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Invariant Differential Operators for Non-Compact Lie Groups: an Introduction*

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Abstract. In the present paper we review the progress of the project of classification and construction of invariant differential operators for non-compact semisimple Lie groups. Our starting points is the class of algebras, which we called earlier 'conformal Lie algebras' (CLA), which have very similar properties to the conformal algebras of Minkowski space-time, though our aim is to go beyond this class in a natural way. For this we introduced recently the new notion of *parabolic relation* between two non-compact semisimple Lie algebras $\mathcal{G}$ and $\mathcal{G}'$ that have the same complexification and possess maximal parabolic subalgebras with the same complexification. In the present paper we consider in detail the orthogonal algebras $so(p,q)$ all of which are parabolically related to the conformal algebra $so(n,2)$ with $p+q=n+2$, the parabolic subalgebras including the Lorentz subalgebra $so(n-1,1)$ and its analogs $so(p-1,q-1)$.

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1 Introduction

Invariant differential operators play very important role in the description of physical symmetries – starting from the early occurrences in the Maxwell, d'Allembert, Dirac, equations, to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory (for reviews, cf. e.g., [1, 2]). Thus, it is important for the applications in physics to study systematically such operators. For other relevant references cf., e.g., [3–6], and others throughout the text.

In a recent paper [7] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the *parabolic subgroups and subalgebras* from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

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Since the study and description of detailed classification should be done group by group we had to decide which groups to study. One first choice would be non-compact groups that have discrete series of representations. By the Harish-Chandra criterion [8] these are groups where holds

$$\mathrm{rank}\, G = \mathrm{rank}\, K,$$

where K is the *maximal compact subgroup* of the non-compact group G. Another formulation is to say that the Lie algebra $\mathcal{G}$ of G has a compact Cartan subalgebra.

Example : the groups $SO(p,q)$ have discrete series, *except* when both p,q are *odd* numbers.

This class is rather big, thus, we decided to consider a subclass, namely, the class of *Hermitian symmetric spaces*. The practical criterion is that in these cases, the *maximal compact subalgebra* $\mathcal{K}$ is of the form

$$\mathcal{K} = so(2) \oplus \mathcal{K}' \ . \tag{1}$$

The Lie algebras from this class are

$$so(n,2), \ \ sp(n,R), \ \ su(m,n), \ \ so^*(2n), \ \ E_{6(-14)} \ , \ \ E_{7(-25)} \tag{2}$$

These groups/algebras have *highest/lowest weight representations*, and relatedly *holomorphic discrete series representations*.

The most widely used of these algebras are the *conformal algebras* $so(n,2)$ in n-dimensional Minkowski space-time. In that case, there is a maximal *Bruhat decomposition* [9] that has direct physical meaning

$$\begin{aligned} & so(n,2) = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}} \ , \\ & \mathcal{M} = so(n-1,1) \ , \ \ \dim \mathcal{A} = 1, \ \ \dim \mathcal{N} = \dim \tilde{\mathcal{N}} = n \ , \end{aligned} \tag{3}$$

where $so(n-1,1)$ is the *Lorentz algebra* of n-dimensional Minkowski space-time, the subalgebra $\mathcal{A} = so(1,1)$ represents the *dilatations*, the conjugated subalgebras $\mathcal{N}, \tilde{\mathcal{N}}$ are the algebras of *translations*, and *special conformal transformations*, both being isomorphic to n-dimensional Minkowski space-time.

The subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ ($\cong \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$) is a *maximal parabolic subalgebra*.

There are other special features which are important. In particular, the complexification of the maximal compact subgroup is isomorphic to the complexification of the first two factors of the Bruhat decomposition

$$\mathcal{K}^{\mathbb{C}} = so(n,\mathbb{C}) \oplus so(2,\mathbb{C}) \cong so(n-1,1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}} \ . \tag{4}$$

In particular, the coincidence of the complexification of the semi-simple subalgebras

$$\mathcal{K}'^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \qquad (*)$$

means that the sets of finite-dimensional (nonunitary) representations of $\mathcal{M}$ are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $\mathcal{K}'$. The latter leads to the fact that the corresponding induced representations are representations of finite $\mathcal{K}$-type [8].

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of $so(n,2)$. This subclass consists of:

$$so(n,2), \;\; sp(n,\mathbb{R}), \;\; su(n,n), \;\; so^*(4n), \;\; E_{7(-25)} \qquad (5)$$

the corresponding analogs of Minkowski space-time V being

$$\mathbb{R}^{n-1,1}, \;\; \mathrm{Sym}(n,\mathbb{R}), \;\; \mathrm{Herm}(n,\mathbb{C}), \;\; \mathrm{Herm}(n,\mathbb{Q}), \;\; \mathrm{Herm}(3,\mathbb{O})\,. \qquad (6)$$

In view of applications to physics, we proposed to call these algebras *conformal Lie algebras*, (or groups).

We have started the study of the above class in the framework of the present approach in the cases: $so(n,2)$ [10], $su(n,n)$ [11], $sp(n,\mathbb{R})$ [12], $E_{7(-25)}$ [13], $so^*(12)$ [14], resp., and we have considered also the algebra $E_{6(-14)}$ [15].

Lately, we discovered an efficient way to extend our considerations beyond this class introducing the notion of 'parabolically related non-compact semisimple Lie algebras' [16].

• ***Definition*** : Let $\mathcal{G},\mathcal{G}'$ be two non-compact semisimple Lie algebras with the same complexification $\mathcal{G}^{\mathbb{C}} \cong \mathcal{G}'^{\mathbb{C}}$. We call them *parabolically related* if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, such that: $\mathcal{M}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}}$ ($\Rightarrow \mathcal{P}^{\mathbb{C}} \cong \mathcal{P}'^{\mathbb{C}}$). $\diamondsuit$

Certainly, there are many such parabolic relationships for any given algebra $\mathcal{G}$. Furthermore, two algebras $\mathcal{G},\mathcal{G}'$ may be parabolically related via different parabolic subalgebras.

We summarize the algebras parabolically related to conformal Lie algebras with maximal parabolics fulfilling $(*)$ in Table 1, where we display only the semisimple part $\mathcal{K}'$ of $\mathcal{K}$; $sl(n,\mathbb{C})_{\mathbb{R}}$ denotes $sl(n,\mathbb{C})$ as a real Lie algebra, (thus, $(sl(n,\mathbb{C})_{\mathbb{R}})^{\mathbb{C}} = sl(n,\mathbb{C}) \oplus sl(n,\mathbb{C})$); e_6 denotes the compact real form of E_6 ; and we have imposed restrictions to avoid coincidences or degeneracies due to well known isomorphisms: $so(1,2) \cong sp(1,\mathbb{R}) \cong su(1,1)$, $so(2,2) \cong so(1,2) \oplus so(1,2)$, $su(2,2) \cong so(4,2)$, $sp(2,\mathbb{R}) \cong so(3,2)$, $so^*(4) \cong so(3) \oplus so(2,1)$, $so^*(8) \cong so(6,2)$.

After this extended introduction we give the outline of the paper. In Section 2 we give the preliminaries, actually recalling and adapting facts from [7]. In Section 3 we consider the case of the pseudo-orthogonal algebras $so(p,q)$ which are parabolically related to the conformal algebra $so(n,2)$ for $p+q=n+2$ [17].

Table 1. Conformal Lie algebras (CLA) $\mathcal{G}$ with $\mathcal{M}$-factor fulfilling $(*)$ and the corresponding parabolically related algebras $\mathcal{G}'$

$\mathcal{G}$	$\mathcal{K}'$	$\mathcal{M}$ $\dim V$	$\mathcal{G}'$	$\mathcal{M}'$
$so(n,2)$ $n \geq 3$	$so(n)$	$so(n-1,1)$ n	$so(p,q)$ $p+q=n+2$	$so(p-1,q-1)$
$su(n,n)$ $n \geq 3$	$su(n) \oplus su(n)$	$sl(n,\mathbb{C})_{\mathbb{R}}$ n^2	$sl(2n,\mathbb{R})$ $su^*(2n), n=2k$	$sl(n,\mathbb{R}) \oplus sl(n,\mathbb{R})$ $su^*(2k) \oplus su^*(2k)$
$sp(2r,\mathbb{R})$ $\mathrm{rank} = 2r \geq 4$	$su(2r)$	$sl(2r,\mathbb{R})$ $r(2r+1)$	$sp(r,r)$	$su^*(2r)$
$so^*(4n)$ $n \geq 3$	$su(2n)$	$su^*(2n)$ $n(2n-1)$	$so(2n,2n)$	$sl(2n,\mathbb{R})$
$E_{7(-25)}$	e_6	$E_{6(-26)}$ 27	$E_{7(7)}$	$E_{6(6)}$

2 Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an *Iwasawa decomposition* $G = KA_0N_0$, where A_0 is Abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the subgroup $P_0 = M_0A_0N_0$ is a *minimal parabolic subgroup* of G. A *parabolic subgroup* $P = M'A'N'$ is any subgroup of G which contains a minimal parabolic subgroup.

Further, let $\mathcal{G}, \mathcal{K}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$ denote the Lie algebras of G, K, P, M, A, N, resp.

For our purposes we need to restrict to *maximal parabolic subgroups* $P = MAN$, i.e. $\mathrm{rank}\, A = 1$, resp. to *maximal parabolic subalgebras* $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ with $\dim \mathcal{A} = 1$.

Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$, parameterized by a real number d, called the *conformal weight* or energy.

Further, let μ fix a discrete series representation D^μ of M on the Hilbert space V_μ, or the finite-dimensional (non-unitary) representation of M with the same Casimirs.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [18]. (These are called *generalized principal series representations* (or *limits thereof*) in [19].) Their spaces of functions are

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) | \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\, \mathcal{F}(g)\}, \qquad (7)$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation action is the *left regular action*

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G\,. \qquad (8)$$

• An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}^{\mathbb{C}}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [20].

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu\, v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

• Another main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [20]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the *vertices* of which correspond to the reducible ERs and the *lines (arrows)* between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation. The notion of multiplets was introduced in [21, 22] and applied to representations of $SO_o(p,q)$ and $SU(2,2)$, resp., induced from their minimal parabolic subalgebras. Then it was applied to (infinite-dimensional) (super-)algebras, quantum groups and other symmetry objects.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the *BGG Verma module reducibility condition* (for highest weight modules) is fulfilled

$$(\Lambda + \rho, \beta^\vee) = m\,, \quad \beta^\vee \equiv 2\beta/(\beta, \beta)\,. \qquad (9)$$

ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. When the above holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [23]. More explicitly, [20], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs). Then there exists [20] an *intertwining differential operator*

$$\mathcal{D}_{m,\beta} : \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{10}$$

given explicitly by

$$\mathcal{D}_{m,\beta} = \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-})\,, \tag{11}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right action* on the functions $\mathcal{F}$.

In most of these situations the invariant operator $\mathcal{D}_{m,\beta}$ has a non-trivial invariant kernel in which a subrepresentation of $\mathcal{G}$ is realized. Thus, studying the equations with trivial RHS

$$\mathcal{D}_{m,\beta}\, f = 0\,, \qquad f \in \mathcal{C}_{\chi(\Lambda)} \tag{12}$$

is also very important. For example, in many physical applications in the case of first order differential operators, i.e., for $m = m_\beta = 1$, these equations are called *conservation laws*, and the elements $f \in \ker \mathcal{D}_{m,\beta}$ are called *conserved currents*.

The above construction works also for the *subsingular vectors* v_{ssv} of Verma modules. Such a vector is also expressed by a polynomial $\mathcal{P}_{ssv}(\mathcal{G}^-)$ in the universal enveloping algebra $v^s_{ssv} = \mathcal{P}_{ssv}(\mathcal{G}^-)\, v_0$, cf. [24]. Thus, there exists a *conditionally invariant differential operator* given explicitly by $\mathcal{D}_{ssv} = \mathcal{P}_{ssv}(\widehat{\mathcal{G}^-})$, and a *conditionally invariant differential equation*, for many more details, see [24]. (Note that these operators (equations) are not of first order.)

Below in our exposition we shall use the so-called Dynkin labels

$$m_i \equiv (\Lambda + \rho, \alpha_i^\vee)\,, \quad i = 1, \ldots, n\,, \tag{13}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$.

We shall use also the so-called Harish-Chandra parameters

$$m_\beta \equiv (\Lambda + \rho, \beta)\,, \tag{14}$$

where β is any positive root of $\mathcal{G}^{\mathbb{C}}$. These parameters are redundant, since they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms. (Clearly, both the Dynkin labels and Harish-Chandra parameters have their origin in the BGG reducibility condition (9).)

3 Conformal Algebras $so(n,2)$ and Parabolically Related

Let $\mathcal{G} = so(n,2)$, $n > 2$. We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\begin{aligned} &\chi = \{n_1, \dots, n_{\tilde{h}}\,;\, c\}, \quad n_j \in \mathbb{Z}/2\,, \quad c = d - \tfrac{n}{2}\,, \quad \tilde{h} \equiv [\tfrac{n}{2}] \\ &|n_1| < n_2 < \cdots < n_{\tilde{h}}\,, \quad n \ \ \text{even}\,, \\ &0 < n_1 < n_2 < \cdots < n_{\tilde{h}}\,, \quad n \ \ \text{odd}\,, \end{aligned} \tag{15}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first $\tilde{h}$ entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M} \cong so(n-1,1)$.

The reason to use the parameter c instead of d is that the parametrization of the ERs in the multiplets is given in a simple intuitive way (cf. [10, 25])

$$\begin{aligned} \chi_1^{\pm} &= \{\epsilon n_1\,, \dots,\, n_{\tilde{h}}\,;\, \pm n_{\tilde{h}+1}\}\,, \quad n_{\tilde{h}} < n_{\tilde{h}+1}\,, \\ \chi_2^{\pm} &= \{\epsilon n_1\,, \dots,\, n_{\tilde{h}-1}\,,\, n_{\tilde{h}+1}\,;\, \pm n_{\tilde{h}}\} \\ \chi_3^{\pm} &= \{\epsilon n_1, \dots, n_{\tilde{h}-2}, n_{\tilde{h}}, n_{\tilde{h}+1}\,;\, \pm n_{\tilde{h}-1}\} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \chi_{\tilde{h}}^{\pm} &= \{\epsilon n_1\,, n_3\,, \dots,\, n_{\tilde{h}}\,,\, n_{\tilde{h}+1}\,;\, \pm n_2\} \\ \chi_{\tilde{h}+1}^{\pm} &= \{\epsilon n_2\,, \dots,\, n_{\tilde{h}}\,,\, n_{\tilde{h}+1}\,;\, \pm n_1\} \\ \epsilon &= \begin{cases} \pm\,, & n \ \ \text{even} \\ 1, & n \ \ \text{odd} \end{cases} \end{aligned} \tag{16}$$

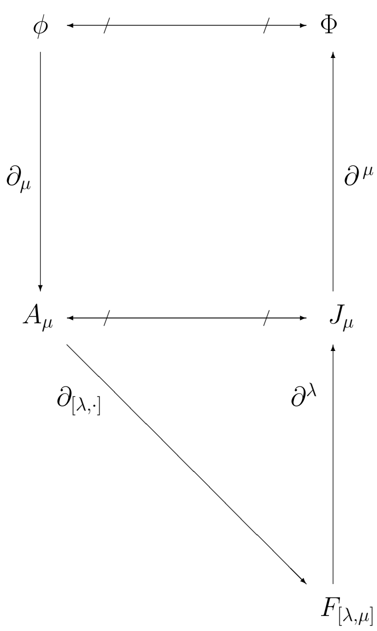

Figure 1. Simplest example of diagram with conformal invariant operators (arrows are differential operators, dashed arrows are integral operators).

$\partial_\mu = \dfrac{\partial}{\partial \mu}$,

A_μ – electromagnetic potential,

$\partial_\mu \phi = A_\mu$,

F – electromagnetic field,

$\partial_{[\lambda, A_\mu]} = \partial_\lambda A_\mu - \partial_\mu A_\lambda = F_{\lambda\mu}$,

J_μ – electromagnetic current,

$\partial^\mu F_{\lambda\mu} = J_\mu$, $\partial^\mu J_\mu = \Phi$.

Further, we denote by $\tilde{\mathcal{C}}_i^{\pm}$ the representation space with signature $\chi_i^{\pm}$.

The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})| = 2(1+\tilde{h})\,, \qquad (17)$$

where $\mathcal{H}^{\mathbb{C}}$, $\mathcal{H}_m^{\mathbb{C}}$ are Cartan subalgebras of $\mathcal{G}^{\mathbb{C}}$, $\mathcal{M}^{\mathbb{C}}$, resp. This formula is valid for the main multiplets of all conformal Lie algebras.

We show some examples of diagrams of invariant differential operators for the conformal groups $so(5,1)$, resp. $so(4,2)$, in 4-dimensional Euclidean, resp. Minkowski, space-time. In Figure 1 we show the simplest example for the most common using well known operators. In Figure 2 we show the same example but using the group-theoretical parity splitting of the electromagnetic current, cf. [26]. In Figure 3 we show the general classification for $so(5,1)$ given in [26]. These diagrams are valid also for $so(4,2)$ [27] and for $so(3,3) \cong sl(4,\mathbb{R})$ [16].

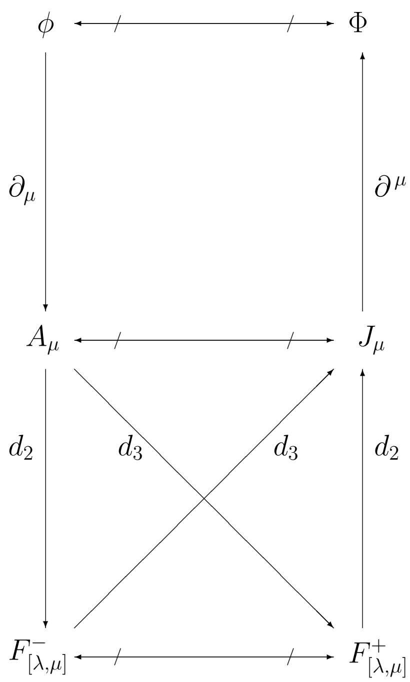

Figure 2. More precise showing of the simplest example, $F = F^+ \oplus F^-$ shows the parity splitting of the electromagnetic field, d_2, d_3 – linear invariant operators.

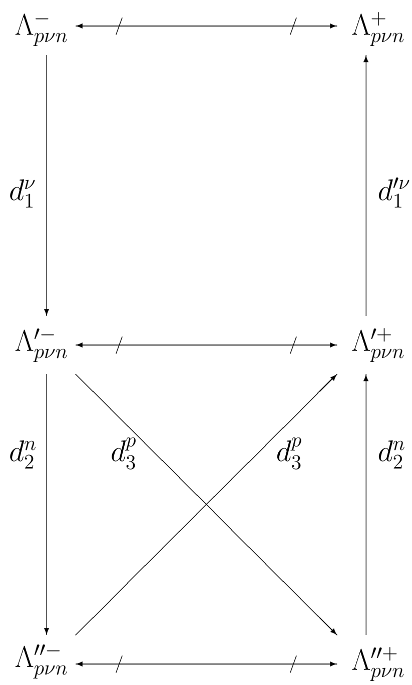

Figure 3. The general classification of invariant differential operators valid for $so(4,2)$, $so(5,1)$, and $so(3,3) \cong sl(4,\mathbb{R})$. p,ν,n are three natural numbers, the shown simplest case is when $p=\nu=n=1$, d_1^ν is a linear differential operator of order ν, similarly d'^{ν}_1, d_2^n, d_3^p.

Next in Figure 4 we show the general even case $so(p,q)$, $p+q = 2h+2$, [10,25], while in Figure 5. we show an alternative view of the same case.

Next in Figure 6 we show the general odd case $so(p,q)$, $p+q = 2h+3$, [10,25], while in Figure 7 we show an alternative view of the same case.

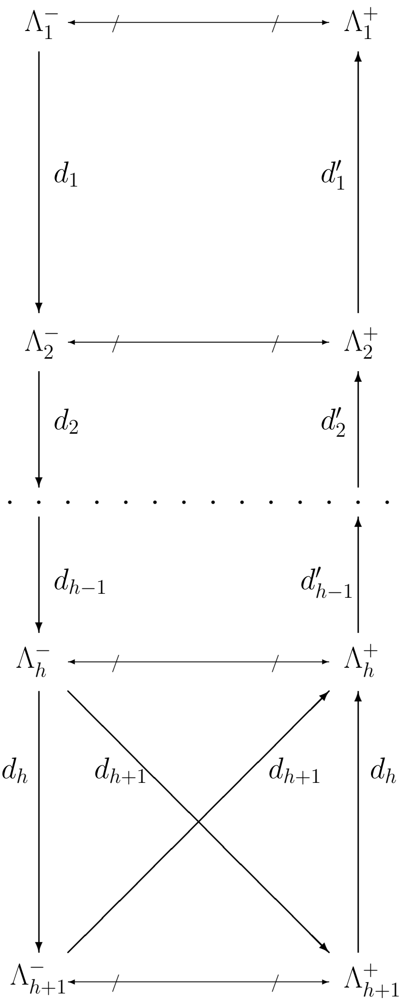

Figure 4. The general classification of invariant differential operators in $2h$-dimensional space-time. By parabolic relation the diagram above is valid for all algebras $so(p,q)$, $p+q = 2h+2$, even.

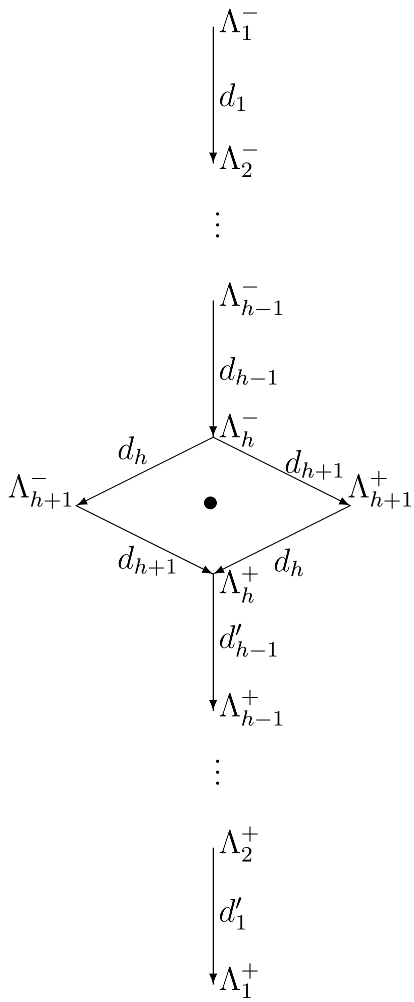

Figure 5. Alternative showing of the case $so(p,q)$, $p+q = 2h+2$, showing only the differential operators, while the integral operators are assumed as symmetry w.r.t. the bullet in the centre.

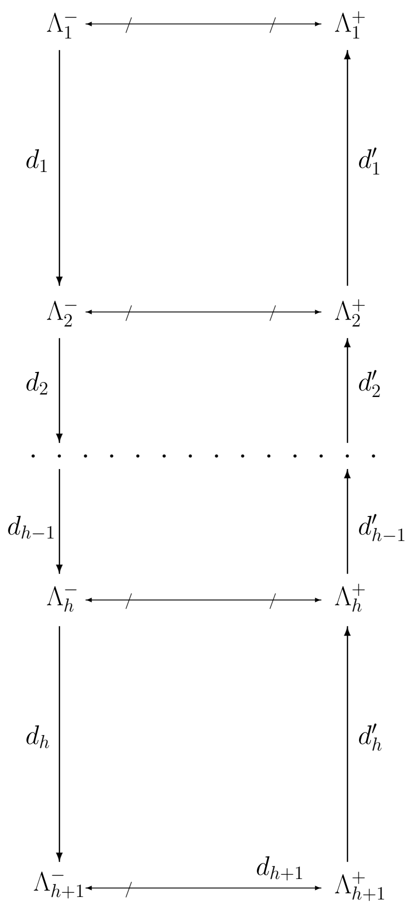

Figure 6. The general classification of invariant differential operators in $2h+1$ dimensional space-time. By parabolic relation the diagram above is valid for all algebras $so(p,q)$, $p+q=2h+3$, odd.

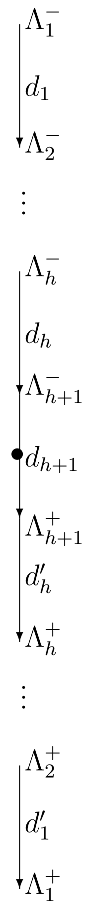

Figure 7. Alternative showing of the case $so(p,q)$, $p+q=2h+3$, showing only the differential operators, while the integral operators are assumed as symmetry w.r.t. the bullet in the centre.

The ERs in the multiplet are related by *intertwining integral and differential operators*. The *integral operators* were introduced by Knapp and Stein [28]. They correspond to elements of the restricted Weyl group of $\mathcal{G}$. These operators intertwine the pairs $\tilde{\mathcal{C}}_i^{\pm}$

$$G_i^{\pm} : \tilde{\mathcal{C}}_i^{\mp} \longrightarrow \tilde{\mathcal{C}}_i^{\pm} , \quad i = 1, \ldots, 1+\tilde{h} \tag{18}$$

The *intertwining differential operators* correspond to non-compact positive roots of the root system of $so(n+2,\mathbb{C})$, cf. [20]. [In the current context, compact roots of $so(n+2,\mathbb{C})$ are those that are roots also of the subalgebra $so(n,\mathbb{C})$, the rest of the roots are non-compact.] The degrees of these intertwining differential operators are given just by the differences of the c entries [25]

$$\begin{aligned} &\deg d_i = \deg d'_i = n_{\tilde{h}+2-i} - n_{\tilde{h}+1-i}\,, \qquad i = 1,\ldots,\tilde{h}\,, \quad \forall n \\ &\deg d_{\tilde{h}+1} = n_2 + n_1\,, \quad n \ \text{even} \end{aligned} \tag{19}$$

where d'_h is omitted from the first line for $(p+q)$ even.

Matters are arranged so that in every multiplet only the ER with signature χ_1^- contains a *finite-dimensional nonunitary subrepresentation* in a subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional unitary irrep of $so(n+2)$ with signature $\{n_1\,,\ldots,\,n_{\tilde{h}}\,,\,n_{\tilde{h}+1}\}$. The subspace $\mathcal{E}$ is annihilated by the operator G_1^+, and is the image of the operator G_1^-.

Although the diagrams are valid for arbitrary $so(p,q)$ ($p+q \geq 5$) the contents is very different. We comment only on the ER with signature χ_1^+. In all cases it contains an UIR of $so(p,q)$ realized on an invariant subspace $\mathcal{D}$ of the ER χ_1^+. That subspace is annihilated by the operator G_1^-, and is the image of the operator G_1^+. (Other ERs contain more UIRs.)

If $pq \in 2\mathbb{N}$ the mentioned UIR is a discrete series representation. (Other ERs contain more discrete series UIRs.)

And if $q = 2$ the invariant subspace $\mathcal{D}$ is the direct sum of two subspaces $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$, in which are realized a *holomorphic discrete series representation* and its conjugate *anti-holomorphic discrete series representation*, resp. Note that the corresponding *lowest weight GVM* is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate *highest weight GVM* is infinitesimally equivalent to the anti-holomorphic discrete series.

Note that the $\deg d_i$, $\deg d'_i$, are Harish-Chandra parameters corresponding to the non-compact positive roots of $so(n+2,\mathbb{C})$. From these, only $\deg d_1$ corresponds to a simple root, i.e., is a Dynkin label.

Above we considered $so(n,2)$ for $n>2$. The case $n=2$ is reduced to $n=1$ since $so(2,2) \cong so(1,2) \oplus so(1,2)$. The case $so(1,2)$ is special and must be treated separately. But in fact, it is contained in what we presented already. In that case the multiplets contain only *two ERs* which may be depicted by the *top pair* $\chi_1^\pm$ in the pictures that we presented. And they have the properties that we described for $so(n,2)$ with $n>2$. The case $so(1,2)$ was given already in 1946-7 independently by Gel'fand et al [29] and Bargmann [30].

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