between some features of identical twins they can conclude that these features are determined by identical chromosomes. We are thus led to admit that there is some common property whose value determines the result of polarization. But such a property, which may differ from one pair to another, is not taken into account by the quantum state which is the same for all pairs. One can thus conclude with EPR that Quantum Mechanics is not complete." It was Bell [4, 7] who realized that even without specifying the nature of the hidden parameters, just assuming that they are not affected by changes in the distant experimental arrangement, their existence implies certain inequalities in the probability distribution of the polarization that are violated if the quantum mechanical predictions hold. Our survey below of this landmark work takes into account subsequent development by Clauser et al. [14] and by Bell himself [5].

4.1 Bell-CHSH inequalities for (classical) hidden variables

One starts with a pair of linearly polarized photons emitted by a source characterized by some supplementary parameters λ, and two analyzers, A in orientation $\mathbf{a}$ and B in orientation $\mathbf{b}$ which may depend on some additional parameters λ'. The photons are assumed to have opposite momenta; the polarizations $\mathbf{a}$ and $\mathbf{b}$ are then represented by two unit vectors in a plane (say (x, z)) orthogonal to their common line of propagation (y) (see Figure 1). The possible outcomes of the polarization measurement $A(\mathbf{a}, \lambda)$ will be taken 1 for a polarization along $\mathbf{a}$ and -1 for a polarization in the orthogonal direction in the same plane (and similarly for $B(\mathbf{b}, \lambda)$ and $\mathbf{b}$, respectively). If we consider, following [5], averaging with respect to the analyzers' parameters λ' we should replace $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$ by their mean values which satisfy

$$-1 \leq \bar{A}(\mathbf{a}, \lambda) \leq 1 \, , \qquad -1 \leq \bar{B}(\mathbf{b}, \lambda) \leq 1 \, . \tag{4.1}$$

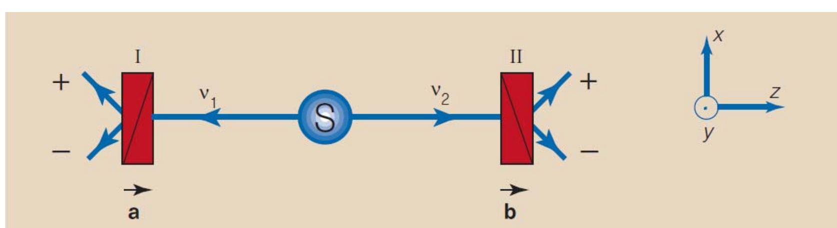

Figure 1. EPR-Bohm Gedanken experiment with photons [2]. The two photons travelling in opposite directions away from a source are analyzed by linear polarizers in orientations $\mathbf{a}$ and $\mathbf{b}$. One measures the probabilities of joint detections in the output channels at various orientations of the polarizers.

(To quote from [5]: "In practice, there will be some occasions on which one or both instruments simply fail to register either way. One might then count A and/or B as zero in defining $\bar{A}$, $\bar{B}$." Then (4.1) still holds.) Introduce further a normalized probability measure

$$d\,\mu(\lambda) \geq 0, \qquad \int d\,\mu(\lambda) = 1\ . \tag{4.2}$$

Knowing $\bar{A}$, $\bar{B}$ (4.1) and the measure (4.2) one can compute the probabilities for various outcomes. Assuming that of the measurements A and B are independent we deduce that the joint probability of a pair of outcomes is equal to the product of the separate probabilities for each of them, so that the statistical correlation function (expectation value) would be given by the *bounded* mean value of their product:

$$-1 \leq E(\mathbf{a},\mathbf{b}) = \langle\, A(\mathbf{a})\, B(\mathbf{b})\,\rangle_{HV} := \int d\mu(\lambda)\, \bar{A}(\mathbf{a},\lambda)\, \bar{B}(\mathbf{b},\lambda) \leq 1\ . \tag{4.3}$$

Remark 4.1 Here we stick to the traditional notation A, B (reminiscent to the "Alice and Bob" used in cryptography) which involves a redundancy: for a fixed vector $\mathbf{a}$, $A(\mathbf{a}) = B(\mathbf{a})$ so that we are dealing with a single physical quantity. We shall make this explicit in our treatment of the quantum case.

Let $\mathbf{a}'$ and $\mathbf{b}'$ be alternative directions of polarization. We shall prove the following inequality for the sum of absolute values of particular linear combination of correlations in any classical (HV) theory:

$$|E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{b}')| + |E(\mathbf{a}',\mathbf{b}) + E(\mathbf{a}',\mathbf{b}')| \leq 2\ . \tag{4.4}$$

Indeed (4.4) holds as a consequence of the following chain of inequalities:

$$\begin{aligned}
&|E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{b}')| = \\
&\quad = \Big| \int d\mu(\lambda) \bar{A}(\mathbf{a},\lambda)\bar{B}(\mathbf{b},\lambda)\,(1 \pm \bar{A}(\mathbf{a}',\lambda)\bar{B}(\mathbf{b}',\lambda)) - \\
&\qquad - \bar{A}(\mathbf{a},\lambda)\bar{B}(\mathbf{b}',\lambda)\,(1 \pm \bar{A}(\mathbf{a}',\lambda)\bar{B}(\mathbf{b},\lambda))\Big| \leq \\
&\quad \leq \int d\mu(\lambda)\,(1 \pm \bar{A}(\mathbf{a}',\lambda)\bar{B}(\mathbf{b}',\lambda)) \\
&\qquad + \int d\mu(\lambda)\,(1 \pm \bar{A}(\mathbf{a}',\lambda)\bar{B}(\mathbf{b},\lambda)) = \\
&\quad = 2 \pm (E(\mathbf{a}',\mathbf{b}') + E(\mathbf{a}',\mathbf{b}))\ .
\end{aligned} \tag{4.5}$$

Introducing the linear combination of 2-point correlation functions

$$\begin{aligned}
S(\mathbf{a},\mathbf{b},\mathbf{a}',\mathbf{b}') &= E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{b}') + E(\mathbf{a}',\mathbf{b}) + E(\mathbf{a}',\mathbf{b}') = \\
&= \int d\mu(\lambda)\, S(\lambda;\mathbf{a},\mathbf{b},\mathbf{a}',\mathbf{b}')\ ,
\end{aligned} \tag{4.6}$$

we deduce from (4.4) the result first obtained (under slightly more restrictive conditions) by CHSH:

$$|S(\mathbf{a},\mathbf{b},\mathbf{a}',\mathbf{b}')| \leq 2\ . \tag{4.7}$$

We stress that the assumption of positivity of the measure $d\mu$ is essential for the validity of the Bell-CHSH inequalities – a point also emphasized by Feynman [18]. Admitting "negative probabilities" one can reproduce all quantum mechanical results!

4.2 Quantum mechanical treatment of pairs of entangled photons

It is customary to consider a 2-dimensional Hilbert space $\mathcal{H}$ of (a single) photon polarization with a basis $|\varepsilon\rangle\ ,\ \varepsilon = \pm$ where $+(-)$ corresponds to a linear polarization along the $z\,(x)$-axis, respectively. In fact, this labeling is not quite complete. Dealing with a pair of entangled photons one should also indicate the sign of the photon momentum p – along the positive or the negative y axis. Observing that changing the sign of p amounts to complex conjugation of the wave function we shall put a bar over the state vector of the second photon that moves in the opposite direction to the first. (This amounts to introducing a real structure in our Hilbert space in terms of a linear isomorphism between the dual space $\mathcal{H}'$ of "bra vectors" and the space $\bar{\mathcal{H}}$ of complex conjugate "kets".) A maximally entangled state in the 4-dimensional complex Hilbert space $\mathcal{H}\otimes\bar{\mathcal{H}}$ is given by

$$\Psi = \frac{1}{\sqrt{2}}\sum_{\varepsilon=\pm}|\varepsilon\rangle\otimes|\bar{\varepsilon}\rangle \quad (= u\otimes\bar{u}\ \Psi \quad \text{for} \quad u\in U(2)\,)\ . \tag{4.8}$$

It is independent of the choice of basis $|\varepsilon\rangle$ being the unique $U(2)$-invariant pure state in $\mathcal{H}\otimes\bar{\mathcal{H}}$. (Note that this is not true for the customarily used real $O(2)$-invariant substitute of (4.8).)

A state $|\theta,\varepsilon\rangle\in\mathcal{H}$ of polarization ε in a direction of angle θ with respect to the z-axis (in the (z,x)-plane) is given by:

$$|\,\theta,\varepsilon\rangle := \cos\theta\,|\varepsilon\rangle + \varepsilon\,\sin\theta\,|-\varepsilon\rangle\ , \qquad \varepsilon=\pm\ . \tag{4.9}$$

The operators $A_i(\theta)\ ,\ i=1,2$ corresponding to the analyzers of the first and the second photon are given by

$$A_1(\theta)=A(\theta)\otimes 1,\quad A_2(\theta)=1\otimes A(\theta),\quad A(\theta)=\cos 2\theta\sigma_3+\sin 2\theta\sigma_1,$$
$$\sigma_3\,|\varepsilon\rangle=\varepsilon\,|\varepsilon\rangle\ ,\quad \sigma_1\,|\varepsilon\rangle=|-\varepsilon\rangle\ . \tag{4.10}$$

Here $A(\theta)$ is the operator with eigenvectors $|\theta,\varepsilon\rangle$ corresponding to eigenvalues ε:

$$A(\theta)\,|\theta,\varepsilon\rangle=\varepsilon\,|\theta,\varepsilon\rangle\ . \tag{4.11}$$

The 2-point correlation function of the product $A_1 A_2$ in the state Ψ (4.8) is given by

$$E(\theta_1, \theta_2) := \langle\, \Psi | A_1(\theta_1) A_2(\theta_2) | \Psi \rangle = \cos 2(\theta_1 - \theta_2)\ . \qquad (4.12)$$

Remark 4.2 This result can also be expressed as a linear combination of individual probabilities $P_{\varepsilon_1 \varepsilon_2}(\theta_1 - \theta_2)$ defined below (cf. [2, 3]):

$$E(\theta_1, \theta_2) := \langle \Psi | A_1(\theta_1) A_2(\theta_2) | \Psi \rangle = \sum_{\varepsilon_1, \varepsilon_2} \varepsilon_1 \varepsilon_2 |\langle \theta_1, \varepsilon_1 | \otimes \langle \theta_2, \varepsilon_2 | \Psi \rangle|^2 =$$
$$= P_{++}(\theta_{12}) + P_{--}(\theta_{12}) - P_{+-}(\theta_{12}) - P_{-+}(\theta_{12})\ , \qquad (4.13)$$

where

$$\theta_{ij} := \theta_i - \theta_j,$$
$$P_{\varepsilon_1 \varepsilon_2}(\theta_{12}) := |\langle \theta_1, \varepsilon_1 | \otimes \langle \theta_2, \varepsilon_2 | \Psi \rangle|^2 = \frac{1}{4}(1 + \varepsilon_1 \varepsilon_2 \cos 2\theta_{12})\ .$$

Remark 4.3 Another way to express the quantum mechanical 2-point correlation function is to use the fact that

$$\mathrm{tr}_2(A_2(\theta_2)|\Psi\rangle\langle\Psi|) = A(\theta_2)\,\rho = \frac{1}{2} A(\theta_2), \quad \rho = \frac{1}{2}\boldsymbol{I} = \frac{1}{2}\sum_{\varepsilon} |\varepsilon\rangle\langle\varepsilon| \qquad (4.14)$$

to write

$$E(\theta_1, \theta_2) = \mathrm{tr}\,(A_1(\theta_1) A_2(\theta_2) |\Psi\rangle\langle\Psi|) =$$
$$= \mathrm{tr}\,(A(\theta_1) A(\theta_2)\,\rho) = \frac{1}{2}\,\mathrm{tr}\,(A(\theta_1) A(\theta_2))\ . \qquad (4.15)$$

The corresponding joint probability distributions $P_{\varepsilon_1 \varepsilon_2}(\theta_{12})$ are given by R. Stora's formula

$$P(a, b) = \frac{1}{2}\ (\langle a|b\rangle\langle b|\rho|a\rangle + \langle a|\rho|b\rangle\langle b|a\rangle) \qquad (4.16)$$

(cf. Eq. (4.7) of Sect. 4.1 of [36]) for $|a\rangle = |\theta_1, \varepsilon_1\rangle$, $|b\rangle = |\theta_2, \varepsilon_2\rangle$ from (4.9).

The matrix ρ is a particular case of *reduced density matrix* (corresponding to a pure state of the composite system), a notion introduced in [15]; it describes the state as viewed by an observer attached to one of the subsystems.

Comment. Any state in a 2-dimensional Hilbert space can be realized as a density matrix $\rho(\underline{a})$ labeled by a vector $\underline{a}$ in the unit ball:

$$\rho = \rho(\underline{a}) = \frac{1}{2}(\boldsymbol{I} + \underline{a} \cdot \underline{\sigma}) = \frac{1}{2}\begin{pmatrix} 1 + a_3 & a_1 - i a_2 \\ a_1 + i a_2 & 1 - a_3 \end{pmatrix},$$
$$\underline{a}^2 = a_1^2 + a_2^2 + a_3^2 \le 1\ . \qquad (4.17)$$

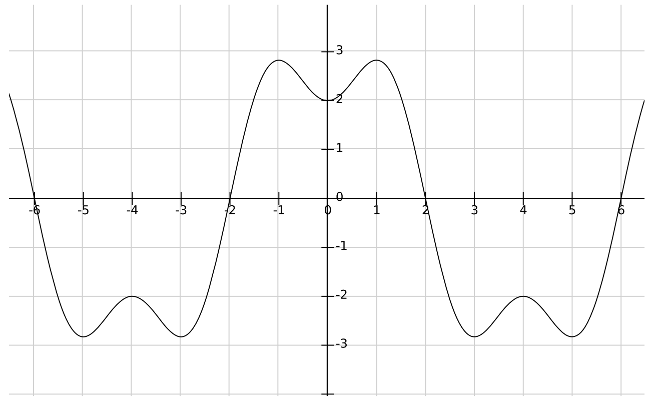

Figure 2. Plot of the function $3\cos\frac{\pi}{4}x - \cos\frac{3\pi}{4}x$.

The pure states belong to the boundary of this domain, the *Bloch sphere*

$$\mathbb{S}^2 \simeq \mathbb{S}^3/\mathbb{S}^1 = SU(2)/U(1) . \tag{4.18}$$

The *von Neumann entropy* $S = -\mathrm{tr}\,\rho\,\log_2\rho$ varies between 0 (for a pure state) and 1 , for the maximally mixed state $\rho = \frac{1}{2}\boldsymbol{I}$.

It follows that the quantum mechanical counterpart of (4.6) is

$$\begin{aligned} S(\theta_1,\theta_2,\theta_3,\theta_4) &= E(\theta_1,\theta_2) - E(\theta_1,\theta_4) + E(\theta_3,\theta_2) + E(\theta_3,\theta_4) = \\ &= \cos 2\,\theta_{12} - \cos 2\,\theta_{14} + \cos 2\,\theta_{23} + \cos 2\,\theta_{34} . \end{aligned} \tag{4.19}$$

Setting the consecutive differences among the angles equal to each other, $\theta_{ii+1} = \theta$, $i = 1,2,3$ we find the following global extremal points of the function (4.19) (see Figure 2):

$$\theta = \frac{\pi}{8}\,(2k+1) \,, \quad k \in \mathbb{Z} . \tag{4.20}$$

For $\theta = \frac{\pi}{8} = 22.5^o$ and $\theta = \frac{3\pi}{8} = 67.5^o$ Eq. (4.19) gives

$$S(0,\theta,2\theta,3\theta) = \begin{cases} 3\,\cos\frac{\pi}{4} - \cos\,\frac{3\pi}{4} = 4\,\frac{1}{\sqrt{2}} = 2\sqrt{2} \\ 3\,\cos\frac{3\pi}{4} - \cos\,\frac{9\pi}{4} = -4\,\frac{1}{\sqrt{2}} = -2\sqrt{2} \end{cases} . \tag{4.21}$$

At these angles the predictions for the quantum mechanical entangled two photon system violates the Bell-CHSH inequalities (4.7) by more than 40% , a fact that has been confirmed by numerous precision tests (see [2, 3]).

5 Feynman and Quantum Computing. Discussion

The story does not end with the beautiful experiments confirming the strange predictions of quantum mechanics. In his “keynote talk” at the 1st conference on Physics and Computation at MIT in 1981 Richard Feynman (1918-1988) demonstrated that he not only had finally appreciated the work on quantum entanglement (albeit he did not cite any names) but proposed to use it in order to simulate quantum physics with computers [18]. During half a century scientists were convinced that any computer is a realization of an universal machine described in 1936 by Alan Turing (1912-1954). Feynman noted that the behavior of entangled photons cannot be imitated by such a classical machine and should be used to construct a new *quantum computer*. If predecessors-mathematicians (such as Manin, 1980) were interested in new possibilities for calculations (using the "greater capacity of quantum states"), Feynman thinks of simulating quantum phenomena which do not admit a classical realization in order to better understand quantum theory: *we never really understood how lousy our understanding of languages was, the theory of grammar and all that stuff, until we tried to make a computer which would be able to understand language* ([18] Sect. 8 p. 486).

One way or another, the cold reception of the first steps revealing the quantum entanglement is been replaced by a hectic activity with pretense for a new science. Books with titles like *Quantum Computers and Quantum Information* are advertised by prestigious publishers (Cambridge University Press, 2010). The publicity is impressive, the progress is modest. A serious mathematical result is Shor's (1994) algorithm for decomposing large positive integers into prime factors [25]. According to it, the time needed to factor the number N does not exceed a multiple of $(\log N)^2 \log\log N \log\log\log N$. It is believed on the other hand (albeit not proven) that the time needed for a classical factoring algorithm grows faster than any power of $\log N$. (The problem of factoring large integers is relevant for cryptography.) In practice, the realization of a quantum computer is hindered by the phenomenon of *decoherence* in large systems. After some twenty years of efforts (and a few billion dollars invested) the record achieved (in 2012) by a real quantum computer using Shor's algorithm is the factoring of $21(= 3 \times 7)$. In the words of the renowned computer scientist Leonid Levin "The present attitude [of quantum computing researchers] is analogous to, say, Maxwell selling the Daemon of his famous thought experiment as a path to cheaper electricity from heat." ([1]). Noticeable applications come after a long quiet development. Quantum mechanics, created during the first quarter of XX century is finding wide applications only after the invention of the transistor in 1948 and the development of the laser in the late 1950's. The true applications of the "second quantum revolution" are yet to come.

If the glory of "quantum computers" has been overblown, the advance in our understanding and appreciation of quantum entanglement can be hardly overstated. It had an impact even on the public awareness of the significance of quantum theory. Here is how Jeremy Bernstein answers his question "why people who seem to have an aversion to more conventional science are drawn to the quantum theory?" He believes that *the present widespread interest in the quan-*

tum theory can be traced to a single paper with the nontransparent title "On the Einstein-Podolsky-Rosen Paradox", which was written in 1964 by the then thirty-four-year-old Irish physicist John Bell. It was published in the obscure journal Physics, which expired after a few issues. ([8] p. 7). "The philosophical discussions of the old outsiders" (Einstein, Bohr, Schrödinger) lead to a new development in quantum physics. The categorical opinion, expressed by Lev Landau (1908-1968), the leader of the Moscow school of theoretical physics after World War II, that "quantum mechanics was completed by 1930 and was only questioned later by crackpots", was shared by the majority of active physicists worldwide. In his famous *Lectures on Physics*, published in 1963 Feynman writes that all the 'mystery' of Quantum Mechanics is in the wave-particle duality and finds nothing special in the EPR situation. It took him another 20 years (and the work of Bohm, Bell, Clauser, Shimony, Aspect) to realize that there was another quantum mystery...

The precise meaning of the violation of the Bell-CHSH inequalities is a matter of continuing discussion. In fact, the framework of nonrelativistic quantum mechanics is not appropriate for testing relativistic locality and causality. The proper playground to discuss these concepts is relativistic quantum field theory (QFT). The great majority of authors speak of "quantum nonlocality". Indeed, the mere notion of a particle spin or energy-momentum in QFT requires integrating a conserved current over an entire 3-dimensional hypersurface. Shimony recalls [34] that Arthur Wightman (1922-2013) asked him "to read the paper by Einstein, Podolsky and Rosen on an argument for hidden variables, and find out what's wrong with the argument". Shimony did not find anything wrong in the argument but later figured out that the EPR framework was not appropriate to test relativistic locality. Bell [6] realized that local commutativity of quantum fields is consistent with the entanglement (and hence with a violation of what he calls "local causality of quantum beables" – but not with sending faster than light [information carrying] signals). Twelve years later it was demonstrated [35] that *maximal violation Bell's inequality is generic in (local) quantum field theory*. The continued unqualified talk of violation of locality in quantum physics provoked S. Doplicher [16] to reiterate, after another 22 years, that there is no EPR paradox in the measurement process in local quantum field theory. His careful treatment of the subject seems to be ignored and the discussion is still going on unconstrained (see [12, 19, 26] among many others).

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