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Quadrupole Shape Phase Transitions in the γ-Rigid Regime

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Abstract. Quadrupole shape phase transitions have been studied in the frame of the Bohr-Mottelson model for which a sextic oscillator potential in the β variable is considered, while the γ degree of freedom is frozen either to 0° or 30°. The corresponding solutions are called X(3)-Sextic ($\gamma = 0^{\circ}$) and Z(4)-Sextic ($\gamma = 30^{\circ}$) in connection with the previous ones X(3) and Z(4) for which an infinite square well potential is used. Both energy spectra and $E2$ transitions are given in analytical form and up to some scale factors depend on a single free parameter. In particular situations when the β^2 or β^4 term vanishes, parameter free solutions are obtained. By varying the free parameter, for both X(3)-Sextic and Z(4)-Sextic, first order shape phase transitions occur from an approximately spherical shape to a well deformed one crossing a critical point where the potential is flat. The structure of the states in the critical points can offer answers concerning the unknown dynamical symmetries of X(5) or Z(5). Regarding the applications to experimental data, the parameter free solutions can be used initially as reference points. Finally, the agreement between the theory and experiment is improved by fitting the free parameter, whose values are used to verify if a shape phase transition appears within an isotopic chain.

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1 Introduction

Once with the proposal of the critical point "symmetries" for quadrupole shape phase transitions, namely E(5) [1], X(5) [2], Y(5) [3] and Z(5) [4], respectively, a great interest of the researchers from the field manifested toward finding the best candidate nuclei for these "symmetries", to improve their agreement with the experimental data by solving the Bohr hamiltonian [5, 6] for more sophisticated potentials, to verify if some dynamical symmetries can be associated to their corresponding hamiltonians as for E(5), to study the evolution of the shape phase transition within many isotopic chains with different models, etc. As it was pointed out in some recent review papers [7–13] most of these objectives were achieved, but also other questions remained or new ones occurred.

In the present work the attention is focused on the study of the quadrupole shape phase transitions in the γ-rigid regime [14–17]. This subfield was opened by the proposal of the γ-rigid versions of Z(5) and X(5), called Z(4) [18] and X(3) [19], respectively. As in the case of X(5) and Z(5) an infinite square well potential in the β variable is used for X(3) and Z(4) excepting that here the γ degree of freedom is frozen to $0°$ and $30°$. By fixing the γ variable, the energy potential depends only on the β variable allowing an exact separation of the remaining variables. Moreover, the infinite square well potential leads to analytical and parameter-free solutions. On the other hand all these constraints restrict the range of applications for X(3) and Z(4). Anyway, these two solutions should be seen from the point of view of their proposal, namely, to describe critical points of some quadrupole shape phase transitions in the frame of the γ-rigid regime. In order to see if such transitions exist we need to replace the infinite square well potential with more flexible potentials. A step toward this direction has been done in Refs. [14–17], where quartic [20] and sextic [21, 22] type potentials have been used for the β variable. For these potentials the solutions, up to a scale factor, depend on a single free parameter. By varying the free parameter, first order shape phase transitions have been observed in Refs. [11, 16] with numerical results in the critical points close to those of X(3) and Z(4). According to these descriptions, X(3) and Z(4) would describe the critical points of some shape phase transitions taking place from an approximately spherical shape to a pronounced deformed one, the γ deformation being preserved during the phase transition. Good candidates for the critical points have been found to be ^{130}Xe, ^{196}Pt for $\gamma = 30°$ and ^{104}Ru, ^{120,126,128}Xe, ^{172}Os, ^{196}Pt, ^{148}Nd for $\gamma = 0°$.

In the following, we will present the main results obtained using a quasi exactly solvable sextic potential in the description of shape phase transitions [11,16]. In Section 2, the solutions of the Bohr hamiltonian with sextic potential [22] for γ fixed to $0°$ and $30°$ will be briefly exposed. In Section 3, the main results will be discussed, while Section 4 is dedicated for conclusions.

2 The X(3)-Sextic and Z(4)-Sextic Solutions

By imposing a γ-rigidity to the quadrupole collective motion, described in the frame of the Bohr-Mottelson model [5,6], the corresponding eigenvalue problem is reduced to that of the Davydov-Chaban hamiltonian [23], while for a prolate shape the problem is furthermore simplified [19]. In these special cases, the potential is γ-independent allowing an exact separation of the remaining variables, namely, the β deformation and the rotational angles. Free parameter solutions were obtained by choosing an infinite square well potential for the β variable, while $\gamma = 30°$ [18] and $\gamma = 0°$ [19]. The former is called Z(4) and the later X(3) after their γ-soft analogs Z(5) [4] and X(5) [2]. Using a sextic potential [22], in Refs. [11, 16] it is shown that Z(4) and X(3) can be seen as good approaches for the description of critical points of some first order shape phase transitions. The sextic potential [22] was previously used to describe the critical points of the shape phase transitions from a spherical to a γ-unstable system [24–27], from a prolate to an oblate system [28–32] and from a spherical to a prolate system [31–33].

In order to adapt here the β-equation to that of the sextic potential [22], for both $\gamma = 0°$ and $30°$, first this is brought to a Schrödinger form by choosing adequate forms for the wave functions [11, 16]:

$$\left[-\frac{d^2}{d\beta^2} + \frac{\Delta}{\beta^2} + v(\beta)\right]\varphi(\beta) = \varepsilon\varphi(\beta), \tag{1}$$

where,

$$\Delta = \frac{L(L+1)}{3} \text{ for } \gamma = 0° \text{ and } \Delta = L(L+1) - \frac{3}{4}\left(R^2 - 1\right) \text{ for } \gamma = 30°. \tag{2}$$

Here, $L(L+1)$ and R are the eigenvalues of the total angular momentum and of its intrinsic projection on the x-axis. Eq. (1) is reduced to that of the sextic potential involving the equalities:

$$\Delta = \left(2s - \frac{1}{2}\right)\left(2s - \frac{3}{2}\right), \tag{3}$$

$$v(\beta) = \left[b^2 - 4a\left(s + \frac{1}{2} + M\right)\right]\beta^2 + 2ab\beta^4 + a^2\beta^6, \tag{4}$$

where a and b are parameters, while M is a natural number. Because s depends on L and even on R through Δ, the sextic potential (4) will depend on these quantum numbers. The condition for a state independent potential is obtained from:

$$s + \frac{1}{2} + M = \text{constant} \equiv c. \tag{5}$$

Eq. (5) is satisfied if for an increase of L with four units, M is decreased with one unit [11, 16]. Moreover, the number of parameters involved in Eq. (1) with the sextic potential (4) can be reduced if the variable is changed as $\beta = ya^{-1/4}$ and the notations $\alpha = b/\sqrt{a}$ and $\varepsilon_y = \varepsilon/\sqrt{a}$ are introduced:

$$\left[-\frac{d^2}{dy^2} + \frac{\Delta}{y^2} + (\alpha^2 - 4c)y^2 + 2\alpha y^4 + y^6\right]\eta(y) = \varepsilon_y\eta(y). \tag{6}$$

From Eq. (6) it is observed that free-parameter solutions result for $\alpha = 0$ and $\alpha^2 = 4c$, when the terms y^4 and y^2 vanish. Moreover, by varying α between $-\infty$ to $+\infty$ a shape phase transition from an approximately spherical shape to a well deformed one is covered crossing a point $\alpha_c = 2\sqrt{c}$ for which the potential is completely flat as an infinite square well. A general solution of Eq. (6) is difficult to find, its potential containing simultaneously y^2, y^4 and y^6 terms. Fortunately, *a quasi-exact solution* is possible by considering the ansatz function:

$$\eta(y) \sim P(y^2) y^{2s-\frac{1}{2}} e^{-\frac{y^4}{4}-\alpha\frac{y^2}{2}}, \tag{7}$$

where $P(y^2)$ are polynomials in y^2 of order M. The resulted equation is:

$$\left[-\left(\frac{d^2}{dy^2}+\frac{4s-1}{y}\frac{d}{dy}\right)+2\alpha y\frac{d}{dy}+2y^2\left(y\frac{d}{dy}-2M\right)\right]P(y^2)=\lambda P(y^2), \tag{8}$$

where with λ is denoted the eigenvalue of $P(y^2)$ which will contribute to the final expression of the total energy [11, 16]. Here by *quasi-exactly solvable* is understood that Eq. (6) is exactly solved only for $M+1$ solutions. This method permits also to determine the expression for the wave functions in a compact form, and therefore to compute $E2$ transition probabilities. To simplify the corresponding matrix elements, a zero order approximation in $\gamma = 0°$ and $\gamma = 30°$ of the quadrupole transition operator is adopted. The solutions presented above for the Bohr hamiltonian with sextic potential and γ fixed to 0° and 30° have been called X(3)-Sextic [11] and Z(4)-Sextic [16] in connection with the previous X(3) and Z(4) solutions. The important contributions of these solutions to the field of the quadrupole shape phase transitions will be presented in the next section.

3 Main Results

From Figure 1 it can be seen that by varying the free parameter, a shape phase transition is described from an approximately spherical shape to a well deformed one crossing through a critical region (the narrow area) where the two shapes coexist. Along the transition, the axial deformation is preserved, namely, $\gamma = 0°$ and $\gamma = 30^0$ for X(3)-Sextic and Z(4)-Sextic, respectively. In Refs. [11, 16] it has been shown that the numerical results obtained with X(3)-Sextic and Z(4)-Sextic in the critical points are close to those of X(3) and Z(4). Therefore, if X(5) and Z(5) are associated to critical points of shape phase transitions in the γ-soft regime, X(3) and Z(4) also correspond to some critical points, but in the γ-rigid regime. Other important remark is that parameter free solutions are possible in two cases, namely for $\alpha = 0$ and $\alpha_c = 2\sqrt{c}$ when the term β^4 and β^2, respectively, cancels. These results, which can be used as signatures in the finding of nuclei candidates, are shown in Figure 2, for $K = 2$ which denotes the maximum value of M. An interesting result is that from Figure 2 (d) corresponding to X(3)-Sextic for α_c, where an approximately degeneracy of the states belonging to different bands is observed. An interpretation of this behavior in terms of some dynamic symmetries would be of great interest taking

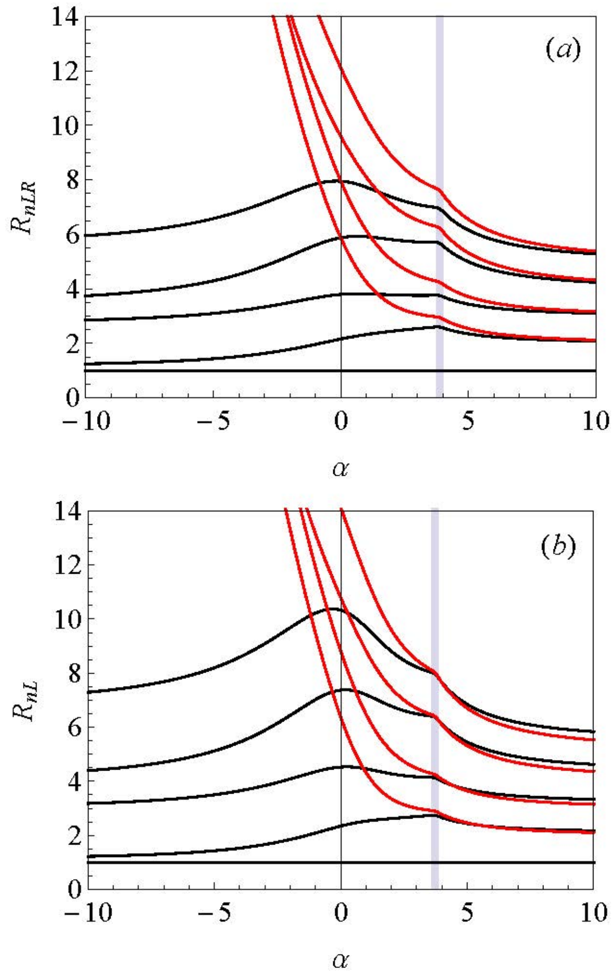

Figure 1. (Color online) The energies of the ground band and of the β band (the curves going to infinity) normalized to the energy of the first excited state are plotted as a function of the free parameter α for Z(4)-Sextic (a) and X(3)-Sextic (b).

into account that the dynamic symmetry of X(5) is still unknown. Moreover, the applications of X(3) and X(3)-Sextic in Refs. [11, 19] to some X(5) candidate nuclei as ^{150}Nd, ^{152}Sm, ^{154}Gd and ^{156}Dy suggest that the β bands of these nuclei prefer a γ-rigid structure rather than a soft one. Thus, it seems that a more appropriate description of these nuclei would be a combination between X(5) and X(3) solutions. A step toward this direction has already been done in Refs. [34, 35] where the hamiltonians of ES-X(5) [36] and X(3) are linked by a free parameter which controls the γ rigidity.

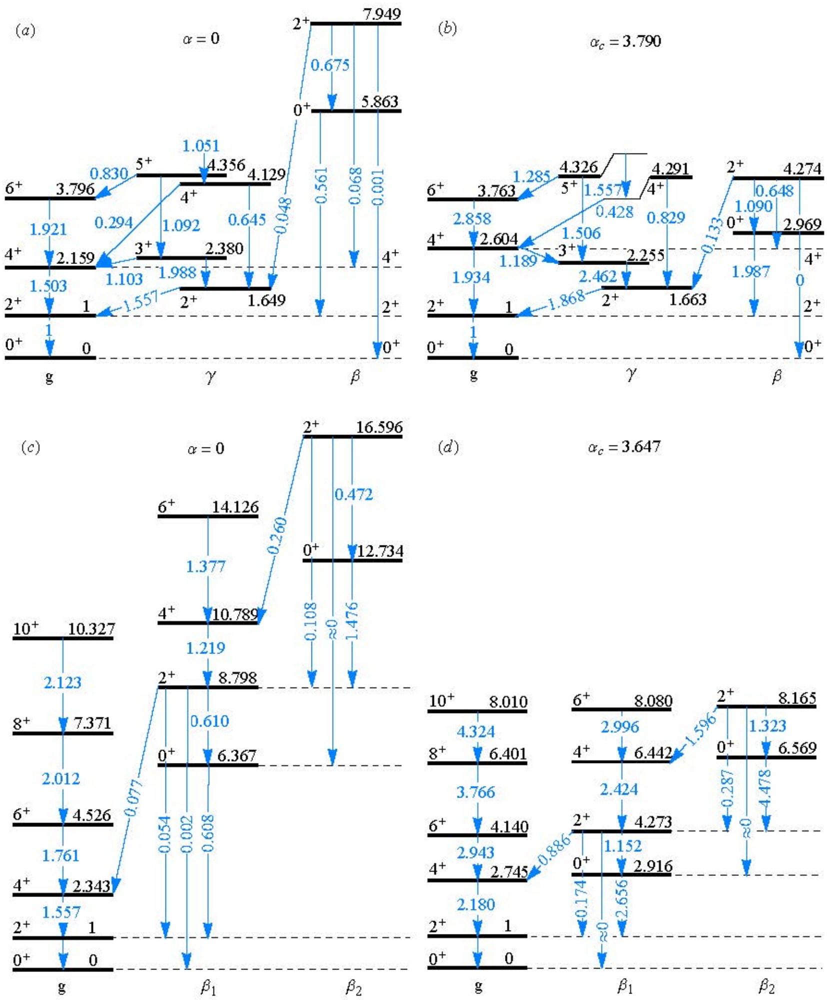

Figure 2. (Color online) The energies and $E2$ transitions in the parameter free cases $\alpha = 0$ and $\alpha = \alpha_c$ for Z(4)-Sextic (a,b) and X(3)-Sextic (c,d).

Numerical applications of Z(4)-Sextic [16] have been done for $^{128-132}$Xe and $^{192-196}$Pt, while of X(3)-Sextic [11] for $^{98-108}$Ru, ^{100,102}Mo, $^{116-130}$Xe, ^{132,134}Ce, $^{146-150}$Nd, ^{150,152}Sm, ^{152,154}Gd, ^{154,156}Dy, ^{172}Os, $^{180-196}$Pt, ^{190}Hg and ^{222}Ra. Besides the good agreement in general with the corresponding experimental data for the energy spectra and $E2$ transitions, the predicted shape phase transitions have been observed in the isotopic chains of Xe, Ru, Nd and Pt. From Figures 3 and 4, it is observed that in the framework of Z(4)-Sextic, good candidates for the critical point are $^{130}_{54}\mathrm{Xe}_{76}$ and $^{196}_{78}\mathrm{Pt}_{118}$, while in case

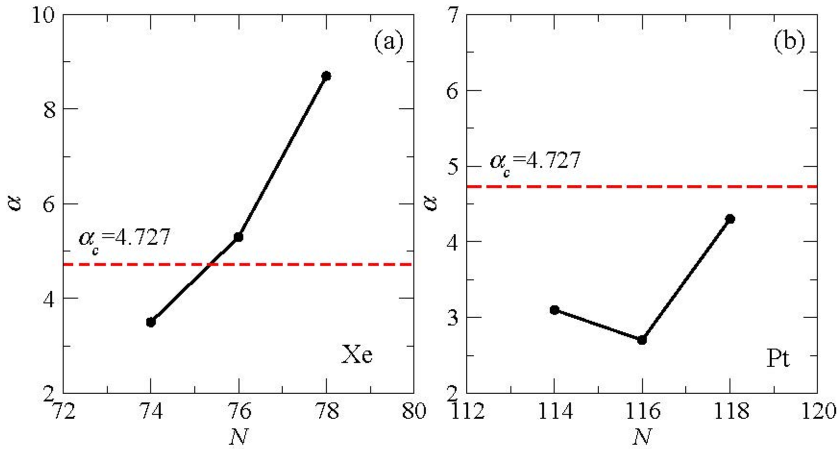

Figure 3. (Color online) The free parameter α of Z(4)-Sextic as a function of the neutron number N for isotopes of Xe (a) and Pt (b). The horizontal dashed line indicates the critical point of the shape phase transition. The α values correspond to $K = 4$.

of X(3)-Sextic are ${}^{104}_{44}\mathrm{Ru}_{60}$, ${}^{120,126}_{54}\mathrm{Xe}_{66,72}$, ${}^{196}_{78}\mathrm{Pt}_{118}$ and ${}^{148}_{60}\mathrm{Nd}_{88}$. Other possible candidates of the critical point of X(3)-Sextic are considered ${}^{128}_{54}\mathrm{Xe}_{74}$ and ${}^{172}_{76}\mathrm{Os}_{96}$. It is interesting that ${}^{196}_{78}\mathrm{Pt}_{118}$ is close to the critical point in both descriptions, while the rest of the considered Pt isotopes prefer the deformed region. By far, the best candidate for the critical point of X(3)-Sextic is ${}^{104}_{44}\mathrm{Ru}_{60}$, which was given as good candidate also for the spherical to γ-unstable shape phase transition [25, 37] and for a situation intermediate between the triaxial and the γ-unstable limit [38–40]. Nevertheless, the present description seems to be more appropriate for ${}^{104}_{44}\mathrm{Ru}_{60}$ than the previous ones.

Even if X(3)-Sextic and Z(4)-Sextic are γ-rigid solutions, their results show that this subfield deserves more attention from the researchers which work in the field of the quadrupole shape phase transitions.

4 Conclusions

Two new analytical solutions of the Bohr hamiltonian with a sextic potential [22] and γ fixed to 30° [16] and 0° [11] are presented. The corresponding energies and $E2$ transitions, up to some scale factors, depend on a single free parameter and in special cases parameter free expressions of these are possible. Due to the special properties of the sextic potential, first order shape phase transitions between an approximately spherical shape and a deformed one are described. The potential in the critical point is flat leading to numerical results close to those given by Z(4) and X(3) for which an infinite square well potential was used. Thus, Z(4) and X(3) can be seen as good approaches for these critical points.

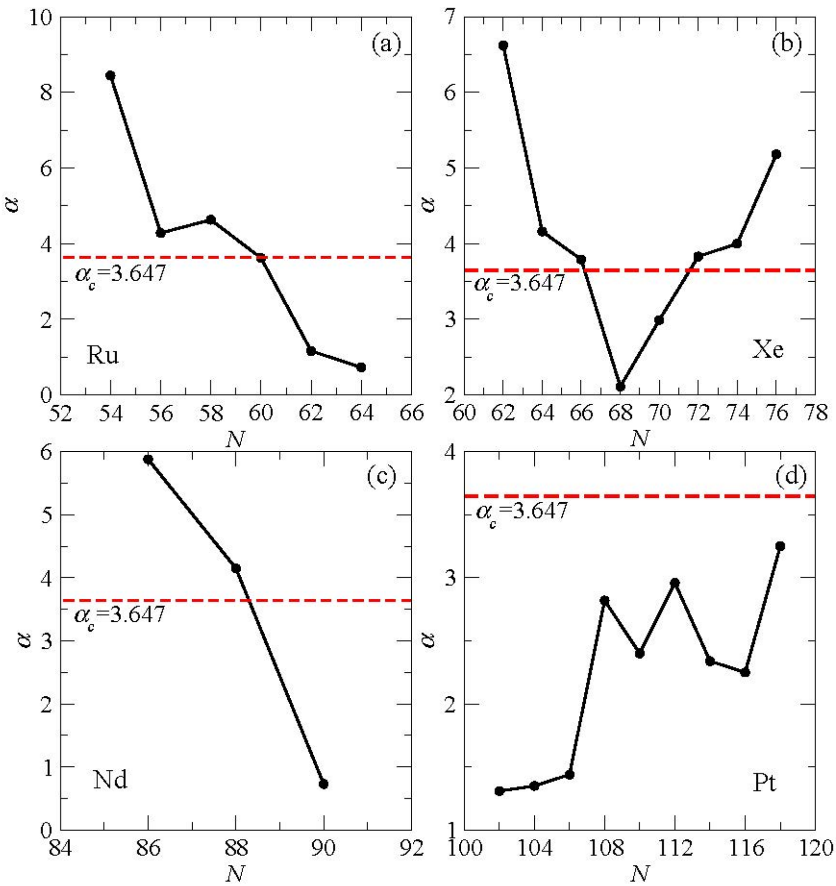

Figure 4. (Color online) The free parameter α of X(3)-Sextic as a function of the neutron number N for isotopes of Ru (a), Xe (b), Nd (c) and Pt (d). The horizontal dashed line indicates the critical point of the shape phase transition. These numerical results correspond to K=2.

Applications of X(3) and X(3)-Sextic to some X(5) candidate nuclei revealed that the β bands are better described in the γ-rigid regime than in the γ-soft one. An important result of X(3)-Sextic is the approximate degeneracy of the states belonging to different bands obtained in the critical point, which can be useful in the association of a dynamic symmetry to X(5). A confirmation of the predicting power of Z(4)-Sextic and X(3)-Sextic is provided by their applications to 6 and respectively 39 nuclei. Moreover, the shape phase transitions described by these solutions have been found in the isotopic chains of Xe, Ru, Nd and Pt.

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