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Bound States of the Dirac-Kratzer-Fues Problem with Spin and Pseudo-Spin Symmetry via Laplace Transform Approach

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Abstract. Bound state solutions of the Dirac equation for the Kratzer-Fues potential with spin and pseudo-spin symmetry are studied in this paper. To obtain the exactly normalized bound state wave function and energy expressions we have used the Laplace transform approach.

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1 Introduction

The exact energy eigenvalues of the bound state play an important role in quantum mechanics. In particular, the Dirac equation which describe the motion of a spin-$1/2$ particle has been used in the relativistic treatment of nuclear phenomena to describe the behavior of the nuclei in nucleus and also in solving many problems of high-energy physics and chemistry. For this reason, the Dirac equation has been used extensively to study the relativistic heavy ion collisions, heavy ion spectroscopy, and more recently in laser-matter interaction [1] and condensed matter physics [2].

In recent years, there has been a renewed interest in obtaining the solutions of the Dirac equations for some typical potentials under spin symmetry and pseudo-spin symmetry cases. The idea about spin symmetry and pseudo-spin symmetry with the nuclear shell model has been introduced in Ref. [3]. This idea has been widely used in explaining a number of phenomena in nuclear physics and related areas. Spin and pseudo-spin symmetric concepts have been used in the studies of certain aspects of deformed and exotic nuclei.

Spin symmetry is relevant to meson with one heavy quark, which is being used to explain the absence of quark spin orbit splitting (spin doublets) observed in heavy-light quark mesons [4] and pseudo-spin symmetry concept has been successfully used to explain different phenomena in nuclear structure including deformation, superdeformation, identical bands, exotic nuclei and degeneracies of some shell model orbitals in nuclei (pseudo-spin doublets) [5,6]. Within this framework, Ginocchio [7-12] deduced that a Dirac Hamiltonian with scalar $S(r)$ and vector $V(r)$ harmonic oscillator potentials when $V(r) = S(r)$ possesses a spin symmetry as well as a U(3) symmetry, whereas a Dirac Hamiltonian for the case of $V(r) + S(r) = 0$ or $V(r) = -S(r)$ possesses a pseudo-spin symmetry and a pseudo-U(3) symmetry. As introduced in nuclear theory, the pseudo-spin symmetry refers to a quasi-degeneracy of the single-nucleon doublets which can be characterized with the non-relativistic quantum mechanics $(n, l, j = l + \frac{1}{2})$ and $(n - 1, l + 2, j = l + \frac{3}{2})$, where n, l and j are the single-nucleon radial, orbital and total angular momentum quantum numbers for a single particle, respectively. The total angular momentum is given as $j = \bar{l} + \bar{s}$, where $\bar{l} = l + 1$ is a pseudo-angular momentum and $\bar{s} = \frac{1}{2}$ is a pseudo-spin angular momentum. The orbital and pseudo-orbital angular momentum quantum numbers for spin symmetry l and pseudo-spin symmetry $\bar{l}$ refer to the upper-and lower-spinor components : $F_{n,k}(r)$ and $G_{n,k}(r)$, respectively.

The spin symmetry occurs when the difference between the repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ in the Dirac Hamiltonian is a constant, that is, $\Delta(r) = V(r) - S(r) =$ const. and pseudo-spin symmetry occurs when the sum of two potential is a constant, that is, $\Sigma(r) = V(r) + S(r) =$ const.

Recently, many researchers have applied spin and pseudo-spin symmetry conditions on a number of potentials. These potentials include: Hulthén potential [13], Eckart potential [14], Pöschl-Teller potential [15,16], the Rosen-Morse potential [17], harmonic potential [18], pseudoharmonic potential [19], Manning-Rosen potential [20], Wood-Saxon potential [21], Kratzer potential with angle dependent potential [22], Scarf potential [23], the Hua potential [24]. In this study we consider the Kratzer-Fues potential [25,26] given in the slightly modified form [27] as

$$V(r) = V_0 \left(1 - \frac{r_0}{r}\right)^2, \tag{1}$$

where r_0 is an equilibrium distance and V_0 is a constant related to the dissociation energy of a molecule.

Several researchers have used various methods to solve spin and pseudo-spin symmetry problems including the centrifugal approximation ranging from the Asymptotic Iteration Method (AIM) [14,28], the Nikiforov-Uvarov method (N-U), supersymmetric and shape invariance method [29]. Recently, solutions of Schrödinger and K-G equations with several potentials have been investigated using various methods[30-44]. Here we adopt the Laplace transform approach (LTA) to cultivate the bound state energy eigenvalues and the corresponding eigenfunctions for Kratzer-Fues potential with the repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ for the Dirac equation.

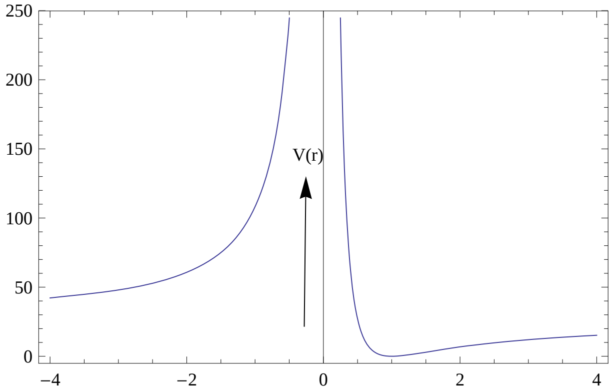

Figure 1. Graphical representation of the Kratzer-Fues potential for $V_0 = 27$ and $r_0 = 1$ for different values of r.

In order to obtain the relativistic bound state energy eigenvalues and the corresponding Dirac spinors, we use a different and very economical method, called the Laplace transform approach within the framework of the pseudo-spin and the spin symmetry concept. The LTA is an integral transform which has been used by many authors [45,46].

This work is organized as follows: To make it self-contained, we present Laplace transform approach with necessary formulas to perform our calculations in the next section. In Section 3, we have formulated the Dirac Equation to Schrödinger-like equation for suitable application of LTA. In Sections 4 and 5, we consider the spin symmetric and the pseudo-spin symmetric solutions of the Kratzer-Fues potential for any k state respectively. And the last section is kept for conclusive remark.

2 Some Important Formula related to Laplace Transform Approach

Suppose the differential equation contains a term of the form $t^m y^{(n)}(t)$, i.e., $t^m \dfrac{d^n y(t)}{dt^n}$. Then the Laplace transform of the term is represented by

$$L\left\{t^m \frac{d^n y(t)}{dt^n}\right\} = (-1)^m \frac{d^m}{ds^m} L\left\{y^{(n)}(t)\right\}. \tag{2}$$

So,

$$L\{ty''(t)\} = (-1)\frac{d}{ds} L\{y''(t)\}. \tag{3}$$

Again, another important theorem for Laplace transform of first order and second order derivative for continuous $y(t)$ and $y'(t)$ with $t \geq 0$ of exponential order σ as $t \to \infty$ and if $y'(t)$ and $y''(t)$ is of class A, then Laplace transform of $y'(t)$ and $y''(t)$ for $s > \sigma$ are given by

$$L\{y'(t)\} = sL\{y(t)\} - y(0) \tag{4}$$

and

$$L\{y''(t)\} = s^2L\{y(t)\} - sy(0) - y'(0)\,. \tag{5}$$

One of the most important formula used in our calculation is: if $y(t)$ is a function of class A, then

$$L\{t^n y(t)\} = (-1)^n \frac{d^n f(s)}{ds^n}\,, \tag{6}$$

where $f(s) = L\{y(t)\} = \int_0^\infty y(t)e^{-ts}dt$ and $n = 1, 2, 3, \ldots$

After conversion of the second order differential equation to a first order one, we further apply the inverse Laplace transform to obtain the wave function. The relevant formulae for inverse Laplace transform are followed from Ref. [47].

3 Formulation of Dirac Equation to Schrödinger-Like Equation

The Dirac equation of a nucleon with mass M moving in moving in an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ for spin-$\frac{1}{2}$ particles in the relativistic unit ($\hbar = c = 1$) is [48]

$$[\alpha p + \beta(M + S(r))]\psi(r) = [E - V(r)]\psi(r)\,, \tag{7}$$

where E is the relativistic energy of the system, $p = -i\nabla$ is the three dimensional momentum operator, M is the mass of the fermion particle, and α, β are the 4×4 Dirac matrices given as [48]

$$\alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\,, \tag{8}$$

where I is a 2×2 unit matrix and σ_i are the Pauli three-vector matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{9}$$

The eigenvalues of the spin-orbit coupling operator are $k = (j + \frac{1}{2}) > 0, k = -(j + \frac{1}{2}) < 0$ for the unaligned spin $j = l - \frac{1}{2}$ and aligned spin $j = l + \frac{1}{2}$, respectively. The set (H, K, J^2, J_z) forms a complete set of conserved quantities. Thus, we can write the spinors as [48],

$$\psi_{nk}(r) = \frac{1}{r}\begin{pmatrix} F_{nk}(r) & Y^l_{jm}(\theta,\varphi) \\ iG_{nk}(r) & Y^{\tilde{l}}_{jm}(\theta,\varphi) \end{pmatrix}\,, \tag{10}$$

where $F_{nk}(r)$, $G_{nk}(r)$ represent the upper and lower components of the Dirac spinors and $\bar{l}$ is pseudo-orbital angular momentum, which is defined as $\bar{l} = l+1$ for the aligned spin $j = \bar{l} - \frac{1}{2}$ and $\bar{l} = l - 1$ for the unaligned spin $j = \vec{l} + \frac{1}{2}$. $Y^l_{jm}(\theta, \varphi)$, $Y^{\bar{l}}_{jm}(\theta, \varphi)$ are the spin and pseudo-spin spherical harmonics and m is the projection on the z-axis. Using well-known identities,

$$
\begin{aligned}
&(\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B), \\
&\sigma \cdot p = \sigma \cdot \hat{r}(\hat{r} \cdot p + i\frac{\sigma \cdot L}{r})
\end{aligned}
\tag{11}
$$

as well as the relations

$$
\begin{aligned}
&(\sigma \cdot L)Y^{\bar{l}}_{jm}(\theta, \varphi) = (k-1)Y^{\bar{l}}_{jm}(\theta, \varphi), \\
&(\sigma \cdot L)Y^{l}_{jm}(\theta, \varphi) = -(k+1)Y^{l}_{jm}(\theta, \varphi), \\
&(\sigma \cdot \hat{r})Y^{l}_{jm}(\theta, \varphi) = -Y^{\bar{l}}_{jm}(\theta, \varphi), \\
&(\sigma \cdot \hat{r})Y^{\bar{l}}_{jm}(\theta, \varphi) = -Y^{l}_{jm}(\theta, \varphi),
\end{aligned}
\tag{12}
$$

we find the following two coupled first-order Dirac equation,

$$
\Big(\frac{d}{dr} + \frac{k}{r}\Big)F_{nk}(r) = (M + E_{nk} - \triangle(r))G_{nk}(r) \tag{13}
$$

$$
\Big(\frac{d}{dr} - \frac{k}{r}\Big)G_{nk}(r) = (M - E_{nk} + \Sigma(r))F_{nk}(r)\,, \tag{14}
$$

where

$$
\Delta(r) = V(r) - S(r) \tag{15}
$$

$$
\Sigma(r) = V(r) + S(r) \tag{16}
$$

Eliminating $F_{nk}(r)$ and $G_{nk}(r)$ in Eqs. (22) and (23), we obtain the second-order Schrödinger-like equation

$$
\begin{aligned}
\Big\{\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} &- (M + E_{nk} - \Delta(r))(M - E_{nk} + \Sigma(r)) \\
&+ \frac{\frac{d\Delta(r)}{dr}\Big(\frac{d}{dr} + \frac{k}{r}\Big)}{(M + E_{nk} - \Delta(r))}\Big\}F_{nk}(r) = 0,
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\Big\{\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} &- (M + E_{nk} - \Delta(r))(M - E_{nk} + \Sigma(r)) \\
&+ \frac{\frac{d\Sigma(r)}{dr}\Big(\frac{d}{dr} - \frac{k}{r}\Big)}{(M - E_{nk} - \Sigma(r))}\Big\}G_{nk}(r) = 0,
\end{aligned}
\tag{18}
$$

where $k(k-1) = \bar{l}(\bar{l}+1)$ and $k(k+1) = l(l+1)$.

We consider bound state solutions that demand the radial components satisfying $F_{nk}(0) = G_{nk}(0) = 0$ and $F_{nk}(\infty) = G_{nk}(\infty) = 0$.

4 Bound State Solutions of the Kratzer-Fues Potential for the Spin Symmetric Case

In the case of exact spin symmetry $\dfrac{d\Delta(r)}{dr} = 0$, i.e., $\Delta(r) = C = \text{const}$, Eq. (17) becomes

$$\left\{\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (M + E_{nk} - \Delta(r))(M - E_{nk} + \Sigma(r))\right\} F_{nk}(r) = 0 \quad (19)$$

where $k = l$ for $k < 0$ and $k = -(l+1)$ for $k > 0$. The energy eigenvalues depend on n and l, i.e., $E_{nk} = E(n, l(l+1))$, which is well known as the exact spin symmetry. We assume that $\Sigma(r)$ is the Kratzer-Fues potential and Eq. (19) takes the form with this potential

$$\left\{\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (M + E_{nk} - C)\left(M - E_{nk} + V_0\left(1 - \frac{r_0}{r}\right)^2\right)\right\} F_{nk}(r) = 0\,. \quad (20)$$

Now defining the function $F_{nk}(r) = \sqrt{r}\varphi(r)$, we get

$$\{r^2\frac{d^2}{dr^2} + r\frac{d}{dr} - (\lambda^2 r^2 + \mu^2 r + \nu^2)\}\varphi(r) = 0\,, \quad (21)$$

where

$$\begin{aligned} \lambda^2 &= (M + E_{nk} - C)(M - E_{nk} + V_0); \\ \mu^2 &= \frac{1}{4} - 2r_0 V_0 (M + E_{nk} - C); \\ \nu^2 &= k(k+1) + V_0 r_0^2 (M + E_{nk} - C). \end{aligned} \quad (22)$$

Setting $\varphi(r) = r^\beta \phi(r)$, where β is a constant. Then Eq. (21) reduces to

$$\left\{r^2\frac{d^2}{dr^2} + (2\beta + 1)r\frac{d}{dr} - (\lambda^2 r^2 + \mu^2 r + \nu^2 - \beta^2)\right\}\phi(r) = 0. \quad (23)$$

In order to obtain a finite wave function when $r \to \infty$, we must take $\beta = -\nu$ in Eq. (23), and then we get

$$\left\{r\frac{d^2}{dr^2} - (2\nu - 1)\frac{d}{dr} - \lambda^2 r - \mu^2\right\}\phi(r) = 0. \quad (24)$$

This form of equation is suitable for the application of LTA, which is described above in Section 2, and applying the LTA to the above equation, we obtain a first order differential equation

$$(t^2 - \lambda^2)\frac{df(t)}{dt} + [(2\nu + 1)t + \mu^2]f(t) = 0\,, \quad (25)$$

where $f(t) = L\{\phi(r)\}$, and the solution is given by

$$f(t) = N(t+\lambda)^{-(2\nu+1)}\Big(\frac{t-\lambda}{t+\lambda}\Big)^{-\frac{\mu^2}{2\lambda}-\frac{2\nu+1}{2}}. \tag{26}$$

The wave functions required to be single-valued but the term $\Big(\frac{t-\lambda}{t+\lambda}\Big)^{-\frac{\mu^2}{2\lambda}-\frac{2\nu+1}{2}}$ is multi-valued. Therefore, we must have to take

$$-\frac{\mu^2}{2\lambda} - \frac{2\nu+1}{2} = n, \quad (n = 0, 1, 2, \ldots). \tag{27}$$

Now applying a simple series expansion to Eq. (26), we obtain

$$f(t) = N' \sum_{m=0}^{n} \frac{(-1)^m n!}{(n-m)!m!}(2\lambda)^m (t+\lambda)^{-(2\nu+1)-m}, \tag{28}$$

where N' is a constant. Using the inverse Laplace transformation in Eq. (28), we get that

$$\begin{aligned}\phi(r) &= N'' r^{2\nu} e^{-\lambda r} \sum_{m=0}^{n} \frac{(-1)^m n!}{(n-m)!m!} \frac{\Gamma(2\nu+1)}{\Gamma(2\nu+1+m)}(2\lambda r)^m \\ &= N'' r^{2\nu} e^{-\lambda r} {}_1F_1(-n; 2\nu+1; 2\lambda r)\,, \end{aligned}\tag{29}$$

where ${}_1F_1(-n; 2\nu+1; 2\lambda r)$ is the notation of confluent hypergeometric function [49].

We also obtain

$$\varphi(r) = N''' r^{\nu} e^{-\lambda r} {}_1F_1\big(-n; 2\nu+1; 2\lambda r\big). \tag{30}$$

Finally, we obtain the upper component of the Dirac spinor as

$$F_{nk}(r) = \mathbf{N} r^{\nu+\frac{1}{2}} e^{-\lambda r} {}_1F_1\big(-n; 2\nu+1; 2\lambda r\big), \tag{31}$$

where **N** is the normalization constant.

In order to find the lower component spinor, the recurrence relation of the confluent hypergeometric function

$$\frac{d}{dr} {}_1F_1(a; b; r) = \frac{a}{b} {}_1F_1(a+1; b+1; r) \tag{32}$$

is used to evaluate Eq. (13), and this is obtained for spin symmetry case (i.e. for $\Delta(r) = C = \text{const}$) as

$$\begin{aligned}G_{nk}(r) = \frac{N r^{\nu+\frac{1}{2}} e^{-\lambda r}}{M + E_{nk} - C}\Big\{&\frac{-n}{2\lambda(2\nu+1)} {}_1F_1\big(-n+1; 2(\nu+1); 2\lambda r\big) \\ &+ \Big(\frac{k+\nu+\frac{1}{2}}{r} - \lambda\Big) {}_1F_1\big(-n; 2\nu+1; 2\lambda r\big)\Big\}. \end{aligned}\tag{33}$$

Using the Eqs. (22) and (27), an explicit expression for the energy eigenvalues of the Dirac equation with the Kratzer-Fues potential under the spin symmetry condition is obtained as

$$2r_0 V_0 \tilde{E_{nk}} = \frac{1}{4} + \left(1 + 2n + 2\sqrt{k(k+1) + V_0 r_0^2 \tilde{E}_{nk}}\right) \times \sqrt{\tilde{E}_{nk}(2M + V_0 - C - \tilde{E}_{nk})}\,, \quad (34)$$

where $\tilde{E}_{nk} = M + E_{nk} - C$. According to Ginocchio [12] there are only positive energy eigenvalues and no bound state negative energy eigenvalues exist in the spin limit. Therefore, in the spin limit, only positive energy eigenvalues are chosen.

5 Bound State Solutions of the Kratzer-Fues Potential for the Pseudo-Spin Symmetric Case

In the case of exact pseudo-spin symmetry $\dfrac{d\Sigma(r)}{dr} = 0$, i.e., $\Sigma(r) = C_{ps} =$ const, Eq. (18) becomes

$$\left\{\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - (M + E_{nk} - \Delta(r))(M - E_{nk} + C_{ps})\right\} G_{nk}(r) = 0, \quad (35)$$

where k is related to the pseudo-orbital angular quantum number $\bar{l}$ as $k(k-1) = \bar{l}(\bar{l}+1)$, $k = -\bar{l}$ for $k < 0$ and $k = (\bar{l}+1)$ for $k > 0$, which implies that $j = \bar{l} \pm \frac{1}{2}$ are degenerate for $\bar{l} \neq 0$. It is required that the upper and lower spinor components must satisfy the following boundary conditions $F_{nk}(0) = G_{nk}(0) = 0$ and $F_{nk}(\infty) = G_{nk}(\infty) = 0$ for bound state solutions. The energy eigenvalues depend on n and $\bar{l}$, i.e., $E_{nk} = E(n, \bar{l}(\bar{l}+1))$, which is well known as the exact pseudo-spin symmetry. We assume that $\Sigma(r)$ is the Kratzer-Fues potential and Eq. (35) takes the form with this potential

$$\left\{\frac{d^2}{dr^2} - \frac{\bar{l}(\bar{l}+1)}{r^2} - \left(M + E_{nk} - V_0\left(1 - \frac{r_0}{r}\right)^2\right)\left(M - E_{nk} + C_{ps}\right)\right\} G_{nk}(r) = 0. \quad (36)$$

Now applying same procedure as above by defining the function $G_{nk}(r) = \sqrt{r}\varphi(r)$, we get

$$\left\{r^2 \frac{d^2}{dr^2} + r\frac{d}{dr} - (\lambda^2 r^2 + \mu^2 r + \nu^2)\right\} \varphi(r) = 0\,, \quad (37)$$

where,

$$\begin{aligned}\lambda^2 &= \tilde{E}(2M - \tilde{E} + C_{ps} - V_0);\\ \mu^2 &= 2r_0 V_0 \tilde{E};\\ \nu^2 &= \bar{l}(\bar{l}+1) - V_0 r_0^2 \tilde{E}\,.\end{aligned} \tag{38}$$

Here we assume $\tilde{E} = M - E + C_{ps}$.

Equation (37) is similar to Eq. (21) and so, via the calculations like the above one, the lower component of the Dirac spinor can be obtained as

$$G_{nk}(r) = \tilde{N} r^{\nu+\frac{1}{2}} e^{-\lambda r}{}_1F_1\big(-n; 2\nu+1; 2\lambda r\big), \tag{39}$$

where $\tilde{N}$ is the normalization constant and ${}_1F_1\big(-n; 2\nu+1; 2\lambda r\big)$ is the confluent hypergeometric function.

In order to find the upper component spinor, the recurrence relation of the confluent hypergeometric function

$$\frac{d}{dr}{}_1F_1(a;b;r) = \frac{a}{b}{}_1F_1(a+1;b+1;r) \tag{40}$$

is used to evaluate Eq. (14) and this is obtained for pseudo-spin symmetry case (i.e. for $\Sigma(r) = C_{ps} = \text{const}$) as

$$\begin{aligned}F_{nk}(r) = \frac{\tilde{N} r^{\nu+\frac{1}{2}} e^{-\lambda r}}{\tilde{E}} \Big\{ &\frac{-n}{2\lambda(2\nu+1)}{}_1F_1\big(-n+1; 2(\nu+1); 2\lambda r\big)\\ &+ \Big(\frac{\nu+\frac{1}{2}-k}{r} - \lambda\Big){}_1F_1\big(-n; 2\nu+1; 2\lambda r\big)\Big\},\end{aligned} \tag{41}$$

where $\tilde{N}$ is the normalization constant.

Also, in the similar fashion as obtained in the case of the spin symmetry condition, an explicit expression for the energy eigenvalues of the Dirac equation with the Kratzer-Fues potential under the pseudospin symmetry is obtained as

$$-2r_0 V_0 \tilde{E} = \Big\{1+2n+2\sqrt{\bar{l}(\bar{l}+1) - V_0 r_0^2 \tilde{E}}\Big\}\sqrt{\tilde{E}(2M - \tilde{E} + C_{ps} - V_0)}\,, \tag{42}$$

where, $\tilde{E} = M - E + C_{ps}$ and $\bar{l}(\bar{l}+1) = k(k-1)$. It has been shown that there are only negative energy eigenvalues and no bound positive energy eigenvalues exist in the pseudo-spin limit [12] . Therefore, in the pseudo-spin limit, only negative energy eigenvalues are chosen.

6 Conclusions

In this work, we have obtained the bound state solutions of the Dirac equation with spin and pseudo-spin symmetry for scalar and vector Kratzer-Fues potential of the form $V(r) = V_0\left(1 - \frac{r_0}{r}\right)^2$ depending on the spatially coordinate r. The variation of the above potential according to coordinate r is given in Figure 1. The two-component spinors and the corresponding energy equation have been obtained within the framework of the LTA which is a powerful algebraic treatment for solving the second-order differential equation via conversion of it into a more simpler form. The upper and lower component spinors have been expressed in terms of the confluent hypergeometric functions.

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