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Einstein-Rosen "Bridge" Revisited and Lightlike Thin-Shell Wormholes*

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Abstract. We study in some detail the properties of the mathematically correct formulation of the classical Einstein-Rosen "bridge" as proposed in the original 1935 paper, which was shown in a series of previous papers of ours to represent the simplest example of a static spherically symmetric traversable lightlike thin-shell wormhole. Thus, the original Einstein-Rosen "bridge" is *not equivalent* to the concept of the *dynamical* and *non-traversable* Schwarzschild wormhole, also called "Einstein-Rosen bridge" in modern textbooks on general relativity. The original Einstein-Rosen "bridge" requires the presence of a special kind of "exotic" matter source located on its throat which was shown to be the simplest member of the previously introduced by us class of lightlike membranes. We introduce and exploit the Kruskal-Penrose description of the original Einstein-Rosen "bridge". In particular, we explicitly construct closed timelike geodesics on the pertinent Kruskal-Penrose manifold.

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1 Introduction

The celebrated Einstein-Rosen "bridge" in its original formulation from 1935 [1] is historically the first example of a traversable gravitational wormhole spacetime. However, the traditional presentation of the Einstein-Rosen "bridge" in modern textbooks in general relativity (*e.g.* [2]) *does not* correspond to its original formulation [1]. The "textbook" version of the Einstein-Rosen "bridge" is physically inequivalent to the original 1935 construction as it represents both a non-static spacetime geometry as well as it is *non-traversable*.

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Based on earlier works of ours [3] we revisit the original Einstein-Rosen formulation from 1935 [1]. We find that the originally used by Einstein and Rosen local spacetime coordinates suffer from a serious problem – the pertinent spacetime metric in these coordinates possesses an essential unphysical singularity at the wormhole "throat" – the boundary between the two "universes" of the Einstein-Rosen "bridge" manifold.

We proposed instead a different set of local coordinates for the Einstein-Rosen "bridge" such that its spacetime geometry becomes well-defined everywhere, including on the wormhole "throat".

On the other hand, this reveals a very important *new feature of the correctly defined* Einstein-Rosen "bridge" [3], which was *overlooked in the original* Einstein-Rosen paper [1]. Namely, we show that the correct construction of the Einstein-Rosen "bridge" as self-consistent solution of the corresponding Einstein equations requires the presence of a "thin-shell" "exotic" matter source on the wormhole "throat" – a special particular member of the originally introduced in other papers of ours [3, 4] class of *lightlike membranes*[1].

In the present note we first briefly review the basics of our construction of the original Einstein-Rosen "bridge" as a specific well-defined solution of wormhole type of gravity interacting self-consistently with a dynamical lightlike membrane matter based on explicit Lagrangian action principle for the latter [3, 4]. Also we present the maximal analytic Kruskal-Penrose extension of the original Einstein-Rosen "bridge" wormhole manifold significantly different from the Kruskal-Penrose manifold of the corresponding Schwarzschild black hole [2].

Next, we discuss in some detail the dynamics of test particles (massless and massive ones) in the gravitational background of Einstein-Rosen "bridge" wormhole. Apart from exhibiting the traversability of the Einstein-Rosen wormhole w.r.t. proper-time of travelling observers, we explicitly construct a closed timelike geodesics.

2 Deficiency of the Original 1935 Formulation of Einstein-Rosen Bridge

The Schwarzschild spacetime metric – the simplest static spherically symmetric black hole metric – is given in standard coordinates (t, r, θ, φ) as (*e.g.* [2])

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \quad A(r) = 1 - \frac{r_0}{r} . \quad (1)$$

$r_0 \equiv 2m$ (m – black hole mass parameter) is the horizon radius, where $A(r_0) = 0$ ($r = r_0$ is a non-physical coordinate singularity of the metric (1), unlike the physical spacetime singularity at $r = 0$). Here $r > r_0$ defines the exterior

[1]For a detailed discussion of *timelike* thin-shell wormholes, see the book [5].

Schwarzschild spacetime region, whereas the region $r < r_0$ is the black hole interior.

In spacetime geometries of static spherically symmetric type (like (1) with generic $A(r)$) special role is being played by the so called "tortoise" coordinate r^* defined as

$$\frac{dr^*}{dr} = \frac{1}{A(r)} \longrightarrow r^* = r + r_0 \log\bigl(|r - r_0|/r_0\bigr) , \qquad (2)$$

such that for radially moving light rays we have $t \pm r^* = \text{const}$ (curved spacetime generalization of Minkowski's $t \pm r = \text{const}$).

In constructing the maximal analytic extension of the Schwarzschild spacetime geometry – the *Kruskal-Szekeres* coordinate chart – essential intermediate use is made of "tortoise" coordinate r^* (2), where the Kruskal-Szekeres ("light-cone") coordinates (v, w) are defined as follows (*e.g.* [2]):

$$v = \pm \frac{1}{\sqrt{2k_h}} e^{k_h \left(t + r^*\right)} , \quad w = \mp \frac{1}{\sqrt{2k_h}} e^{-k_h \left(t - r^*\right)} \qquad (3)$$

with all four combinations of the overall signs. Here $k_h = \frac{1}{2}\partial_r A(r) \bigm|_{r=r_0} = \frac{1}{2r_0}$ denotes the so called "surface gravity", which is related to the Hawking temperature as $\frac{k_h}{2\pi} = k_B T_{\text{hawking}}$. Equivalently, Eqs.(3) can be writtes as

$$\mp vw = \frac{1}{k_h} e^{2k_h r^*} , \quad \mp \frac{v}{w} = e^{2k_h t} , \qquad (4)$$

wherefrom t and r^*, as well as r, are determined as functions of vw.

The various combination of the overall signs in Eqs.(3) define a doubling of the two regions of the standard Schwarzschild geometry [2]:

(i) $(+,-)$ – exterior Schwarzschild region $r > r_0$ (region I);

(ii) $(+,+)$ – black hole $r < r_0$ (region II);

(iii) $(-,+)$ – second copy of exterior Schwarzschild region $r > r_0$ (region III);

(iv) $(-,-)$ – "white" hole region $r < r_0$ (region IV).

In the classic paper [1] Einstein and Rosen introduced in (1) a new radial-like coordinate u via $r = r_0 + u^2$

$$ds^2 = -\frac{u^2}{u^2 + r_0} dt^2 + 4(u^2 + r_0) du^2 + (u^2 + r_0)^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) , \qquad (5)$$

and let $u \in (-\infty, +\infty)$. Therefore, (5) describes two identical copies of the exterior Schwarzschild spacetime region ($r > r_0$) for $u > 0$ and $u < 0$, respectively, which are formally glued together at the horizon $u = 0$.

However, there is a very serious problem with (5) (apart from the coordinate singularity at $u = 0$, where $\det \|g_{\mu\nu}\|_{u=0} = 0$). The Einstein-Rosen metric (5) *does not* satisfy the vacuum Einstein equations at $u = 0$. The latter acquire an *ill-defined* non-vanishing "matter" stress-energy tensor term on the r.h.s., which was overlooked in the original 1935 paper [1]!

Indeed, as explained in [3], using Levi-Civita identity $R^0_0 = -\frac{1}{\sqrt{-g_{00}}}\nabla^2_{(3)}\left(\sqrt{-g_{00}}\right)$ (where $\nabla^2_{(3)}$ is the 3-dimensional spatial Laplacian) we deduce that (5) solves vacuum Einstein equation $R^0_0 = 0$ for all $u \neq 0$. However, since $\sqrt{-g_{00}} \sim |u|$ as $u \to 0$ and since $\frac{\partial^2}{\partial u^2}|u| = 2\delta(u)$, Levi-Civita identity tells us that

$$R^0_0 \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2)\,, \tag{6}$$

and similarly for the scalar curvature $R \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2)$.

3 Original Einstein-Rosen Bridge is a Lightlike Thin-Shell Wormhole

In Refs. [3] we proposed a correct reformulation of the original Einstein-Rosen bridge as a mathematically consistent traversable lightlike thin-shell wormhole. This is achieved via introducing a different radial-like coordinate $\eta \in (-\infty, +\infty)$, by substituting $r = r_0 + |\eta|$ in (1)

$$ds^2 = -\frac{|\eta|}{|\eta| + r_0}dt^2 + \frac{|\eta| + r_0}{|\eta|}d\eta^2 + (|\eta| + r_0)^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\,. \tag{7}$$

Obviously, Eq.(7) again describes two "universes" – two identical copies of the exterior Schwarzschild spacetime region for $\eta > 0$ and $\eta < 0$, respectively. However, unlike the ill-behaved original 1935 metric (5), now both "universes" are correctly glued together at their common horizon $\eta = 0$.

Namely, the metric (7) solves Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T^{(brane)}_{\mu\nu}\,, \tag{8}$$

where on the r.h.s. $T^{(brane)}_{\mu\nu} = S_{\mu\nu}\delta(\eta)$ is the energy-momentum tensor of a special kind of a *lightlike membrane* located on the common horizon $\eta = 0$ – the wormhole "throat". As shown in [3], the lightlike analogues of W.Israel's junction conditions on the wormhole "throat" are satisfied. The resulting lightlike thin-shell wormhole is traversable (see Sections 4,6 below).

The energy-momentum tensor of lightlike membranes $T^{(brane)}_{\mu\nu}$ is self-consistently derived as $T^{(brane)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm LL}}{\delta g^{\mu\nu}}$ from the following manifestly reparametrization invariant world-volume Polyakov-type lightlike membrane action (written for arbitrary $D = (p+1)+1$ embedding spacetime dimension and $(p+1)$-dimensional brane world-volume)

$$S_{\rm LL} = -\frac{1}{2}\int d^{p+1}\sigma\, T b_0^{\frac{p-1}{2}}\sqrt{-\gamma}\left[\gamma^{ab}\bar{g}_{ab} - b_0(p-1)\right] , \tag{9}$$

$$\bar{g}_{ab} \equiv g_{ab} - \frac{1}{T^2}\partial_a u \partial_b u \,, \quad g_{ab} \equiv \partial_a X^{\mu} g_{\mu\nu}(X)\partial_b X^{\nu} \,. \tag{10}$$

Here the following notations are used:

(a) γ_{ab} is the *intrinsic* Riemannian metric on the world-volume with $\gamma = \det\|\gamma_{ab}\|$; b_0 is a positive constant measuring the world-volume "cosmological constant". g_{ab} (10) is the *induced* metric on the world-volume which becomes *singular* on-shell – manifestation of the lightlike nature of the p-brane.

(b) $(\sigma) \equiv (\sigma^a)$ with $a = 0, 1, \ldots, p$; $\partial_a \equiv \frac{\partial}{\partial\sigma^a}$.

(c) $X^{\mu}(\sigma)$ are the p-brane embedding coordinates in the bulk D-dimensional spacetime with Riemannian metric $g_{\mu\nu}(x)$ ($\mu, \nu = 0, 1, \ldots, D-1$).

(d) u is auxiliary world-volume scalar field defining the lightlike direction of the induced metric g_{ab} (10) and it is a non-propagating degree of freedom.

(e) T is *dynamical (variable)* membrane tension (also a non-propagating degree of freedom).

The Einstein Eqs.(8) imply the following relation between the lightlike membrane parameters and the Einstein-Rosen bridge "mass" ($r_0 = 2m$)

$$-T = \frac{1}{8\pi m} \,, \quad b_0 = \frac{1}{4} \,, \tag{11}$$

i.e., the lightlike membrane dynamical tension T becomes *negative* on-shell – manifestation of the "exotic matter" nature of the lightlike membrane.

4 Test Particle Dynamics and Traversability in the Original Einstein-Rosen Bridge

As already noted in [3] traversability of the original Einstein-Rosen bridge is a particular manifestation of the traversability of lightlike "thin-shell" wormholes[2]. Here for completeness we will present the explicit details of the traversability within the proper Einstein-Rosen bridge wormhole coordinate chart (7) which are needed for the construction of the pertinent Kruskal-Penrose diagram in Section 4.

[2]Subsequently, traversability of the Einstein-Rosen bridge has been studied using Kruskal-Szekeres coordinates for the Schwarzschild black hole [6], or the 1935 Einstein-Rosen coordinate chart (5) [7].

The motion of test-particle ("observer") of mass m_0 in a gravitational background is given by the reparametrization-invariant world-line action

$$S_{\text{particle}} = \frac{1}{2} \int d\lambda \left[\frac{1}{e} g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu - e m_0^2 \right] , \qquad (12)$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$, e is the world-line "einbein" and in the present case $(x^\mu) = (t, \eta, \theta, \varphi)$.

For a static spherically symmetric background such as (7) there are conserved Noether "charges" – energy E and angular momentum $\mathcal{J}$. In what follows we will consider purely "radial" motion ($\mathcal{J} = 0$) so, upon taking into account the "mass-shell" constraint (the equation of motion w.r.t. e) and introducing the world-line proper-time parameter τ ($\frac{d\tau}{d\lambda} = e m_0$), the timelike geodesic equations (world-lines of massive point particles) read

$$\left(\frac{d\eta}{d\tau}\right)^2 = \frac{E^2}{m_0^2} - A(\eta)\, , \quad \frac{dt}{d\tau} = \frac{E}{m_0 A(\eta)}\, , \quad A(\eta) \equiv \frac{|\eta|}{|\eta| + r_0}\, . \qquad (13)$$

where $A(\eta)$ is the "$-g_{00}$" component of the proper Einstein-Rosen bridge metric (7).

The first radial η-equation (13) exactly resembles classical energy-conservation equation for a "non-relativistic" particle with mass $\frac{1}{2}$ moving in an effective potential $\mathcal{V}_{\text{eff}}(\eta) \equiv \frac{1}{2} A(\eta)$ graphically depicted in Figure 1 below

$$\frac{d\eta}{d\tau} = \epsilon \sqrt{\frac{E^2}{m_0^2} - A(\eta)}\, , \quad \epsilon = \pm 1\, , \qquad (14)$$

depending whether $\eta(\tau)$ moves towards larger values ($\epsilon = +1$) or lower values ($\epsilon = -1$).

For a test-particle starting for $\tau = 0$ at initial position $\eta_0 = \eta(0)$, $t_0 = t(0)$ the solutions of Eqs.(13) read

$$\epsilon \frac{E}{2k_h m_0} \int\limits_{2k_h\eta_0}^{2k_h\eta(\tau)} dy \sqrt{(1 + |y|)\Big[(1 + \big(1 - \frac{m_0^2}{E^2}\big)|y|\Big]^{-1}} = \tau\, , \qquad (15)$$

$$\epsilon \frac{1}{2k_h} \int\limits_{2k_h\eta_0}^{2k_h\eta(\tau)} dy \frac{1}{|y|} \sqrt{(1 + |y|)^3 \Big[(1 + \big(1 - \frac{m_0^2}{E^2}\big)|y|\Big]^{-1}} = t(\tau) - t_0\, . \qquad (16)$$

In particular, Eq.(15) shows that the particle will cross the wormhole "throat" ($\eta = 0$) within a finite proper-time $\tau_0 > 0$

$$\tau_0 = \epsilon \frac{E}{2k_h m_0} \int\limits_{2k_h\eta_0}^{0} dy \sqrt{(1 + |y|)\Big[(1 + \big(1 - \frac{m_0^2}{E^2}\big)|y|\Big]^{-1}} \qquad (17)$$

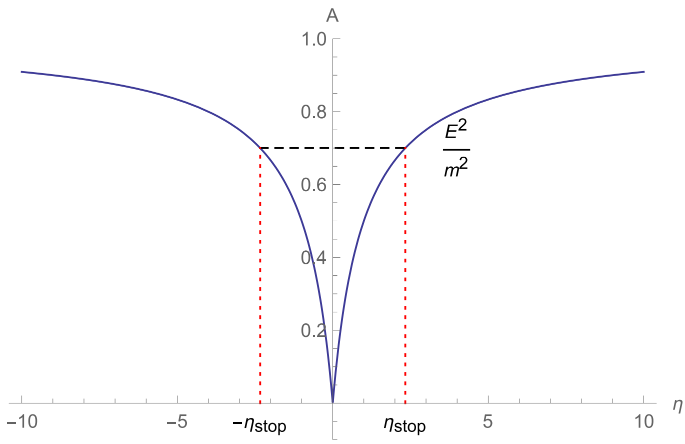

Figure 1. Graphical representation of $A(\eta)$ (13) with "reflection" points $\pm\eta_{\rm stop}$ (18) indicated.

(here $\epsilon = -1$ for $\eta_0 > 0$ and $\epsilon = +1$ for $\eta_0 < 0$).

Concerning the "laboratory" time t, it follows from (16) that $t(\tau_0 - 0) = +\infty$, *i.e.*, from the point of view of a static observer in "our" (right) universe it will take infinite "laboratory" time for the particle to reach the "throat" – the latter appears to the static observer as a future black hole horizon.

Eq.(16) also implies $t(\tau_0 + 0) = -\infty$, which means that from the point of view of a static observer in the second (left) universe, upon crossing the "throat", the particle starts its motion in the second (left) universe from infinite past, so that it will take an infinite amount of "laboratory" time to reach a point $\eta_1 < 0$ – *i.e.* the "throat" now appears as a past black hole horizon.

For small energies $E < m_0$ according to (15) the particle is trapped in an effective potential well and shuttles within finite proper-time intervals between the "reflection" points (see Figure 1)

$$\pm\eta_{\rm stop} = \Bigl(2k_H\bigl[m_0^2/E^2 - 1\bigr]\Bigr)^{-1} . \tag{18}$$

In Section 6 we will show that for a special value of m_0/E the pertinent particle geodesics is a closed timelike curve on the extended Kruskal-Penrose manifold.

In analogy with the usual "tortoise" coordinate r^* for the Schwarzschild black hole geometry (2) we now introduce Einstein-Rosen bridge "tortoise" coordinate η^* (recall $r_0 = \frac{1}{2k_h}$)

$$\frac{d\eta^*}{d\eta} = \frac{|\eta| + r_0}{|\eta|} \longrightarrow \eta^* = \eta + \mathrm{sign}(\eta) r_0 \log\bigl(|\eta|/r_0\bigr) . \tag{19}$$

Let us note here an important difference in the behavior of the "tortoise" coordinates r^* (2) and η^* (19) in the vicinity of the horizon. Namely,

$$r^* \to -\infty \quad \text{for} \;\; r \to r_0 \pm 0 \,, \tag{20}$$

i.e., when r approaches the horizon either from above or from below, whereas when η approaches the horizon from above or from below:

$$\eta^* \to \mp\infty \quad \text{for} \;\; \eta \to \pm 0 \,. \tag{21}$$

For infalling/outgoing massless particles (light rays) Eqs.(15)-(19) imply

$$t \pm \eta^* = \text{const} \,. \tag{22}$$

For infalling/outgoing massive particles we obtain accordingly

$$\begin{aligned} \big[t \pm \eta^*\big](\tau) = \frac{1}{2k_h} \int\limits_{2k_h\eta_0}^{2k_h\eta(\tau)} & dx \Big(1 + \frac{1}{|x|}\Big) \\ & \times \left[\epsilon \sqrt{(1+|x|)\Big[\big(1+\big(1-\frac{m_0^2}{E^2}\big)|x|\Big]^{-1}} \pm 1\right] \,. \end{aligned} \tag{23}$$

5 Kruskal-Penrose Manifold of the Original Einstein-Rosen Bridge

Following [8] we now introduce the maximal analytic extension of original Einstein-Rosen wormhole geometry (7) through the following Kruskal-like coordinates (v,w):

$$v = \pm\frac{1}{\sqrt{2k_h}} e^{\pm k_h(t+\eta^*)} \,, \quad w = \mp\frac{1}{\sqrt{2k_h}} e^{\mp k_h(t-\eta^*)} \,, \tag{24}$$

implying

$$-vw = \frac{1}{2k_h} e^{\pm 2k_h\eta^*} \,, \quad -\frac{v}{w} = e^{\pm 2k_h t} \,. \tag{25}$$

Here and below η^* is given by (19).

The upper signs in (24)-(25) correspond to region I $(v > 0, w < 0)$ describing "our" (right) universe $\eta > 0$, whereas the lower signs in (24)-(25) correspond to region II $(v < 0, w > 0)$ describing the second (left) universe $\eta < 0$ (see Figure 2).

Using the explicit expression (19) for η^* in (25) we find two "throats" (horizons) – at $v = 0$ or $w = 0$ corresponding to $\eta = 0$:

(a) In region I the "throat" $(v > 0, w = 0)$ is a future horizon $(\eta = 0, t \to +\infty)$, whereas the "throat" $(v = 0, w < 0)$ is a past horizon $(\eta = 0, t \to -\infty)$.

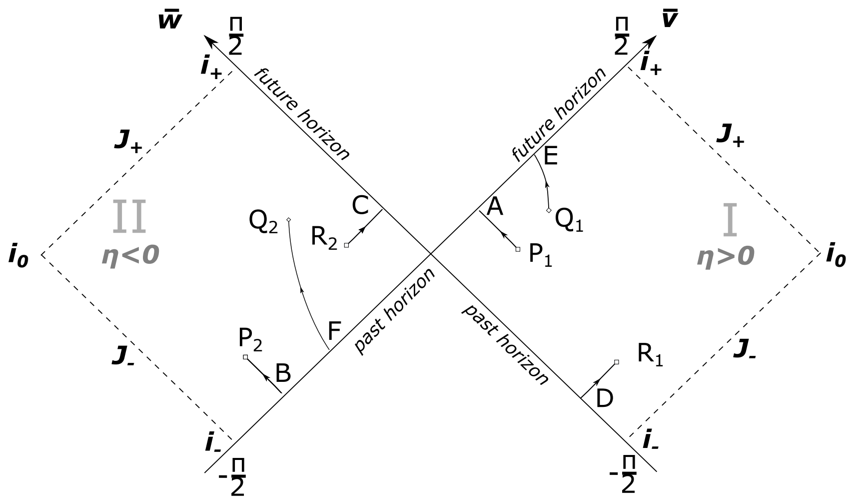

Figure 2. Kruskal-Penrose Manifold of the Original Einstein-Rosen Bridge

(b) In region II the "throat" $(v = 0, w > 0)$ is a future horizon $(\eta = 0, t \to +\infty)$, whereas the "throat" $(v < 0, w = 0)$ is a past horizon $(\eta = 0, t \to -\infty)$.

As usual one replaces Kruskal-like coordinates (v, w) (24) with compactified Penrose-like coordinates $(\bar{v}, \bar{w})$:

$$\bar{v} = \arctan(\sqrt{2k_h}\, v) = \pm \arctan\big(e^{\pm k_h (t+\eta^*)}\big) ,$$
$$\bar{w} = \arctan(\sqrt{2k_h}\, w) = \mp \arctan\big(e^{\mp k_h (t-\eta^*)}\big) , \tag{26}$$

mapping the various "throats" (horizons) and infinities to finite lines/points:

- In region I: future horizon $(0 < \bar{v} < \frac{\pi}{2}, \bar{w} = 0)$;
  past horizon $(\bar{v} = 0, -\frac{\pi}{2} < \bar{w} < 0)$.
- In region II: future horizon $(\bar{v} = 0, 0 < \bar{w} < \frac{\pi}{2})$;
  past horizon $(-\frac{\pi}{2} < \bar{v} < 0, \bar{w} = 0)$.
- i_0 – spacelike infinity ($t = \text{fixed}, \eta \to \pm\infty$):
  $i_0 = (\frac{\pi}{2}, -\frac{\pi}{2})$ in region I; $i_0 = (-\frac{\pi}{2}, \frac{\pi}{2})$ in region II.
- $i_\pm$ – future/past timelike infinity ($t \to \pm\infty, \eta = \text{fixed}$):
  $i_+ = (\frac{\pi}{2}, 0),\ i_- = (0, -\frac{\pi}{2})$ in region I;
  $i_+ = (0, \frac{\pi}{2}),\ i_- = (-\frac{\pi}{2}, 0)$ in region II.
- J_+ – future lightlike infinity ($t \to +\infty, \eta \to \pm\infty,\ t \mp \eta^* = \text{fixed}$):
  $J_+ = (\bar{v} = \frac{\pi}{2}, -\frac{\pi}{2} < \bar{w} < 0)$ in region I;
  $J_+ = (-\frac{\pi}{2} < \bar{v} < 0, \bar{w} = \frac{\pi}{2})$ in region II.
- J_- – past lightlike infinity ($t \to -\infty, \eta \to \pm\infty),\ t \pm \eta^* = \text{fixed}$):
  $J_- = (0 < \bar{v} < \frac{\pi}{2}, \bar{w} = -\frac{\pi}{2})$ in region I:
  $J_- = (\bar{v} = -\frac{\pi}{2}, 0 < \bar{w} < \frac{\pi}{2})$ in region II.

In Ref. [8], using the continuity of the light ray geodesics (22) when starting in one of the regions I or II and crossing the horizon ("throat") into the other one, we have exhibited the following mutual identification of a future horizon of one region with the past horizon of the second region (see Figure 2):

- Future horizon in region I is identified with past horizon in region II as:

$$(\bar{v}, 0) \sim (\bar{v} - \frac{\pi}{2}, 0) \,. \tag{27}$$

  Infalling light rays cross from region I into region II via paths $P_1 \to A \sim B \to P_2$ – all the way within finite world-line time intervals (the symbol $\sim$ means identification according to (27)). Similarly, infalling massive particles cross from region I into region II via paths $Q_1 \to E \sim F \to Q_2$ within finite proper-time interval.

- Future horizon in II is identified with past horizon in I:

$$(0, \bar{w}) \sim (0, \bar{w} - \frac{\pi}{2}) \,. \tag{28}$$

  Infalling light rays cross from region II into region I via paths $R_2 \to C \sim D \to R_1$ where $C \sim D$ is identified according to (28).

6 Closed Timelike Geodesics

Let us consider again massive test-particle dynamics with small energies $E < m_0$, which according to (15) means that the particle is trapped within an effective potential well (Fig.1).

Let the particle starts in "universe" I at $\eta = 0$ (past horizon) (at some point A of the Kruskal-Penrose coordinate chart as indicated on Fig.3) and then moves radially forward in η until it reaches the "reflection" point $\eta_{\rm stop}$ (18) (indicated by Q_1 in Figure 3) within the finite proper-time interval according to (15)

$$\begin{aligned}\Delta\tau = \frac{E}{2k_h m_0} \int_0^{2k_h \eta_{\rm stop}} dx \sqrt{(1+|x|)\Big[1 + \big(1 - \frac{m_0^2}{E^2}\big)|x|\Big]^{-1}} \\ = \frac{1}{b} + \frac{b+1}{b^{3/2}}\Big(\frac{\pi}{2} - \arctan(\sqrt{b})\Big) \,,\end{aligned} \tag{29}$$

where a short hand notation b is introduced

$$b \equiv \frac{m_0^2}{E^2} - 1 > 0 \,. \tag{30}$$

Then the particle proceeds by returning backward in η from $\eta_{\rm stop}$ (18) towards $\eta = 0$ (future horizon of Kruskal-Penrose region I) within the same proper-time interval (29) and it crosses the horizon at the point B in Figure 3. Thus, the particle enters Kruskal-Penrose region II ($\eta = 0$ – past horizon in II) at the point C

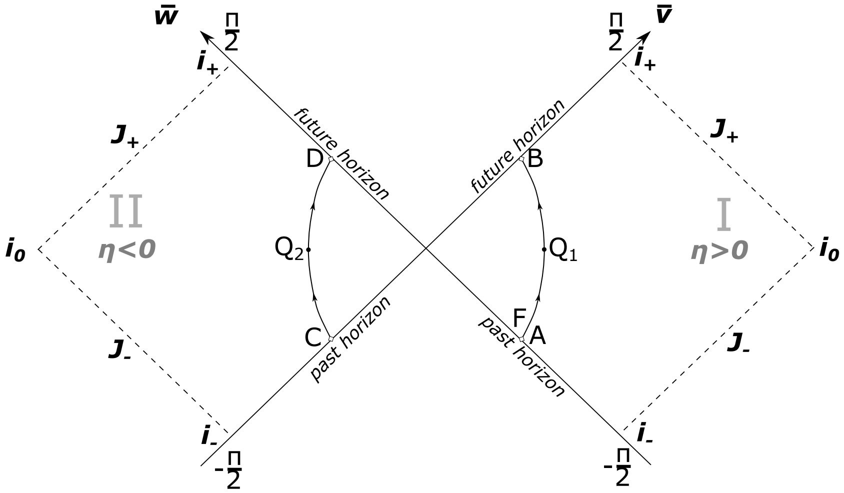

Figure 3. Geodesic test-particle trajectory $A \to Q_1 \to B \sim C \to Q_2 \to D \sim F$. Eq.(33) is the condition for $F = A$, *i.e.*, the geodesics is a CTC.

in Figure 3 which is "dual" to point B on the future horizon of I according to the future/past horizon identification (27)-(28). Now the particle continues towards negative η until it reaches within the same proper-time interval (29) the other "reflection point" $-\eta_{\rm stop}$ (18) (indicated by Q_2 in Figure 3). Finally, the particle returns from $-\eta_{\rm stop}$ (18) towards $\eta = 0$ (the future horizon of II) and reaches it at the point D on the Kruskal-Penrose chart (Fig.3) within the same proper-time interval (29). Then it crosses into region I at the point F on the past horizon of I, which is dually equivalent to D on the future horizon of II according to the horizon identification (27)-(28). Afterwards the particle continues again from F towards the first "reflection" point Q_1 in Figure 3.

We want now to find the conditions for the coincidence of the points $F = A$, *i.e.*, to find conditions for the existence of a closed timelike curve (CTC) meaning that the particle travels from some starting spacetime point in "universe" I, crosses into "universe" II, then crosses back into "universe" I and returns to the *same* starting spacetime point for a *finite* proper-time interval equal to $4\Delta\tau$ (29).

To describe the above geodesic curve $\big(\bar{v}(\tau), \bar{w}(\tau)\big)$ on the Kruskal-Penrose coordinate chart (recall Eqs.(26)) we need the explicit expressions for the integrals (23) (regarded as functions of $x \equiv 2k_h \eta$)

$$2k_h\big(t \pm \eta^*\big)(x) = \int dx\Big(1 + \frac{1}{|x|}\Big)\left[\epsilon\sqrt{\frac{1+|x|}{1+\big(1-\frac{m_0^2}{E^2}\big)|x|}} \pm 1\right]$$
$$= f_{\pm}^{(\epsilon)}(x) + c_{\pm}^{\epsilon}\,, \quad f_{\pm}^{(-1)}(x) = -f_{\mp}^{(1)}(x)\,. \quad (31)$$

Here

$$
\begin{aligned}
f^{(1)}_{\pm}(x) = \pm x - \frac{1}{b}\sqrt{(1+x)(1-bx)} \\
+ \log\Big[2 + x - bx - 2\sqrt{(1+x)(1-bx)}\Big] \\
+ \frac{1+3b}{2b^{3/2}}\arctan\Big(\frac{2bx+b-1}{b(1+x)(1-bx)}\Big) + \begin{pmatrix} 0 \\ -2\log x \end{pmatrix} ,
\end{aligned} \tag{32}
$$

where the short hand notation b (30) was used, and $c^{\epsilon}_{\pm}$ are integration constants determined from the matching conditions at the points of return (18) in regions I and II: $x = \pm 2k_h \eta_{\rm stop} = \pm 1/b$ (using notation (30)).

Using Eqs.(31)-(32) the condition for existence of CTC – coincidence on the past horizon of region I of the starting point of the particle geodesics A with the endpoint of the same geodesics F – yields the following condition on the value of the parameter b (30):

$$
\frac{1}{b} - \log(b+1) + \log 4 + \frac{1+3b}{2b^{3/2}}\Big[\frac{\pi}{2} - \arctan\Big(\frac{b-1}{\sqrt{b}}\Big)\Big] = 0 \tag{33}
$$

with a solution $b \approx 5.5876$, *i.e.* $m_0 \approx 2.5666\, E$.

Eq. (33) implies for the intergration constants

$$
\begin{aligned}
f^{(1)}_{(-)}(0) + c^{(1)}_{(-)} = 2k_h\big(t - \eta^*\big)(0) = 0\,, \\
f^{(-1)}_{(+)}(0) + c^{(-1)}_{(+)} = 2k_h\big(t + \eta^*\big)(0) = 0\,,
\end{aligned} \tag{34}
$$

where in the last equalities in (34) the definition (31) of $f^{(\pm 1)}_{(\mp)}(x)$ has been taken into account.

Relations (34) show that, for the CTC exhibited above, the points A, C and B, D are located exactly at the middle of the past/future horizons of regions I and II

$$
\begin{aligned}
\big(\bar{v}_A, \bar{w}_A\big) = \big(0, -\frac{\pi}{4}\big) \quad , \quad \big(\bar{v}_B, \bar{w}_B\big) = \big(\frac{\pi}{4}, 0\big)\,, \\
\big(\bar{v}_C, \bar{w}_C\big) = \big(-\frac{\pi}{4}, 0\big) \quad , \quad \big(\bar{v}_A, \bar{w}_A\big) = \big(0, \frac{\pi}{4}\big)\,.
\end{aligned} \tag{35}
$$

Existence of CTC's (also called "time machines") turns out to be quite typical phenomenon in wormhole physics (not necessarily of thin-shell wormholes) [9] (for a review, see [10]). This is due to the violation of the null-energy conditions in general relativity because of the presence of an "exotic matter" at the "throat". In the present case of the original Einstein-Rosen bridge as a specific example of a lightlike thin-shell wormhole, the violation of the null-energy conditions is manifesting itself via the negativity of the dynamical lightlike membrane tension T (11), *i.e.*, the lightlike membrane residing on the wormhole throat is an "exotic" lightlike thin-shell matter source.

7 Conclusions

We have discussed in some detail the basic properties of the mathematically consistent formulation of the original "Einstein-Rosen bridge" proposed in their classic 1935 paper. We have stressed a crucial feature (overlooked in the 1935 paper) of the correctly formulated original Einstein-Rosen bridge – it is *not* a solution of the vacuum Einstein equations but rather it the simplest example of a static spherically symmetric *traversable* lightlike "thin-shell" wormhole solution in general relativity. The consistency of the latter is guaranteed by the remarkable special properties of the world-volume dynamics of the lightlike membrane located at the wormhole "throat", which serves as an "exotic" thin-shell matter source of gravity.

- We have described the Kruskal-Penrose diagram representation of the original Einstein-Rosen bridge.
- The Kruskal-Penrose manifold of the original Einstein-Rosen bridge differs significantly from the well-known Kruskal-Penrose extension of the standard Schwarzschild black hole. Namely, Kruskal-Penrose coordinate chart of the original Einstein-Rosen bridge has only two regions corresponding to "our" (right) and the second (left) "universes" (two identical copies of the exterior Schwarzschild spacetime region) unlike the four regions in the standard Kruskal-Penrose chart of the Schwarzschild black hole, *i.e.*, now there are no black/white hole regions.
- There is a special pairwise identification between the future and past horizons of the neighbouring Kruskal-Penrose regions.
- We have explicitly exhibited traversability of the original Einstein-Rosen bridge w.r.t. "proper-time" of test-particles (travelling observers). In particular, we have found for a special relation between the energy and the mass of the test-particle a solution for the pertinent geodesics which is a *closed timelike curve*. The latter is a typical feature in wormhole physics with "exotic" matter sources.

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