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Algebraic Collective Model and Its Applications

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Abstract. An algebraic collective model (ACM) is applied to describe single and multiphonon excitations in ^{106}Mo and ^{164}Dy. These two nuclei have been chosen because they present different degrees of anharmonicity of double-γ vibrations. In ^{106}Mo a state at 1435 keV has been identified as a best candidate for a harmonic double-γ $K^\pi = 4^+$ state while in ^{164}Dy a strongly anharmonic double-γ $K^\pi = 4^+$ state has been observed. The purpose of the present study is to see how the ACM model is capable of describing diverse degrees of anharmonicity of those double-γ excitations and what are the successes and weak points of such a description.

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1 Introduction

Collective quadrupole vibrational modes in deformed nuclei are described within the Bohr model (BM) as β and γ vibrations [1]. The one-phonon β vibrations preserve axial symmetry and are characterised by $K^\pi = 0^+$ where K is the projection of the angular momentum on the symmetry axis of the nucleus. A one-phonon γ vibration, on the other hand, dynamically breaks the axial symmetry and gives rise to $K^\pi = 2^+$ rotational bands. While γ-vibrational bands have been identified in many deformed nuclei and their structure seems to be well understood, most of the low-lying excited 0^+ states do not have properties expected for a β vibration.

The existence and properties of two-phonon excited states in deformed nuclei has been the subject of many theoretical and experimental studies. Erbium isotopes were the first examples of nuclei where double-γ vibrational states have been found. The discovery of the $K^\pi = 4^+$ double-γ state in ^{168}Er [2] was followed by discoveries of $K^\pi = 0^+$ and $K^\pi = 4^+$ double-γ states in ^{166}Er. Their identification has been based on their strong, collective transitions to the γ band [3], as expected for two-phonon states.

Another example is the neutron-rich nucleus ^{106}Mo studied by prompt γ -ray spectroscopy where a double-γ excitation at 1435 keV has been identified as a bandhead of a rotational band characterised by $K^\pi = 4^+$ [4]. A striking feature of the ^{106}Mo level scheme is that the double-γ excitation at 1435 keV is located well below the neutron and the proton pairing gap and it is thus built on an intrinsic state of collective structure. A partial decay scheme of this nucleus obtained in an earlier work [5] has suggested a rigid triaxial description of ^{106}Mo with a large nonaxiality parameter $\gamma = 19^o$. An assymetric shape of this nucleus has also been assumed in Ref. [6]. However such an interpretation is unable to describe the high-spin states obtained in [7]. We will show that within the ACM a large degree of triaxiality would lead to a large anharmonicity of the $K^\pi = 4^+$ double-γ vibration, in contrast to its harmonic character.

A search for a double-γ vibration in ^{164}Dy has been realised using a thermal neutron capture on ^{163}Dy [8]. A level at 2173 keV has been found to exhibit a collective enhancement in its decay to the single-γ vibration, suggesting a contribution from a double-γ $K^\pi = 4^+$ excitation with a possible admixture of a two-quasiparticle state. This state presents a strong anaharmonicity, the single-γ vibration lying at 762 keV.

The above mentioned examples of nuclei possesing double-γ excitations point to a large range of anharmonicities of those excitations [4, 8–10]. The existence of those states and the origin of their anharmonicities is not yet well understood by nuclear models. For example, according to the Quasiphonon Nuclear Model no $K^\pi = 0^+$ double-γ vibrations below 2.5 MeV should be expected in ^{166}Er [11], contrary to the observations. A particularly interesting for us are results obtained within the related Interacting Boson Model (IBM) which show that large anharmonicities of double-γ excited states can only be obtained for a finite number of bosons combined with a three-body term in the Hamiltonian that can give rise to triaxial shapes [12]. However, with only three-body terms, the moments of inertia, in particular for the γ bands, are not correctly described.

The theoretical description of the structure of single and multiphonon excitations in nuclei is far from satisfactory as the existing controversy between various model predictions shows. In the present ACM analysis we try to identify limitations as much as successes of the present model in such a kind of description [13–15]. We focus on a description of the excitation energies and relative positions of ground bands and excited β and γ bands. We also compare the calculated and experimental quasi-γ band staggerings that serve as a useful signature of γ softness or rigid triaxiality in low-energy and low-spin nuclear spectra [16].

The ACM is described in the next section. In third section the ACM results are shown and compared to the data. In the discussion section we discuss some possible directions in which the present model could be improved.

2 ACM Model Calculations

The ACM may be viewed as a computationally tractable version of the BM [17] restricted to rotational and quadrupole vibrational degrees of freedom. Similarly to the IBM and BM, the ACM is characterized by a well defined algebraic structure [see ref. [18] for a review of this model]. Unlike the conventional harmonic-oscillator $U(5) \supset SO(5) \supset SO(3)$ dynamical subgroup chain used, for example, in the Frankfurt program [19, 20] and not very useful for calculations of deformed nuclei, the ACM makes use of the subgroup chain

$$SU(1,1) \times SO(5) \supset U(1) \times SO(3) \supset SO(2).$$

The basis wave functions of this subgroup chain are defined as direct products of $SU(1,1)$ β wave functions and $SO(5)$ spherical harmonics. This choice of dynamical subgroup chain has several advantages: (i) with the now available SO(5) Clebsch-Gordan coefficients [21, 22], and explicit analytical expressions for SO(5) reduced matrix elements, analytical expressions for matrix elements of BM operators can be obtained; (ii) SU(1,1) modified oscillator representations with the β basis wave functions ranging from those of the $U(5) \supset SO(5)$ harmonic vibrational model to those of the rigid-β wave function of the $SO(5)$-invariant Wilets-Jean model can be used; and (iii) with these modified SU(1,1) representations, collective-model calculations of deformed nuclei converge an order of magnitude more rapidly than in the conventional $U(5) \supset SO(5)$ bases. Thus, within the ACM the advantages of both the BM and the IBM can be combined to make collective-model calculations a simple routine procedure [13, 14, 23, 24].

A great advantage of the ACM is that fully converged calculations can be performed easily for a range of Hamiltonians [23]. This enables one to determine the extent to which a particular BM Hamiltonian can realistically describe experimental data. Such calculations for various BM Hamiltonians prepare the way for more general, but still solvable, algebraic collective models that include intrinsic degrees of freedom, similarly to the unified model of Bohr and Mottelson [1, 25–28].

A general purpose ACM Hamiltonian is given, for example, in the form

$$\hat{H}(M,\alpha,\kappa,\chi) = \frac{-\nabla^2}{2M} + \frac{1}{2}M[(1-2\alpha)\beta^2 + \alpha\beta^4] - \chi\beta\cos 3\gamma + \kappa\cos^2 3\gamma, \quad (1)$$

where

$$\nabla^2 = \frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\hat{\Lambda}$$

is the Laplacian on the 5-dimensional collective model space [18].

The above Hamiltonian, expressed in terms of the quadrupole deformation parameters β and γ is useful because various BM dynamical symmetry limits correspond to simple parameter choices [23]. The value of the parameter α controls

the β deformation of the model. For $\alpha = 0$ the potential is that of a spherical harmonic oscillator, $\frac{1}{2}M\beta^2$, while for values of $\alpha > 0.5$ a deformed minimum appears. The mass parameter M controls the rigidity of the model. When M increases, the kinetic energy decreases and as a result the vibrational β fluctuations of the model about its equilibrium deformation decrease. Thus, by adjusting the parameters α and M a model with any equilibrium value of the β deformation and any degree of rigidity may be obtained. It can be shown that parameter values in the range $0 < \alpha < 2.0$ and $10 < M < 100$ are sufficient to describe the β deformations and rigidities of the observed nuclear collective states [23].

The terms β^2 and $\cos 3\gamma$ in eq.(1) are defined in terms of the quadrupole tensor operator $\hat{Q}$ by

$$\hat{Q} \cdot \hat{Q} = \beta^2, \quad (\hat{Q} \otimes \hat{Q} \otimes \hat{Q})_0 = -\sqrt{\frac{2}{35}}\,\beta^3 \cos 3\gamma.$$

The last term in the Hamiltonian proportional to $cos^2 3\gamma$ can induce a triaxial minimum in the potential energy surface (PES). Such a triaxial minimum results from a competition between all the three terms in the potential and the second and the third one in particular; for a strong term proportional to $cos3\gamma$ the potential energy minimum will remain axial while a strong term proportional to $cos^2 3\gamma$ drives a system to a triaxial minimum.

The calculations have been carried out using a least-square method in a suitable part of the multidimensional parameter space defined above. The procedure was carried out in two steps. First coarse-grain fits were performed to narrow down the parameter space to a physically meaningful subspace. In the second step fine-grain fits were done with steps $\Delta M = 0.5$, $\Delta\alpha = 0.1$, $\Delta\chi = 0.1$ and $\Delta\kappa = 0.1$ to obtain final fits.

3 Results

In this section results obtained with the ACM Hamiltonian (1) are presented. First in Figure 1 calculated energy spectra of ^{106}Mo are compared with the available experimental data taken from Ref. [29]. In general, the calculation provides a fair description of the energies of the ground and first excited band while the B($E2$) transition rates are not well described. We observe that the calculated $K^\pi = 4^+$ double-γ vibration is slightly anharmonic. An even larger anharmonicity of this state has been avoided at the expense of the calculated single-γ-bandhead located above the observed one. One way to obtain a lower energy of the single-γ bandhead would be to increase the parameter κ of the last term that can induce a triaxial minimum in the PES. This would, however, provoke a larger anharmonicity of the double-γ state and at the same time lead to a larger centrifugal stretching in the ground-state band. Another way would be to decrease the stiffness of the potential by choosing a smaller parameters M or

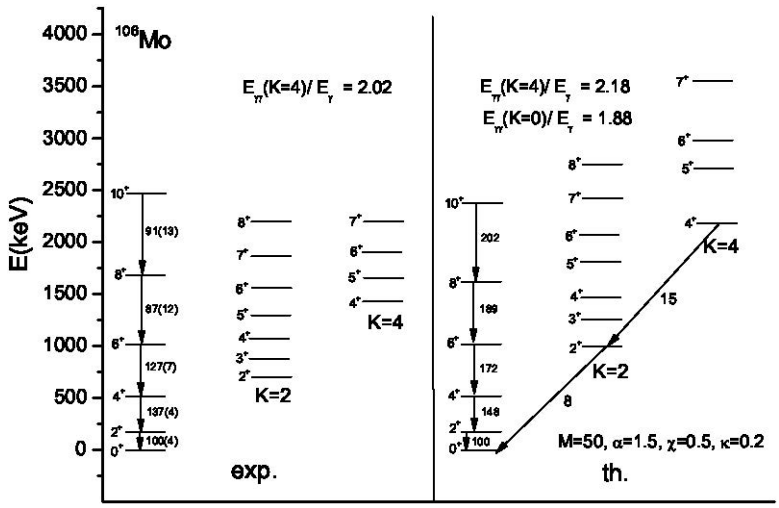

Figure 1. Experimental a) and calculated b) spectra of ^{106}Mo. B$E2$ transition rates are shown as percentages of those for the $2_1 \rightarrow 0_1 = 100$ transitions. Experimental errors are indicated in the parenthesis.

χ which would again result in a large ground-state band centrifugal stretching. These tendencies seem to be irreconcilable in the ACM as can be seen in Figure 2, where anharmonicities of calculated double-γ excitation energies, ratios of calculated and experimental single-γ excitation energies and ratios of calculated and experimental energies of $J = 10$ members of the ground-state band are shown as a function of the parameter κ.

Thus, a significant result that emerges from this calculation is the large amount of centrifugal stretching that occurs in the ground-state band when there are γ or β vibrational bands in the low-energy domain [23,30]. The centrifugal stretching is seen in the ground-state rotational band energies which would systematically fall below the experimental ones and in the B($E2$) transition rates that exceed the observed ones (by a factor of approx. 2 in case of ^{106}Mo) and increase rapidly as the angular momentum increases. It is also interesting to note that a potential candidate for a β vibration would lie, according to the model prediction, at an energy around 7 MeV.

In Figure 3 a PES and a γ-band staggering pattern, that arise in this calculation, are shown. The PES represents the potential part of the Hamiltonian (1) as a function of deformation parameters β and γ. The depth of the potential is represented on the vertical axis (in keV). It is seen that an axially symmetric but very γ-soft minimum arises in this calculation. Such a weakly γ-confining potential (due to small χ and κ) yields both dynamical γ softness and a low γ excitation energy. This is also seen in large amplitudes of the calculated staggering

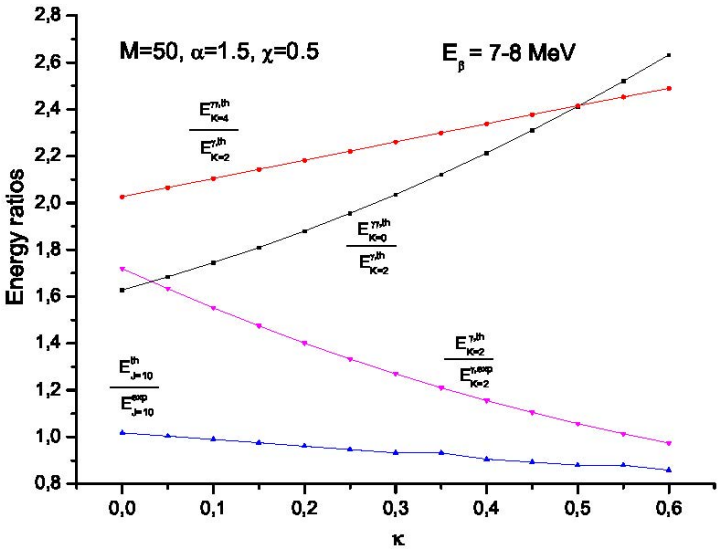

Figure 2. (Color online) Anharmonicities of calculated double-γ excitations, ratios of calculated and experimental single-γ excitation energies and ratios of calculated and experimental energies of $J = 10$ members of the ground-state band are shown as a function of the parameter κ. The experimental values are those of ^{106}Mo.

pattern. A more strongly γ-confining potential would yield a better description of the staggering but would result in an even higher γ excitation energy. At the same time strongly anharmonic double-γ excitations would be obtained. In summary, both the data and the calculation speak in favor of an axially symmetric, γ-soft shape of ^{106}Mo, even if the degree of softness is larger in the model.

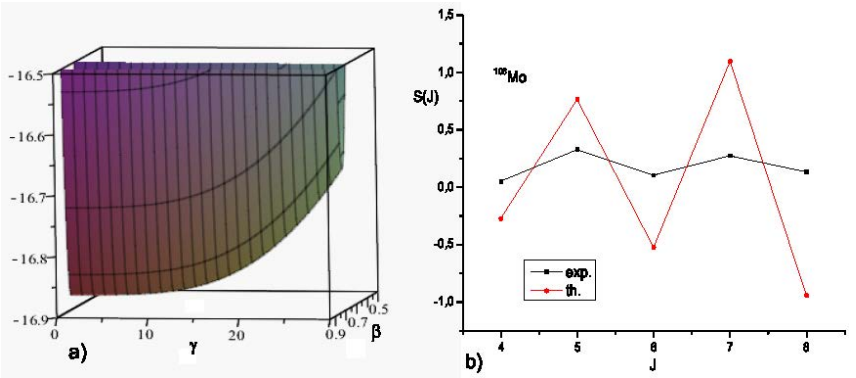

Figure 3. (Color online) Potential energy surface of ^{106}Mo a) and experimental and calculated γ-band staggering of ^{106}Mo b).

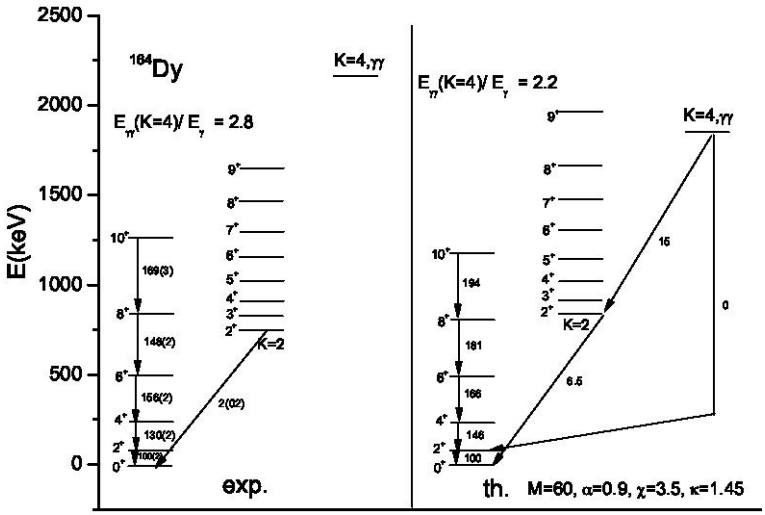

Figure 4. Experimental a) and calculated b) spectra of ^{164}Dy. B$E2$ transition rates are shown as percentages of those for the $2_1 \rightarrow 0_1 = 100$ transitions. Experimental errors are indicated in the parenthesis.

In Figure 4 experimental and calculated energy spectra of ^{164}Dy are shown. In this case, the double-γ $K^\pi = 4^+$ state is strongly anharmonic and appears at an energy 2.8 times higher than the energy of the single-γ band. The study of anharmonicities in nuclear spectra has a long history. Within the IBM-1, it has been shown some time ago by Bohr and Mottelson [31] that large anharmonicities, as observed for instance in ^{166}Er, cannot be described. Subsequently, it was shown that one would need a g boson with $l = 4$ in addition to the s and d bosons [32, 33]. More recently, a study of two-phonon states in the IBM-1 was reported [12] which showed that the IBM-1 is a harmonic model in the limit of large boson number. Anharmonicities can only exist for finite boson number but they are small if only two-body interactions are used. It was also suggested that anharmonicities are strongly linked to triaxiality, which can be induced by three-body terms in the potential. The ACM, capable of giving rise to a stable triaxial minimum, can accomodate large anharmonicities, as is seen, for example, in Figure 2. However, the price to pay is the appearance of the ground band centrifugal stretching mentioned in the previous paragraph. Thus, in fitting the lowest-lying bands, one would typically obtain a fit showing a smaller degree of anharmonicity than the observed one, as is the case of ^{164}Dy. A PES obtained with the parameters of ^{164}Dy shows a shallow triaxial minimum and the staggering pattern is characteristic of a nucleus with an equilibrium triaxial deformation (see Figure 5).

It should be noted though, that a large interval for B$(E2; 2173 \rightarrow 2^+_\gamma)$ values

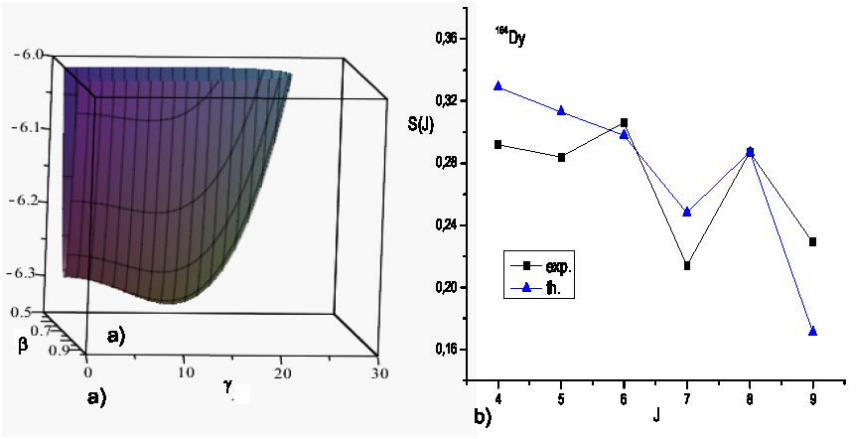

Figure 5. (Color online) Potential energy surface of ^{164}Dy a) and experimental and calculated γ-band staggering of ^{164}Dy b).

covers the values expected for a weakly collective state, to a highly collective one [8]. The uncertainty of the B$(E2)$ value thus does not allow one to determine definitely if this level is predominantly two-phonon in character or if it merely has a minor two-phonon component in its wave function. Deviations from the Alaga rules suggest a certain degree of K mixing in the wave function (in accord with the ACM prediction of a soft triaxial shape). An alternative explanation could be mixing of the $K^\pi = 4^+$ state with the known quasiparticle excitation at 2194 keV [34] which is not included in the ACM.

4 Conclusions

The ACM is a useful tool to perform routine collective-model calculations for practically any collective BM Hamiltonian with potentials corresponding to arbitrary degree of β and γ softness. The purpose of the present analysis is to identify the model's strong and weak points in adressing complex nuclear structure and propose directions in which it could be extended. To acheave this goal, many experimental spectra for various ACM Hamiltonians must analysed and this paper represents an attempt in this direction where a particular choise of the Hamiltonian has been done.

We have shown that the ACM can account for a wide variety of γ-vibrational anharmonicities in nuclei such as those observed in ^{106}Mo or ^{164}Dy but not without substantially changing the moments of inertia and energy positions of various bands. A significant result that emerges from the calculations is the large amount of centrifugal stretching in the ground-state band when there are β and γ vibrational bands in the low-energy domain. These results may call into

question the interpretation of low-lying excited bands as β and γ vibrational bands when the corresponding centrifugal stretching effects are observed to be small [23]. However, various spectra for many possible ACM Hamiltonians have to be explored before any hard conclusion is made.

Another obvious limitation of the BM is its inability to correctly predict observed moments of inertia. Their ratios are fixed by the SO(5) Casimir operator to be those of an irrotational quantum fluid. This is at odds with experiment which requires much larger values. From a liquid-drop model perspective a better description might be acheived by including vorticity degrees of freedom in the collective model [23].

Another important point concerns the nature of the interactions included in the calculations. From the theoretical perspective, all types of three-body interactions should be tested against experimental data. Moreover, it has been shown that within the IBM-1 and with up to three-body interactions that preserve SU(3) symmetry the observed anharmonicities cannot be fully reproduced. A substantial improvement of the IBM predictions has been achieved by including four-body interactions in the Hamiltonian [35]. Similar analysis should be done also within the ACM.

The preliminary ACM calculations may also raise a basic question whether the dicrepancies lie in the details of the ACM in question or point to a fundamental failure of the collective quadrupole degree of freedom to account for the observed nuclear properties.

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