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The Emergent Euclidean Dynamical Symmetry in Nuclear Shape Phase Transitions

Y. Zhang, X.X. Wang, X.R. Yu, X.X. Li, S.Y. An
Department of Physics, Liaoning Normal University, Dalian 116029, P. R. China

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Abstract. Based on the boson realization of the Euclidean algebras, it is shown that the Euclidean dynamical symmetry can naturally emerge at the critical point of the U(5)-SO(6) transition in the interacting boson model. With a nonlinear projection, it is further shown that both the E(5) and the X(5) critical point structures can be realized within the Euclidean dynamical symmetry, which thus provides a unified symmetry-based interpretation of the spherical to deformed shape phase transitions.

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1 Introduction

Dynamical symmetries (DSs) provide considerable insight into the nature of quantum many-body dynamical structures. Generally, DS occurs when the Hamiltonian of a system can be written in terms of Casimir operators of a chain of Lie algebras $G \supset G' \supset G'' \cdots$. A typical example is those associated with the interacting boson model (IBM) [1] for nuclear structure.

The IBM possesses an overall U(6) symmetry with three DSs corresponding to three typical nuclear shapes (or collective modes), namely a spherical vibrator [U(5)], an axially deformed rotor [SU(3)], and a γ-soft rotor [SO(6)]. The three DSs are traditionally placed at the vertices of the so called Casten triangle [2] as shown in Figure 1. Nuclei not only manifest these DSs but also indicate that a first-order shape phase transition (SPT) occurs from U(5) to SU(3) and a second-order SPT happens from U(5) to SO(6) [2, 3]. On the other hand, in his pioneering work [4, 5], Iachello found that SPTs are amenable to analytical descriptions of the states at some critical points, called critical point symmetries (CPSs), which can be derived from the differential equation of the geometric Bohr-Mottelson Model [6]. Especially, the E(5) [4, 7] and X(5) [5] CPSs are applicable to approximately describe, respectively, the second-order (U(5)-SO(6)) and first-order (U(5)-SU(3)) SPTs in nuclei.

In this work, the five-dimensional Euclidean dynamical symmetry (Eu(5) DS) will be introduced, and its relations to the IBM and the CPSs will be further discussed.

2 Underlying Eu(5) DS in the IBM

The Eu(5) DS is associated with the invariance of both translations and rotations in a five-dimensional space [7]. One can use the d-boson operator to construct the Casimir operator of the Eu(5) group [8, 9] as

$$\hat{C}_2[\mathrm{Eu}(5)] = \hat{n}_d + \frac{5}{2} - \frac{1}{2}\left(\hat{P}_d^\dagger + \hat{P}_d\right) , \tag{1}$$

where $\hat{n}_d = \sum_u d_u^\dagger d_u$ and $\hat{P}_d = \sum_u (-)^u d_u d_{-u}$. Accordingly, the d-boson operator can be also used to construct the fifteen generators of the Eu(5) Lie algebra as [9]

$$\begin{aligned} \hat{Q}_u^{(2)} &= \frac{1}{\sqrt{2}}[\tilde{d}_u - d_u^\dagger], \\ \hat{T}_u^{(\lambda)} &= \sqrt{2}(d^\dagger \times \tilde{d})_u^{(\lambda)},\ \lambda = 1, 3, \end{aligned} \tag{2}$$

where $\tilde{d}_u = (-1)^u d_u$. It can be proven that these Eu(5) generators may satisfy the commutation relations

$$\begin{aligned} &[\hat{Q}_u^{(2)}, \hat{Q}_v^{(2)}] = 0, \\ &[\hat{T}_u^{(\lambda)}, \hat{Q}_v^{(2)}] = -\sqrt{\frac{4\lambda+2}{5}}\langle \lambda u 2 v | 2 u+v\rangle \hat{Q}_{u+v}^{(2)} \\ &[\hat{T}_u^{(\lambda)}, \hat{T}_{u'}^{(\lambda')}] = -\sqrt{8(2\lambda+1)(2\lambda'+1)} \\ &\qquad \times \sum_{k=\mathrm{odd}} \begin{Bmatrix} \lambda, \lambda', k \\ 2, 2, 2 \end{Bmatrix} \langle \lambda u \lambda' u' | k u+u'\rangle\, \hat{T}_{u+u'}^{(k)} , \end{aligned} \tag{3}$$

and

$$[\hat{Q}_u^{(2)},\ \hat{C}_2[\mathrm{Eu}(5)]] = [\hat{T}_u^{(\lambda)},\ \hat{C}_2[\mathrm{Eu}(5)]] = 0 . \tag{4}$$

One can further prove that the operators $\hat{T}^{(\lambda)}$ with $\lambda = 1,\ 3$ may generate the SO(5) algebras, in which the angular momentum operators defined by $\hat{L} = \sqrt{5}\hat{T}^{(1)}$ generate the SO(3) algebra. Then the Eu(5) dynamical symmetry situations may be characterized by the group chain [7, 9]

$$\mathrm{Eu}(5) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3) . \tag{5}$$

To directly investigate the relation between the IBM and the Eu(5) DS at the Hamiltonian level, we consider the IBM consistent-Q Hamiltonian [10]

$$\hat{H}(\eta,\ \chi) = \varepsilon\left[(1-\eta)\hat{n}_d - \frac{\eta}{4N}\hat{Q}^\chi \cdot \hat{Q}^\chi\right] , \tag{6}$$

where $\hat{Q}^{\chi} = (d^{\dagger}s + s^{\dagger}\tilde{d})^{(2)} + \chi(d^{\dagger}\tilde{d})^{(2)}$ is the quadrupole operator, η and χ are the control parameters with $\eta \in [0, 1]$ and $\chi \in [-\sqrt{7}/2, 0]$, and ε is a scale factor. It can be proven that the Hamiltonian is in the U(5) DS when $\eta = 0$; it is in the SO(6) DS when $\eta = 1$ and $\chi = 0$; it is in the SU(3) DS when $\eta = 1$ and $\chi = -\frac{\sqrt{7}}{2}$. The whole parameter range of the Hamiltonian (6) may be described by the so called Casten triangle, which is shown in Figure 1. It can be also proven that there are second-order SPT occurring along the U(5)-SO(6) side with the critical point $\eta_c = 0.5$ and first-order SPTs appearing between the U(5) limit and points along the SU(3)-SO(6) leg of the triangle with the critical points $\eta_c = \frac{14}{28+\chi^2}$.

To identify the underlying Eu(5) DS in the IBM, we would examine the commutation relations between the generators of the Eu(5) group defined in (2) and the IBM Hamiltonian $\hat{H}(\eta, \chi)$ given in (6). Firstly, it is easy to know that the IBM Hamiltonian does commute with the angular momentum operator $\hat{L}_u$ thus with the generator $\hat{T}_u^{(1)}$ since the Hamiltonian is a scalar quantity. As a result, one only needs to examine the conditions under which the Hamiltonian may commute (approximately) with the other generators of the Eu(5) group. In view of the facts that $\hat{T}_u^{(3)}$ is the SO(5) generator and $\hat{n}_d$ is known to be an SO(5) scalar, one can derive [11]

$$[\hat{T}_q^{(3)}, \hat{H}(\eta,\chi)] = \frac{3\sqrt{5}\varepsilon\eta\chi}{28N}\Big\{\sqrt{10}[(\hat{B}^{(2)}\hat{A}^{(2)})_q^{(3)} - (\hat{A}^{(2)}\hat{B}^{(2)})_q^{(3)}] - 2[(\hat{B}^{(4)}\hat{A}^{(2)})_q^{(3)} - (\hat{A}^{(2)}\hat{B}^{(4)})_q^{(3)}] - 2\chi[(\hat{B}^{(4)}\hat{B}^{(2)})_q^{(3)} - (\hat{B}^{(2)}\hat{B}^{(4)})_q^{(3)}]\Big\}, \quad (7)$$

where $\hat{A}_q^{(2)} = (s^{\dagger}\tilde{d} + d^{\dagger}s)_q^{(2)}$ and $\hat{B}_q^{(k)} = (d^{\dagger}\tilde{d})_q^{(k)}$ with $k = 2, 4$. Furthermore, by implementing the matrix elements related to s-boson under the $\mathrm{U}(6) \supset \mathrm{U}(5) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)$ basis $\{|Nn_d\tau\Delta L\rangle\}$, where N, n_d, τ and L

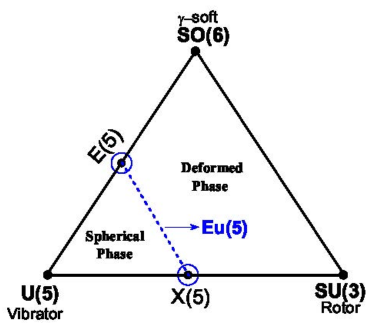

Figure 1. Shape phase diagram of the IBM characterized by the Casten triangle. Note that there are two systems for labeling this paradigm; the geometric language and the IBM.

are the quantum number of U(6), U(5), SO(5) and SO(3), respectively, and Δ is the additional quantum number to characterize the multiplicity of L in τ, one gets the replacements $s^\dagger \to \sqrt{\hat{n}_s + 1}$ and $s \to \sqrt{\hat{n}_s}$ with $\hat{n}_s = N - \hat{n}_d$. Then one can derive in the $n_d/N \ll 1$ limit that

$$[\hat{Q}_q^{(2)}, \hat{H}(\eta, \chi)] = \frac{\sqrt{2}\varepsilon}{2}(1-2\eta)\hat{C}_q^{(2)} - \frac{\sqrt{2}\varepsilon\eta\chi}{8N}\Big[(\hat{A}^{(2)}\hat{C}^{(2)})_q^{(2)} + (\hat{C}^{(2)}\hat{A}^{(2)})_q^{(2)} + 2\hat{B}_q^{(2)} + \chi(\hat{C}^{(2)}\hat{B}^{(2)})_q^{(2)} + \chi(\hat{B}^{(2)}\hat{C}^{(2)})_q^{(2)}\Big], \tag{8}$$

where $\hat{C}_q^{(2)} = (\tilde{d} + d^\dagger)_q^{(2)}$, and $\hat{A}_q^{(2)}$ and $\hat{B}_q^{(2)}$ are those defined above. In order to make the commutators given in (7) and (8) vanish, it is uniquely required $(\eta = 0.5, \chi = 0)$, under which the IBM Hamiltonian just locates at the critical point of the U(5)-SO(6) SPT. The results indicate that the Hamiltonian at this critical point is approximately invariant under the Eu(5) transformations in the $\hat{n}_d/N \ll 1$ limit. It should be noted that the approximation condition $\hat{n}_d/N \ll 1$ may be well satisfied for the low-lying states solved from the IBM Hamiltonian in (6) with $\eta \in [0, 0.5]$ and $\chi = 0$ in a large N case [8].

3 Algebraic Realization of the CPSs Dynamics within the Eu(5) DS

On the other hand, the E(5) and X(5) CPSs built in the Bohr-Mottelson model are both related to the infinite well problem with a difference only in the way of handling the γ degree of freedom [4, 5]. If only $n_\gamma = 0$ states in the X(5) model [5] are considered, which corresponds to the yrast and yrare states, the β dependence in the E(5) and X(5) models can be expressed uniformly by the Bessel equation:

$$\psi''(z) + \frac{\psi'(z)}{z} + \left(1 - \frac{v^2}{z^2}\right)\psi(z) = 0\,, \tag{9}$$

where $\psi(z) \sim z^{-3/2}J_v(z)$ with $J_v(z)$ being a Bessel function of order v, in which z is proportional to the β variable. For the E(5) model, $v = \tau + 3/2$ with τ being the seniority number of the O(5) group, while for the X(5) model, $v = \left[\frac{L(L+1)}{3} + \frac{9}{4}\right]^{1/2}$ with L being the angular momentum quantum number. Accordingly, we can establish a mapping $v = f(L, \chi)$ with $f(L, 0) = \tau$ since $L = 2\tau$ for the yrast states in this case and $f(L, -\frac{\sqrt{7}}{2}) = \left[\frac{L(L+1)}{3} + \frac{9}{4}\right]^{1/2}$. Then a linear mapping connecting the E(5) point with the X(5) point can be given as

$$v = \left(1 + \frac{2}{\sqrt{7}}\chi\right)\frac{L}{2} - \frac{2\chi}{\sqrt{7}}\left[\frac{-3 + \sqrt{9 + 4L(L+1)/3}}{2}\right] + \frac{3}{2} \tag{10}$$

with $\chi \in [0, -\frac{\sqrt{7}}{2}]$. For a given χ, we define the projection $\hat{P}^{\chi}_{\tau',\tau}$ that projects the quantum number τ to be equivalent to $\tau' = v - 3/2$. We found that, after the projection , the Eu(5) Hamiltonian $\hat{H}_{Eu(5)} = \hat{C}_2[\text{Eu}(5)]$ defined in (1) can be rewritten in terms of functionals of the U(5) operators with

$$\begin{aligned}\hat{H}'_{\text{Eu}(5)} &= (\hat{P}^{\chi}_{\tau',\tau})^\dagger \hat{H}_{\text{Eu}(5)} \hat{P}^{\chi}_{\tau',\tau} \\ &= A + \frac{2\chi}{\sqrt{7}}\sqrt{B} - \frac{\chi}{\sqrt{7}}\sqrt{\frac{16}{3}B - \frac{40}{3}\sqrt{B} + 17} + \frac{5}{2} \\ &- \frac{A + (1 + \frac{4\chi}{\sqrt{7}})\sqrt{B} - \frac{2\chi}{\sqrt{7}}\sqrt{\frac{16}{3}B - \frac{40}{3}\sqrt{B} + 17} + \frac{7}{2}}{2(A + \sqrt{B} + \frac{7}{2})} C^\dagger \\ &- C\frac{A + (1 + \frac{4\chi}{\sqrt{7}})\sqrt{B} - \frac{2\chi}{\sqrt{7}}\sqrt{\frac{16}{3}B - \frac{40}{3}\sqrt{B} + 17} + \frac{7}{2}}{2(A + \sqrt{B} + \frac{7}{2})}\,, \end{aligned} \tag{11}$$

where $A = \hat{n}_d$, $B = \hat{n}_d(\hat{n}_d + 3) - 2P^\dagger_d P_d + \frac{9}{4}$, and $C = P_d$. The expression (11) is the Hamiltonian for $\chi \in [0, -\frac{\sqrt{7}}{2}]$, which is well defined when being diagonalized under the $\text{U}(6) \supset \text{U}(5) \supset \text{SO}(5) \supset \text{SO}(3)$ basis, and regains the unprojected form given in (1) as taking $\chi = 0$. The quadrupole operator in this case may be taken simply as $T_u = e(d^\dagger + \tilde{d})^{(2)}_u$ with e being an effective charge. As a result, a symmetry-based realization of the dynamical structural evolution between the E(5) and the X(5) CPSs is provided in the projection scheme. It should be noted that the dimension of the Hilbert subspace is infinite due to the noncompactness of the Eu(5) algebra. However, the analysis in [8] shows that the dynamical structure of the Eu(5) DS may be well kept in the finite-N cases, which indicates that one can diagonalize the Eu(5) Hamiltonian within a finite N subspace.

To test the validity of the model, several typical energy and $B(E2)$ ratios in the related models are listed in Table 1. The results show clearly that the projected Eu(5) DS with $\chi = 0$ and $\chi = -1.32$ in the large N limit reproduces nicely

Table 1. Typical energy ratios and $B(E2)$ ratios calculated for the Eu(5) DS together with those solved from the E(5) and X(5) CPSs.

	E(5)	Eu(5) at $N = 1000$				X(5)
		$\chi = 0.0$	$\chi = -0.8$	$\chi = -1.1$	$\chi = -1.32$	
E_{4_1}/E_{2_1}	2.20	2.19	2.51	2.71	2.89	2.91
E_{6_1}/E_{0_2}	1.18	1.19	1.00	1.00	0.96	0.96
E_{0_2}/E_{2_1}	3.03	3.02	4.22	4.93	5.61	5.67
$\frac{B(E2;\ 4_1 \to 2_1)}{B(E2;\ 2_1 \to 0_1)}$	1.68	1.67	1.63	1.61	1.60	1.58
$\frac{B(E2;\ 0_2 \to 2_1)}{B(E1;\ 2_1 \to 0_1)}$	0.86	0.86	0.72	0.66	0.62	0.63

the results of the E(5) and X(5) models. Furthermore, the calculated quantities increase or decrease monotonously as χ changes from the X(5) limit with $\chi \approx -1.32$ to the E(5) limit with $\chi = 0$. The results indicate that the Eu(5) DS can definitely be considered as the critical DS of the spherical to deformed SPT region as shown in Figure 1. It is remarkable that the bandhead energies of excited 0^+ states for any given N in the Eu(5) scheme are universally independent of χ when normalized to E_{0_2}. For example, $E_{0_3}/E_{0_2} = 2.57$ for $N = 10$ and $E_{0_3}/E_{0_2} = 2.50$ for $N = 1000$, which in the large N limit coincides with the rule of $E_{0_n} = A(n-1)(n+2)$ [12], where A is a χ-dependent parameter. The analysis in Ref. [12] shows that the same law also occurs to the excited 0^+ states around the critical point of the U(5)–SU(3) SPT in the large N limit.

We also investigated the scaling properties of some typical quantities in the projected Eu(5) model for the case with $\chi = 0$ corresponding to the E(5) and with $\chi = -1.32$ corresponding to the X(5) critical point. The results are shown in Figure 2. It is evident from Figure 2 that each excited level scales with N^{-1}, and each $E2$ transition rate scales with N^{1}. Along the analysis in Ref. [13], if a Hamiltonian $H = -\nabla^2/(2M) + k\beta^{2n}$ with $k \propto M^{\,t}$, its spectrum should have a scale factor $M^{(t-n)/(n+1)}$. Therefore, the spectrum of an infinite square well should have a scale factor $M^{(t-n)/(n+1)}\,|_{n\to\infty} = M^{-1}$. The N^{-1} power law of the spectrum in the projected Eu(5) scheme is indeed consistent with the conclusion with $M \propto N$ as shown in Ref. [13]. It is apparent that ratio of two quantities must be an N-independent constant if they obey the same power law.

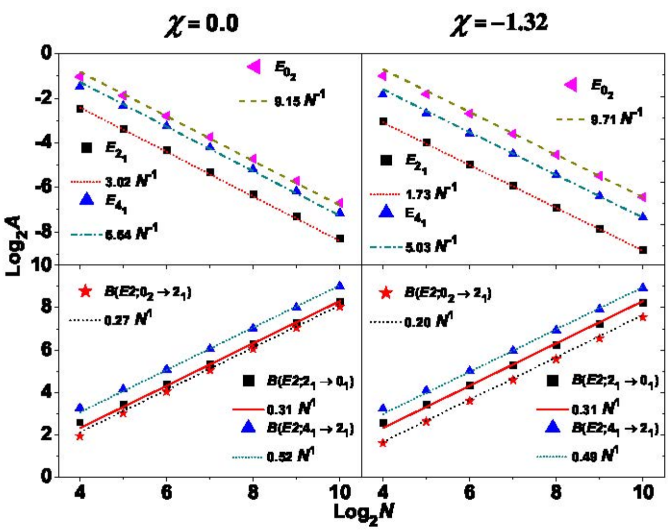

Figure 2. (Color online) Scaling behaviors of some typical energies and $E2$ transition rates with respect to N ($\log_2$ by $\log_2$) for the projected Eu(5) DS with two χ cases.

As a result, the N-scaling laws shown in Figure 2 confirm [8] that the Eu(5) DS is well kept in finite N cases, which in turn suggests that the CPS associated with an infinite well is robust in finite systems.

4 Conclusions

In summary, a boson realization of the Eu(5) algebras has been presented, based on which the relation between the Eu(5) DS and the IBM is revealed. Specifically, it has been proven that the Eu(5) DS may emerge at the critical point of the U(5)-SO(6) SPT in the $n_d/N \ll 1$ limit. With a nonlinear projection, it was further shown that dynamical structures of the E(5) and X(5) CPSs as well as the structural evolution between them can be unified within the Eu(5) DS, which indicate that the Eu(5) DS is indeed dominant but hidden in the whole critical region of the spherical to deformed SPTs.

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References

[1] F. Iachello and A. Arima (1978) *The Interacting Boson Model*, Cambridge University Press, Cambridge.

[2] R. F. Casten, and E.A. McCutchan (2007) *J. Phys. G* **34** R285; R. F. Casten (2009) *Prog. Part. Nucl. Phys.* **62** 183.

[3] P. Cejnar, J. Jolie, and R. F. Casten (2010) *Rev. Mod. Phys.* **82** 2155; P. Cejnar and J. Jolie (2009) *Prog. Part. Nucl. Phys.* **62** 210.

[4] F. Iachello (2000). *Phys. Rev. Lett.* **85** 3580.

[5] F. Iachello (2001) *Phys. Rev. Lett.* **87** 052502.

[6] A. Bohr and B. R. Mottelson (1998) *Nuclear Structure* Vol. 2, World Scientific, Singapore.

[7] M. A. Caprio and F. Iachello (2007) *Nucl. Phys. A* **781** 26.

[8] Y. Zhang, Y. X. Liu, F. Pan, Y. Sun, and J. P. Draayer (2014) *Phys. Lett. B* **732** 55.

[9] Y. Zhang, F. Pan, Y. X. Liu, Y. A. Luo, and J. P. Draayer (2014) *Phys. Rev. C* **90** 064318.

[10] D. D. Warner and R. F. Casten (1983) Phys. Rev. C **28**, 1798.

[11] D. Bonatsos, S. Karampagia, and R. F. Casten (2011) *Phys. Rev. C* **83** 054313.

[12] D. Bonatsos, E. A. McCutchan, and R. F. Casten (2008) *Phys. Rev. Lett.* **101** 022501.

[13] D. J. Rowe, P. S. Turner and G. Rosensteel (2004) *Phys. Rev. Lett.* **93** 232502; D. J. Rowe (2004) *Phys. Rev. Lett.* **93** 122502.