Bulg. J. Phys. 42 (2015) 429–438

Exact Solutions of the Extended Pairing Interactions in Bose and Fermi Many-Body Systems

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Received *7 october 2015*

Abstract. A solvable extended Hamiltonian that includes multi-pair interactions among s- and d-bosons up to infinite order within the framework of the interacting boson model is proposed to gain a better description of E(5) model results for finite-N systems. Similarly, an extended pairing Hamiltonian to describe pairing interactions among valence nucleon monopole pairs up to infinite order in a spherical mean-field can also be built, which includes the extended pairing interaction model within a deformed mean-field theory as a special case. These mean-field plus pairing models are all constructed based on the local $\tilde{E}_2$ algebraic structure.

PACS codes: 21.60.-n

1 Introduction

In atomic nuclei, the quantum phase transitions are often related to different geometrical shapes of the system, which can be described either by the Bohr-Mottelson model (BMM) [1] or by the Interacting Boson Model (IBM) [2]. With suitable simplification, Iachello proposed an exactly solvable model [3] within the BMM, which is called the E(5) critical point symmetry model suitable to describe the critical phenomena in the vibration to γ-unstable (shape) phase transition. Inspired by the E(5) model, Lévai and Arias studied the Bohr Hamiltonian with a sextic potential and a centrifugal barrier, of which quasi-exact solutions can be derived [4], while Bonatsos *et al* explored numerical solutions for the γ-independent Bohr Hamiltonian with β^{2n} potentials for $n \geq 1$ called the confined γ-soft rotor model [5], in which the spectra and transition rates for the β^{2n} potentials for $2 \leq n \leq 4$ are given explicitly and compared with the original E(5) model. In the IBM, usually, the $U(5)$-$O(6)$ transition is described by the consistent-Q Hamiltonian, of which the solutions at the critical point seem similar to those of the E(5) model. However, it has been shown that only lower lying part of the spectrum produced from the consistent-Q formulism is similar to that of the E(5) model, while eigenenergies of higher lying states in the consistent-Q formulism are suppressed in comparison to the E(5) results.

On the other hand, it is well known that the monopole pairing interaction is one of the important residual interactions in any nuclear mean-field theory [6], which is the key to elucidate ground state and low-energy spectroscopic properties of nuclei, such as binding energies, odd-even effects, single-particle occupancies, excitation spectra, and moments of inertia, etc. Though the BCS and the more refined HFB approximations provide simple and clear pictures in demonstrating pairing correlations in nuclei [7,8], tremendous efforts have been made in finding accurate solutions to the problem to overcome serious drawbacks in the BCS and the HFB, such as spurious states, nonorthogonal solutions, etc resulting from particle number-nonconservation effects in these approximations [9, 10]. It is known that either spherical or deformed mean-field plus the standard pairing interaction can be solved exactly by using the Gaudin-Richardson method [11]. It has been shown that the set of Gaudin-Richardson equations can be solved more easily by using the extended Heine-Stieltjes polynomial approach [12,13]. However, numerical work in finding roots of the Gaudin-Richardson equations increases with the increasing of the number of orbits and the number of valence nucleon pairs, which limits the application of the theory within a shell model subspace. In the deformed case, the deformed mean-field plus the extended pairing can be solved more easily than the standard pairing case, especially when both the number of valence nucleon pairs and the number of single-particle orbits are large [14]. However, the extended pairing interaction proposed in [14] is within a deformed mean-field theory, which can not be used directly to describe pairing interactions among valence nucleon monopole pairs in spherical mean-field theories, in which the total angular momentum of the system is always conserved.

In this talk, it will be shown that a Hamiltonian of a mean-field plus the extended pairing interactions up to infinite order in either the Bose or Fermi case mentioned above can be used to describe either the E(5)-like critical point symmetry within the IBM or pairing interactions among valence nucleon monopole pairs in a spherical mean-field similar to the deformed case [14], of which the pairing interactions are constructed from local boson or fermion operators with $\tilde{E}_2$ algebraic structure.

2 A Solvable Hamiltonian Near the $U(5)$-$O(6)$ Critical Point

Similar to the well-known consistent-Q formalism in the IBM [2, 16, 17], using up to two-body interactions the $U(5)$-$O(6)$ Hamiltonian may be schematically written as [15–18]

$$\hat{H} = g\left(x\hat{n}_d + \frac{1-x}{N}\hat{P}^\dagger\hat{P}\right), \tag{1}$$

where g is a real parameter, the control parameter $x \in [0,1]$, $\hat{n}_d = \sum_\mu d_\mu^\dagger d_\mu$ is the d-boson number operator, and

$$P^\dagger = \frac{1}{2}(d^\dagger \cdot d^\dagger - s^{\dagger 2}) \tag{2}$$

is the boson pairing operator. In order to clarify the structure of the $U(5)$-$O(6)$ transitional solutions, similar to [15–17], we introduce the s- and d-boson SU(1,1) pairing algebras with

$$S_d^+ = (S_d^-)^\dagger = \frac{1}{2}d^\dagger \cdot d^\dagger,\ S_d^0 = \frac{1}{2}\hat{n}_d + \frac{5}{4}, \tag{3}$$

$$S_s^+ = (S_s^-)^\dagger = \frac{1}{2}s^{\dagger 2},\ S_s^0 = \frac{1}{2}\hat{n}_s + \frac{1}{4}, \tag{4}$$

where $\hat{n}_s$ are the number operator for s-bosons, which satisfy the following commutation relations:

$$[S_\sigma^+, S_\rho^-] = -2\delta_{\sigma\rho}S_s^0,\ \ [S_\sigma^0, S_\rho^\pm] = \pm\delta_{\sigma\rho}S_\rho^\pm, \tag{5}$$

where ρ stands for d or s. Thus, the pairing operators appearing in (1) can be regarded as the results of the SU(1,1) coupling:

$$P^\dagger = S_d^+ - S_s^+,\ P = S_d^- - S_s^-, \tag{6}$$

which, together with

$$P^0 = S_d^0 + S_s^0, \tag{7}$$

generate SU$_{sd}$(1,1) algebra. Under the $U(6) \supset U(5) \supset O(5) \supset O(3)$ basis, the generators of either $SU_d(1,1)$ or $SU_s(1,1)$ commute with all generators of $O(5)$. Therefore, the basis vectors of $U(6) \supset U(5) \supset O(5) \supset O(3)$ are simultaneously the basis vectors of $SU_d(1,1)$. The orthonormalized basis vectors of $U(6) \supset U(5) \supset O(5) \supset O(3)$ can be equivalently expressed as those of SU$_d$(1,1)$\otimes$SU$_s$(1,1) with

$$|\xi\rangle \equiv |\tau_s;\xi;\tau\alpha LM\rangle = \mathcal{N}(S_s^+)^{\frac{N-\tau-\tau_s}{2}-\xi}(S_d^+)^\xi|\tau_s;\tau\alpha LM\rangle, \tag{8}$$

of which $n_d = 2\xi + \tau$, where N is the total number of bosons, n_d is the number of d-bosons, $\tau_s = 0$ or 1 is the seniority number of s-bosons, τ is the seniority number of d-bosons labeling the irrep of $O(5)$, L is the angular momentum quantum number of $O(3)$, α is an additional quantum number needed to distinguish different states with the same L, M is the quantum number of the third component of the angular momentum, and the normalization constant

$$\mathcal{N} = \left(\frac{2^{N-\tau-\tau_s-2\xi}(2\tau+3)!!}{\xi!(N-\tau-2\xi)!(2\tau+2\xi+3)!!}\right)^{\frac{1}{2}}, \tag{9}$$

in which $\xi = 0, 1, 2, \cdots, \frac{1}{2}(N - \tau - \tau_s)$. The matrix representations of $SU_d(1,1)$ and $SU_s(1,1)$ under the basis vectors (8) are given by

$$\begin{aligned} S_d^+|\xi\rangle &= \frac{1}{2}\sqrt{(2\xi+2)(2\tau+2\xi+5)}|\xi+1\rangle, \\ S_d^-|\xi\rangle &= \frac{1}{2}\sqrt{2\xi(2\tau+2\xi+3)}|\xi-1\rangle, \\ S_d^0|\xi\rangle &= \frac{1}{2}(\tau+2\xi+\frac{5}{2})|\xi\rangle, \end{aligned} \tag{10}$$

and

$$\begin{aligned} S_s^+|\xi\rangle &= \frac{1}{2}\sqrt{(N-\tau-2\xi)(N-\tau-2\xi-1)}|\xi+1\rangle, \\ S_s^-|\xi\rangle &= \frac{1}{2}\sqrt{(N-\tau-2\xi+2)(N-\tau-2\xi+1)}|\xi-1\rangle, \\ S_s^0|\xi\rangle &= \frac{1}{2}(N-\tau-2\xi+\frac{1}{2})|\xi\rangle. \end{aligned} \tag{11}$$

Thus, $P^\dagger P$ under the basis vectors (8) is tridiagonal. The matrix elements of the diagonal part can be expressed as

$$\langle\xi|(S_d^+S_d^- + S_s^+S_s^-)|\xi\rangle = \frac{1}{4}(2\xi)(2\tau+2\xi+3) + \frac{1}{4}(N-\tau-2\xi)(N-\tau-2\xi-1), \tag{12}$$

while those of the nonzero non-diagonal parts are given by

$$\langle\xi+1|S_d^+S_s^-|\xi\rangle = \frac{1}{4}((N-\tau-2\xi)(N-\tau-2\xi-1)(2\xi+2)(2\tau+2\xi+5))^{\frac{1}{2}}, \tag{13}$$

$$\langle\xi-1|S_s^+S_d^-|\xi\rangle = \frac{1}{4}((N-\tau-2\xi+2)(N-\tau-2\xi+1)(2\xi)(2\tau+2\xi+3))^{\frac{1}{2}}. \tag{14}$$

It is clear that the U(5)-O(6) transitional phase is mainly driven by the nonzero non-diagonal part of $P^\dagger P$.

Using the $SU_d(1,1)$ and $SU_s(1,1)$ generators, we can build an extended IBM Hamiltonian (EXT) as

$$\hat{H}_c = \Delta\hat{n}_d + \frac{\lambda}{N}(S_s^+S_s^- + S_d^+S_d^-) - g_2\sum_{k=1}^{\infty}(\tilde{S}_s^{+k}\tilde{S}_d^{-k} + \tilde{S}_d^{+k}\tilde{S}_s^{-k}), \tag{15}$$

where $\Delta = \epsilon_d - \epsilon_s > 0$ is the energy gap of s- and d-bosons, $\lambda > 0$ and $g_2 > 0$ are real parameters. Obviously, (15) becomes (1) when $g_2 = \lambda/N$ and only the $k = 1$ term with the replacement $\tilde{S}_\rho^\pm \to S_\rho^\pm$ is included in the third term of (15). The second term in (15) is the same as the diagonal part of the boson pairing interactions included in (1), while the third term contributes to the non-diagonal part of the boson pairing interactions but not restricted with the tridiagonal form shown by (13) and (14) when it is diagonalized within the subspace spanned by basis vectors shown in (8), in which

$$\tilde{S}_d^+ = S_d^+ \frac{1}{\sqrt{(S_d+S_d^0)(S_d^0-S_d+1)}}, \quad \tilde{S}_d^- = \left(S_d^+\right)^\dagger,$$
$$\tilde{S}_s^+ = S_s^+ \frac{1}{\sqrt{(S_s+S_s^0)(S_s^0-S_s+1)}}, \quad \tilde{S}_s^- = \left(S_s^+\right)^\dagger. \tag{16}$$

Instead of the operators $\{\tilde{S}_\rho^\pm\}$, one can also use the usual boson pairing operators $\{S_\rho^\pm\}$ to construct the multi-pair interactions similar to the third term of (15). In this case, the problem, however, is no longer exactly solvable. The operators appearing in (16) are well-defined under the basis vectors (8). In contrast to (5), the operators $\{\tilde{S}_d^+, \tilde{S}_d^-, S_d^0\}$ and $\{\tilde{S}_s^+, \tilde{S}_s^-, S_s^0\}$ under the corresponding SU(1,1) irreps satisfy the following commutation relations:

$$[\tilde{S}_\sigma^+, \tilde{S}_\rho^-] = -\delta_{\rho\sigma}\delta_{S_\rho S_\rho^0}, \quad [S_\sigma^0, \tilde{S}_\rho^\pm] = \pm\delta_{\sigma\rho} S_\rho^\pm. \tag{17}$$

Hence, $\{\tilde{S}_d^+, \tilde{S}_d^-, S_d^0\}$ and $\{\tilde{S}_s^+, \tilde{S}_s^-, S_s^0\}$ are two copies of the generators of E_2 algebra when $S_\rho^0 \neq S_\rho$, which become those of the Heisenberg algebra only when $S_\rho^0 = S_\rho$. They are called as the $\tilde{E}_2$ algebra.

Similar to the consistent-Q formalism [15] in describing the U(5)-O(6) transitional nuclei, the Hamiltonian (15) is also exactly solvable. To digonalize the Hamiltonian (15), we use the simple algebraic Bethe ansatz with

$$|N, \zeta, \tau\alpha L\rangle = \sum_{\xi=0}^{\frac{1}{2}(N-\tau-\tau_s)} C_\xi^{(\zeta)} |\xi\rangle, \tag{18}$$

where $|\xi\rangle \equiv |\tau_s; \xi; \tau\alpha LM\rangle$ as given in (8), and $C_\xi^{(\zeta)}$ is the expansion coefficient to be determined. Similar to the procedures used in [14], it can be proven that the expansion coefficient $C_\xi^{(\zeta)}$ can be expressed as

$$C_\xi^{(\zeta)} = \frac{1}{F^{(\zeta)}(\xi)}, \tag{19}$$

where

$$F^{(\zeta)}(\xi) = E_{\tau,L}^{(\zeta)} - g_2 - \frac{\lambda}{2N}\xi(2\tau + 2\xi + 3) - \frac{\lambda}{4N}(N - \tau - 2\xi)(N - \tau - 2\xi - 1) - \Delta(\tau + 2\xi), \tag{20}$$

in which $E_{\tau,L}^{(\zeta)}$ is the ζ-th eigen-energy for given τ and L. To show that (18) and (19) are indeed consistent, one may directly apply the Hamiltonian (15) on the N-particle state (18) to establish the eigen-equation $\hat{H}_c|N,\zeta\rangle = E^{(\zeta)}_{\tau,L}|N,\zeta\rangle$. After simple algebraic manipulation, one can easily find that

$$- g_2 \sum_{k=1}^{\infty} (\tilde{S}_s^{+k}\tilde{S}_d^{-k} + \tilde{S}_d^{+k}\tilde{S}_s^{-k})|N,\zeta,\tau\alpha,LM\rangle = \\ g_2|N,\zeta,\tau\alpha,LM\rangle - g_2 \sum_{\xi'} C^{\zeta}_{\xi'} \sum_{\xi} |\xi\rangle. \quad (21)$$

Once the expansion coefficients are chosen as those shown in (19), the eigen-equation $\hat{H}_c|N,\zeta\rangle = E^{(\zeta)}_{\tau,L}|N,\zeta\rangle$ is fulfilled when and only when

$$-g_2 \sum_{\xi} \frac{1}{F^{(\zeta)}(\xi)} = 1. \quad (22)$$

Solutions of (22) provide eigenvalues $E^{(\zeta)}_{\tau,L}$ and the corresponding eigenstates (18) simultaneously.

As shown in Table 1 for $N = 10$ case, not only lower lying level energies, but also higher lying level energies generated by the EXT are all close to those produced by the E(5) model, while the higher lying level energies obtained from the IBM consistent-Q Hamiltonian at the critical point with $x_c = (N-1)/(2N-1)$ (CQ) are too low in energy resulting in a suppressed spectrum in comparison to those of the E(5) model. The situation for other N cases is similar. As shown in [19] for the fits to low-lying level energies and B(E2) values of the transitions among these levels for ^{102}Pd, ^{134}Ba, ^{128}Xe, ^{104}Ru, ^{108}Pd, and ^{114,116}Cd, the overall fitting results of the EXT are the best.

Table 1. Comparison of level energies generated by the extended IBM Hamiltonian (EXT) and the IBM consistent-Q Hamiltonian at the critical point with $x_c = (N-1)/(2N-1) = 0.473684$ (CQ) for $N = 10$ bosons with those provided by the E(5) model.

$\zeta,\ \tau$	E(5)	EXT	CQ	$\zeta,\ \tau$	E(5)	EXT	CQ
1, 0	0.00	0.00	0.00	1, 6	8.88	8.53	8.03
1, 1	1.00	1.00	1.00	2, 3	8.97	8.91	7.95
1, 2	2.20	2.13	2.17	3, 1	10.11	10.31	8.95
2, 0	3.03	3.37	3.16	2, 4	11.36	11.15	9.73
1, 3	3.59	3.47	3.47	3, 2	12.85	12.93	10.91
2, 1	4.80	5.02	4.67	4, 0	13.64	13.92	11.57
1, 4	5.17	4.96	4.89	2, 5	13.95	13.60	11.57
2, 2	6.78	6.86	6.27	3, 3	15.81	15.76	12.94
3, 0	7.58	7.88	7.04	2, 6	16.73	16.21	13.57
1, 5	6.93	6.67	6.41	4, 1	16.93	17.13	13.80

3 The Extended Monopole Pairing Model

Similar to the extended pairing model for deformed nuclei [14], the Hamiltonian of a spherical mean-field plus the extended monopole pairing model may be written as

$$\hat{H} = \sum_j \epsilon_j N_j - \sum_j G_j S_j^\dagger S_j + \hat{H}_P \tag{23}$$

where $\{\epsilon_j\}$ is a set of single-particle energies generated from any spherical mean-field theory, such as those of the spherical shell model,

$$N_j = \sum_m a_{jm}^\dagger a_{jm}, \tag{24}$$

in which $a_{jm}^\dagger$ (a_{jm}) is the creation (annihilation) operator for a nucleon with angular momentum quantum number j and that of its projection m, G_j is the pairing interaction strength in the j-orbit,

$$S_j^\dagger = \sum_{m>0} (-1)^{j-m} a_{jm}^\dagger a_{j-m}^\dagger \tag{25}$$

is the monopole pair creation operator in the j-orbit,

$$\hat{H}_P = -G \sum_{k=1}^{\infty} \sum_{j_1 \le \cdots \le j_k} \tilde{S}_{j_1}^\dagger \cdots \tilde{S}_{j_k}^\dagger \sum_{j_1' \le \cdots \le j_k'} \tilde{S}_{j_1'} \cdots \tilde{S}_{j_k'}, \tag{26}$$

in which G is an overall pairing interaction strength for pairing interactions among different orbits, and two groups of orbit indices $\{j_1, \cdots, j_k\}$ and $\{j_1', \cdots, j_k'\}$ in the restricted summation run over all orbits with the restriction that no any one of $\{j_1, \cdots, j_k\}$ equals to any one of $\{j_1', \cdots, j_k'\}$, but the orbit labels in the same group can be taken as the same, which describes pair hopping from the orbits $\{j_1', \cdots, j_k'\}$ to the orbits $\{j_1, \cdots, j_k\}$ simultaneously, and

$$\tilde{S}_j^\dagger = \frac{1}{\sqrt{(Q_j - \hat{S}_j^0 + 1)(Q_j + \hat{S}_j^0)}} S_j^\dagger, \tag{27}$$

in which $Q_j = \frac{1}{2}(\Omega_j - \nu_j)$ and $\hat{S}_j^0 = \frac{1}{2}(N_j - \Omega_j)$ are the quasi-spin and the third component of the quasi-spin operator in the j-orbit, respectively, with $\Omega_j = j + \frac{1}{2}$ and the seniority number ν_j for the j-orbit. Let $|0\rangle$ be the vacuum state satisfying $a_{jm}|0\rangle = 0 \ \forall\ j$, which is nothing but the lowest weight state for the quasi-spin Q_j when seniority number is zero. For the seniority $\nu_j \neq 0$ case, the lowest weight state for the quasi-spin Q_j is denoted as $|\nu_j \rho_j\rangle$ satisfying $S_j|\nu_j \rho_j\rangle = 0$, where ρ_j stands for the angular momentum quantum number J_j and that of its third component M_{J_j}, and the multiplicity label needed to distinguish different possible ways of ν_j particles coupled to the angular momentum J_j under the anti-symmetric irreducible representation $\langle 1^{\nu_j}\rangle$ of $Sp(2j+1)\downarrow O(3)$. Namely, the lowest weight state with the quasi-spin Q_j, or called the pairing vacuum state, is

$$|Q_j, -Q_j; \nu_j\rho_j\rangle \equiv |\nu_j\rho_j\rangle, \tag{28}$$

while

$$|Q_j, -Q_j+n; \nu_j\rho_j\rangle = \tilde{S}_j^{\dagger n}|\nu_j\rho_j\rangle \tag{29}$$

with $n = 0, 1, \cdots, 2Q_j$, which satisfies

$$\begin{pmatrix} S_j^{\dagger}S_j \\ \\ S_j^0 \end{pmatrix} |Q_j, -Q_j+n; \nu_j\rho_j\rangle = \begin{pmatrix} n(Q_j-n+1) \\ \\ -Q_j+n \end{pmatrix} |Q_j, -Q_j+n; \nu_j\rho_j\rangle. \tag{30}$$

The states with n monopole pairs in the j-orbit (29) are orthonormal with respect to n, ν_j, and ρ_j as long as $|\nu_j\rho_j\rangle$ are orthonormalized. Hence, the operator S_j^0 appearing in (27) is well defined.

A set of operators $\{\tilde{S}_{j_i},\ \tilde{S}_{j_i}^{\dagger},\ N_{j_i}\}$ $(i = 1, 2, \cdots)$ under the tensor product basis constructed from (29) satisfy the commutation relations

$$[N_{j'}/2,\ \tilde{S}_j] = -\delta_{jj'}\tilde{S}_j,\ [N_{j'}/2,\ \tilde{S}_j^{\dagger}] = \delta_{jj'}\tilde{S}_j^{\dagger},\ [\tilde{S}_{j'},\ \tilde{S}_j^{\dagger}] = \delta_{jj'}\delta_{N_j 0}. \tag{31}$$

Therefore, $\{\tilde{S}_j,\ \tilde{S}_j^{\dagger},\ N_j\}$ for given j is isomorphic to the $\tilde{E}_2$ algebra.

To digonalize the Hamiltonian (23) for N monopole pairs over p orbits, we use the simple algebraic Bethe ansatz with

$$|N, \zeta\rangle = \sum_{n_1=0}^{2Q_{j_1}} \cdots \sum_{n_p=0}^{2Q_{j_p}} \delta_{\sum_{i=1}^{p} n_i, N}\, C_{n_1,\cdots,n_p}^{(\zeta)} |n_1, \cdots, n_p\rangle, \tag{32}$$

where $\{|n_1, \cdots, n_p\rangle = \tilde{S}_{j_1}^{\dagger n_1} \cdots \tilde{S}_{j_p}^{\dagger n_p}|\nu_{j_1}\rho_{j_1}, \cdots, \nu_{j_p}\rho_{j_p}\rangle\}$, which are mutually orthonormal, the sum is restricted with $0 \le n_i \le 2Q_{j_i}$ and $\sum_{i=1}^{p} n_i = N$, and $C_{n_1,\cdots,n_p}^{(\zeta)}$ is the expansion coefficient to be determined. Actually, the dimension of the N-pair subspace can be calculated by the counting formula [20]

$$\dim(V_N) = \sum_{n_1=0}^{2Q_{j_1}} \cdots \sum_{n_p=0}^{2Q_{j_p}} \delta_{\sum_{i=1}^{p} n_i, N}. \tag{33}$$

Similar to the procedures used in Sec. 2, it can be proven that the expansion coefficient $C_{n_1,\cdots,n_p}^{(\zeta)}$ can be expressed as

$$C_{n_1,\cdots,n_p}^{(\zeta)} = \frac{1}{F(n_1, \cdots, n_p)}, \tag{34}$$

where

$$F(n_1, \cdots, n_p) = E^{(\zeta)} - G - \sum_{j=1}^{p} \epsilon_{j_i}(2n_{j_i} + \nu_{j_i}) + \sum_{i=1}^{p} G_{j_i} n_i (2Q_{j_i} - n_i + 1), \tag{35}$$

in which $E^{(\zeta)}$ is the ζ-th eigen-energy. To show that (32) and (34) are indeed consistent, one may directly apply the Hamiltonian (23) on the N-pair state (32) to establish the eigen-equation $\hat{H}|N,\zeta\rangle = E^{(\zeta)}|N,\zeta\rangle$. Since the two groups of orbit indices $\{j_1, \cdots, j_k\}$ and $\{j'_1, \cdots, j'_k\}$ in the restricted sum of $\hat{H}_P$ run over all orbits with the restriction that no any one of $\{j_1, \cdots, j_k\}$ equals to any one of $\{j'_1, \cdots, j'_k\}$. After simple algebraic manipulation, one can easily find that

$$\begin{aligned} - G\sum_{k=1}^{\infty} \sum_{j_1 \le \cdots \le j_k} \tilde{S}^{\dagger}_{j_1} \cdots \tilde{S}^{\dagger}_{j_k} \sum_{j'_1 \le \cdots \le j'_k} \tilde{S}_{j'_1} \cdots \tilde{S}_{j'_k} |N,\zeta\rangle = \\ G|N,\zeta\rangle - G \sum_{n'_1, \cdots, n'_p} C^{(\zeta)}_{n'_1, \cdots, n'_p} \sum_{n_1, \cdots, n_p} |n_1, \cdots, n_p\rangle. \end{aligned} \tag{36}$$

Once the expansion coefficient is chosen as that shown in (35), the eigen-equation $\hat{H}|N,\zeta\rangle = E^{(\zeta)}|N,\zeta\rangle$ is fulfilled when and only when

$$-G \sum_{n_1=0}^{2Q_{j_1}} \cdots \sum_{n_p=0}^{2Q_{j_p}} \frac{\delta_{\sum_{i=1}^{p} n_i, N}}{F(n_1, \cdots, n_p)} = 1. \tag{37}$$

Solutions of (37) provide with eigenvalues $E^{(\zeta)}$ and the corresponding eigenstates (32) simultaneously. When ϵ_j $(j = 1, \cdots, p)$ are not equal one another, which is always the case in nuclear shell model, the zeros of the polynomials related with Eq. (37) are either within $\dim(V_N)$ open intervals $(-\infty, x_1)$, (x_1, x_2), $\cdots$, $(x_{\dim(V_N)-1}, x_{\dim(V_N)})$ or within (x_1, x_2), (x_2, x_3),$\cdots$, $(x_{\dim(V_N)}, +\infty)$. In this case, binomials $F(n_1, \cdots, n_M)$ with variable $E^{(\zeta)}$ in the denominators of terms in the sum of (37) are all different. Therefore, (37) in this case results in a polynomial equation with variable $E^{(\zeta)}$. The degree of the polynomial equals exactly to the dimension $\dim(V_N)$ of the concerned Hilbert subspace. There are exactly $\dim(V_N)$ distinct roots $E^{(\zeta)}$ of (37) in this case. Hence, the extended monopole pairing Hamiltonian (23) in this case is exactly solved. It is easy to find the roots of (37) with a suitable computer code for one-variable polynomial equations. Therefore, the model with relatively larger dimension can easily be solved numerically with CPU time far less than that required in the standard pairing model [13]. It should be noted that the Hamiltonian (23) is equivalent to the extended pairing model reported in [14] for the Nilsson type mean-field cases when $\Omega_j = 1 \,\forall\, j$. Thus, the extended pairing interaction reported in [14] becomes a special case of (23). Applications of the model to nuclear systems will be a part of our future work.

4 Summary

In this talk, a solvable extended Hamiltonian that includes multi-pair interactions among s- and d-bosons up to infinite order within the framework of the IBM and an extended pairing Hamiltonian to describe pairing interactions among valence nucleon monopole pairs up to infinite order in a spherical mean-field are proposed, of which the former is suitable to describe the E(5) critical point nuclei in the IBM, while the latter includes the extended pairing interaction model within a deformed mean-field theory as a special case. These mean-field plus pairing models are all constructed based on the local $\tilde{E}_2$ algebraic structure.

Support from the U.S. National Science Foundation (OCI-0904874), the Southeastern Universities Research Association, the Natural Science Foundation of China (11175078; 11375080), and the LSU–LNNU joint research program (9961) is acknowledged.

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