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Multiple Multi-Orbit Fermionic and Bosonic Pairing and Rotational *SU*(3) Algebras

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Abstract. In nuclei with valence nucleons that are identical nucleons and occupy r number of j-orbits, there will be 2^{r-1} number of multiple pairing (quasi-spin) $SU(2)$ algebras with the generalized pair creation operator S_+ being a sum of single-j pair creation operators with arbitrary phases. Also, for each set of phases there will be a corresponding $Sp(2\Omega)$ algebra in $U(2\Omega) \supset Sp(2\Omega)$; $\Omega = \sum(2j+1)/2$. Using this correspondence, derived is the condition for a general one-body operator of angular momentum rank k to be a quasi-spin scalar or a vector vis-a-vis the phases in S_+. These will give special seniority selection rules for electromagnetic transitions. We found that the phase choice advocated by Arvieu and Moszkowski gives pairing Hamiltonians having maximum correlation with well known effective interactions. All the results derived for identical fermion systems are shown to extend to identical boson systems such as sd, sp, sdg and $sdpf$ interacting boson models (IBM's) with $SU(2) \to SU(1,1)$ and $Sp(2\Omega) \to SO(2\Omega)$. Going beyond pairing, for a given set of oscillator orbits, there are multiple rotational $SU(3)$ algebras both in shell model and IBM's. Different $SU(3)$ algebras in IBM's are shown, using sdgIBM as an example, to give different geometric shapes.

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1 Introduction

Pairing force and the related quasi-spin or seniority quantum number continue to play an important role in shell model (SM) in particular and nuclear structure in general [1, 2]. There are several single-j shell nuclei that are known to carry seniority quantum number as a good or useful quantum number [1, 3, 4]. Even when single shell seniority is a broken symmetry, seniority quantum number provides a basis for constructing shell model Hamiltonian matrices [5]. Pairing symmetry with nucleons occupying several j-orbits is more complex and less well understood from the point of view of its goodness and usefulness in nuclei. Restricting to nuclei with valence nucleons that are identical nucleons (protons or neutrons) and occupy several-j orbits, then it is possible to consider pair creation operator S_+ to be a sum of the single-j shell pair creation operators $S_+(j)$ with arbitrary phases α_j. This gives rise to quasi-spin $SU(2)$ algebra with multi-orbit or generalized seniority. However, with r number of j shells there will be 2^{r-1} $SU(2)$ algebras. Also, for each α_j set there is a corresponding sympletic algebra $Sp(2\Omega)$ in $U(2\Omega) \supset Sp(2\Omega)$ with $\Omega = \sum(2j+1)/2$. In this paper we will examine in detail these multiple pairing $SU(2)$ algebras in SM and also the corresponding multiple pairing algebras in the interacting boson models (IBM's) with identical bosons (e.g. sd, sdg, $sdpf$ IBM's). The usefulness or goodness of these multiple pairing algebras in SM is not well known (similarly in IBM's) except that a special situation was studied long time back by Arvieu and Moszkowski (AM) [6] in the context of surface delta interaction. Let us add that more general pairing with α_j's treated as free parameters, with three and higher-body pairing, using pair coupled basis and so on are topics of many recent studies [1, 7–10]. Besides pairing, the rotational $SU(3)$ symmetry is important, both in SM and IBM's, in describing quadrupole collective states in light to heavy nuclei and this symmetry continues to be of great interest; see [11–13] and references therein. The $SU(3)$ algebra will be present when nucleons (bosons) occupy all orbits in an oscillator shell. In an oscillator shell η, there will be $2^{\eta/2}$ number of $SU(3)$ algebras. Now we will give a preview.

Section 2 gives in some detail, algebraic structure of the multiple multi-orbit pairing quasi-spin algebras in SM, selection rules for electromagnetic transitions with multi-orbit seniority and the results for the correlation between realistic effective interactions and pairing operator with a given set of phases in S_+. Section 3 gives details of the multiple multi-orbit pairing algebras in interacting boson models with identical bosons. Applications of multi-orbit pairing algebras is given in Section 4. In Section 5, some results for multiple $SU(3)$ algebras are given with particular reference to geometric shapes. Finally, Section 6 gives conclusions.

2 Multiple Pairing *SU*(2) and Complimentary *Sp*(*N*) Algebras

Let us say there are m number of identical fermions (protons or neutrons) in j orbits j_1, j_2, ..., j_r. Now, it is possible to define a generalized pair creation operator S_+ as

$$S_+ = \sum_j \alpha_j S_+(j) \; ; \; S_+(j) = \frac{\sqrt{2j+1}}{2} \left(a_j^\dagger a_j^\dagger\right)^0 . \tag{1}$$

Here, α_j are free parameters and real. The m used for number of particles should not be confused with the m in jm. Given the S_+ operator, the corresponding pair annihilation operator $S_- = (S_+)^\dagger$ and $S_0 = (\hat{n} - \Omega)/2$ form the generalized quasi-spin SU(2) algebra [hereafter called $SU_Q(2)$] if $\alpha_j^2 = 1$ for all j. Note that $\hat{n}$ is the number operator and $\Omega = \sum_j \Omega_j$ with $\Omega_j = (2j+1)/2$. Thus, in the multi-orbit situation for each $\{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}\}$ with $\alpha_{j_i} = \pm 1$ there is a $SU_Q(2)$ algebra. Then, for r number of j orbits there will be 2^{r-1} number of $SU_Q(2)$ algebras. The consequences of having these multiple pairing $SU_Q(2)$ algebras will be investigated in the following.

Firstly, it is well known that with $SU_Q(2)$, we have seniority classification of m particle states giving $|m, v, \beta\rangle$ states; β is an extra label that is required to specify the m particle state completely. The seniority quantum number $v = m$, $m-2$, …, 0 or 1 for $m \leq \Omega$ and $v = (2\Omega - m)$, $(2\Omega - m) - 2$, …, 0 or 1 for $m \geq \Omega$. Now, the pairing Hamiltonian $H_p = -4GS_+S_-$ and its eigenvalues are $\langle -4GS_+S_- \rangle^{mv} = -G(m-v)(2\Omega - m - v + 2)$. In the $(j_1, j_2, \ldots, j_r)^m$ space, often it is more convenient to start with the $U(N)$ algebra ($N = 2\Omega$) generated by the one-body operators $u^k_q(j_1, j_2) = (a^\dagger_{j_1} \tilde{a}_{j_2})^k_q$. Then, m fermion states belong to the irreducible representation (irrep) $\{1^m\}$ of $U(N)$. The quadratic Casimir invariant of $U(N)$ is easily given by $C_2(U(N)) = \sum_{j_1,j_2} (-1)^{j_1-j_2} \sum_k u^k(j_1, j_2) \cdot u^k(j_2, j_1)$ with eigenvalues $\langle C_2(U(N)) \rangle^m = m(N+1-m)$. More importantly, $U(N) \supset Sp(N)$. The $N(N+1)/2$ number of $Sp(N)$ generators are $u^k_q(j,j)$ with k=odd and $V^k_q(j_1, j_2)$ [with $j_1 > j_2$ and $X(j_1, j_2, k) = \pm 1$],

$$V^k_q(j_1, j_2) = [\mathcal{N}(j_1, j_2, k)]^{1/2} \left[\left(a^\dagger_{j_1} \tilde{a}_{j_2} \right)^k_q + X(j_1, j_2, k) \left(a^\dagger_{j_2} \tilde{a}_{j_1} \right)^k_q \right] . \quad (2)$$

The quadratic Casimir invariant of $Sp(N)$ is $C_2(Sp(N)) = 2\sum_j \sum_{k=odd} u^k(j,j) \cdot u^k(j,j) + \sum_{j_1>j_2;k} V^k(j_1, j_2) \cdot V^k(j_1, j_2)$. Single most important result that can be proved, after some algebra, is that the $Sp(N)$ algebra is complimentary to $SU_Q(2)$ algebra defined for a given set of $\{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}\}$ provided

$$\mathcal{N}(j_1, j_2, k) = (-1)^{k+1} \alpha_{j_1} \alpha_{j_2} \;, \quad X(j_1, j_2, k) = (-1)^{j_1+j_2+k} \alpha_{j_1} \alpha_{j_2} \; . \quad (3)$$

Using Eqs. (2) and (3), it is easy to derive the important relation, $C_2(U(N)) - C_2(Sp(N)) = 4S_+S_- - \hat{n}$ and then $\langle C_2(Sp(N)) \rangle^{m,v} = v(2\Omega + 2 - v)$. As the $Sp(N)$ generators are one-body operators and that $Sp(N) \leftrightarrow SU_Q(2)$, there will be special selection rules for electro-magnetic transition operators connecting m fermion states with good seniority. Their relation to the $\{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}\}$ set, to our best of knowledge, is not discussed before in literature. We will turn to this now.

2.1 Selection rules for electro-magnetic transitions

Electro-magnetic (EM) operators T^{EL} and T^{ML} respectively with $L = 1, 2, 3, \ldots$, are one-body operators (two and higher-body terms are usually not considered). Their general form with $X = E$ or M is,

$$T_q^{XL} = \sum_j \epsilon_{j,j}^{XL} \left(a_j^\dagger \tilde{a}_j\right)_q^L + \sum_{j_1 > j_2} \epsilon_{j_1,j_2}^{XL} \left[\left(a_{j_1}^\dagger \tilde{a}_{j_2}\right)_q^L + \frac{\epsilon_{j_2,j_1}^{XL}}{\epsilon_{j_1,j_2}^{XL}} \left(a_{j_2}^\dagger \tilde{a}_{j_1}\right)_q^L\right] . \quad (4)$$

Now, the commutator of T^{XL} with S_+ will give the seniority selection rules for allowed transitions, the commutator is zero implies that the operator is a scalar T_0^0 with respect to $SU_Q(2)$ and otherwise it will be a quasi-spin vector T_0^1. In either situation the S_z component of T is zero as a one-body operator can not change particle number. For $j_1 \neq j_2$ we have

$$\begin{aligned} &\left[S_+ \, , \, \left(a_{j_1}^\dagger \tilde{a}_{j_2}\right)_q^k + X(a_{j_2}^\dagger \tilde{a}_{j_1})_q^k\right] \\ &\qquad\qquad = -\alpha_{j_2} \left(a_{j_1}^\dagger a_{j_2}^\dagger\right)_q^k \left\{1 - X \, \alpha_{j_1} \alpha_{j_2} \, (-1)^{j_1+j_2+k}\right\} . \end{aligned} \quad (5)$$

Therefore,

$$\begin{aligned} \left[\epsilon_{j_2,j_1}^{XL}\right] / \left[\epsilon_{j_1,j_2}^{XL}\right] &= \alpha_{j_1} \alpha_{j_2} (-1)^{j_1+j_2+L} \to T_0^0 \;\; \text{w.r.t.} \;\; SU_Q(2) \; , \\ \left[\epsilon_{j_2,j_1}^{XL}\right] / \left[\epsilon_{j_1,j_2}^{XL}\right] &= -\alpha_{j_1} \alpha_{j_2} (-1)^{j_1+j_2+L} \to T_0^1 \;\; \text{w.r.t.} \;\; SU_Q(2) \; . \end{aligned} \quad (6)$$

Similarly, for $j_1 = j_2$ we have $(a_j^\dagger \tilde{a}_j)_q^k$ with k odd will be T_0^0 and k even (except for $k = 0$) will be T_0^1 w.r.t. $SU_Q(2)$. These results are consistent with the $Sp(N) \leftrightarrow SU_Q(2)$ equivalence. Thus, the $SU_Q(2)$ tensorial nature of T^{XL} depend on the α_i choice. For T_0^0 we have $v \to v$ and for T_0^1 we have $v \to v, \; v{\pm}2$ transitions. It is well known [1, 6] that for T^{EL} and T^{ML} operators,

$$\frac{\epsilon_{j_2,j_1}^{EL}}{\epsilon_{j_1,j_2}^{EL}} = -(-1)^{\ell_1+\ell_2+j_1+j_2+L} , \qquad \frac{\epsilon_{j_2,j_1}^{ML}}{\epsilon_{j_1,j_2}^{ML}} = (-1)^{\ell_1+\ell_2+j_1+j_2+L} . \quad (7)$$

In Eq. (7), ℓ_i is the orbital angular momentum of the j_i orbit. Therefore, combining results in Eqs. (6) with (7) together with parity selection rule will give seniority selection rules, in the multi-orbit situation, for electro-magnetic transition operators when the observed states carry seniority quantum number as a good quantum number. The selection rules with the choice $\alpha_{j_i} = (-1)^{\ell_i}$ for all i are as follows. (i) T^{EL} with L even will be T_0^1 w.r.t. $SU_Q(2)$. (ii) T^{EL} with L odd will be T_0^1 w.r.t. $SU_Q(2)$. However, if all j orbits have same parity, then T^{EL} with L odd will not exist. (iii) T^{ML} with L odd will be T_0^0 w.r.t. $SU_Q(2)$. (iv) T^{ML} with L even will be T_0^0 w.r.t. $SU_Q(2)$. However, if all j orbits have same parity, then T^{ML} with L even will not exist. These rules were given already by AM [1, 6] and as stated in [6], AM introduced the choice $\alpha_i = (-1)^{\ell_i}$ "for convenience ". It is important to note that for $SU_Q(2)$ generated by $\alpha_i \neq (-1)^{\ell_i}$, the above rules (i)-(iv) will be violated and then Eq. (6) has to be applied. This is a new result not reported before, to our knowledge, in the literature.

2.2 Correlation between pairing operator and effective interactions

In m particle spaces it is possible to define, using the spectral distribution method of French [14], a geometry with norm (or size or length) of an operator $\mathcal{O}$ given by $|| \mathcal{O} ||_m = [< [\tilde{\mathcal{O}}]^\dagger \tilde{\mathcal{O}} >^m]^{1/2}$; $\tilde{\mathcal{O}}$ is the traceless part of $\mathcal{O}$. Then, the correlation coefficient $\zeta(\mathcal{O}_1, \mathcal{O}_2) = \{|| \mathcal{O}_1 ||_m || \mathcal{O}_2 ||_m]\}^{-1} < \left[\widetilde{\mathcal{O}_1}\right]^\dagger \widetilde{\mathcal{O}_2} >^m$ gives the cosine of the angle between the operators $\mathcal{O}_1$ and $\mathcal{O}_2$. Most recent application of norms and correlation coefficients, in understanding the structure of effective interactions, is due to Draayer et al [15].

Given a realistic effective interaction Hamiltonian H, ζ defined above can be used as a measure for its closeness to the pairing Hamiltonian $H_P = S_+S_-$ for a given set of α_j's defining S_+. Following this, $\zeta(H, H_P)$ is evaluated for effective interactions in (${}^0f_{7/2}$, ${}^0f_{5/2}$, ${}^1p_{3/2}$, ${}^1p_{1/2}$), (${}^0f_{5/2}$, ${}^1p_{3/2}$, ${}^1p_{1/2}$, ${}^0g_{9/2}$) and (${}^0g_{7/2}$, ${}^1d_{5/2}$, ${}^1d_{3/2}$, ${}^2s_{1/2}$, ${}^0h_{11/2}$) spaces using GXPF1 [16], JUN45 [17] and jj55-SVD [18] interactions respectively. As we are considering only identical particle systems and also as we are studying correlation of H's with H_P's, only the $T = 1$ part of the interactions is considered. Results presented in Table 1 clearly show that the choice $\alpha_j = (-1)^{\ell_i}$ gives the largest value for ζ and hence it should be the most preferred choice. This is a significant result justifying the choice made by AM [6], although the magnitude of ζ is not more

Table 1. Correlation coefficient ζ between a realistic interaction (H) and the pairing Hamiltonian H_p for various particle numbers (m) in three different spectroscopic spaces. The phases α_j for each orbit in the generalized pair creation operator are given in column #4 (the order is same as the sp orbits listed in column #1). The variation in ζ with particle number m is given in column #5. Results for the phase choices that give $|\zeta| < 0.1$ for all m values are not shown in the table.

sp orbits	interaction	m	α_j	$\zeta(H, H_p)$
${}^0g_{7/2}$, ${}^1d_{5/2}$, ${}^1d_{3/2}$, ${}^2s_{1/2}$, ${}^0h_{11/2}$	jj55-SVD	$2-30$	$(+,+,+,+,-)$	0.33-0.11
			$(+,+,+,-,-)$	0.26-0.09
			$(+,+,-,+,-)$	0.17-0.06
			$(+,+,-,-,-)$	0.13-0.04
			$(+,-,+,+,-)$	0.11-0.04
${}^0f_{5/2}$, ${}^1p_{3/2}$, ${}^1p_{1/2}$, ${}^0g_{9/2}$	jun45	$2-20$	$(+,+,+,-)$	0.42-0.21
			$(+,+,-,-)$	0.27-0.13
			$(+,-,+,-)$	0.15-0.07
			$(+,-,-,-)$	0.12-0.06
${}^0f_{7/2}$, ${}^1p_{3/2}$, ${}^0f_{5/2}$, ${}^1p_{1/2}$	gxpf1	$2-18$	$(+,+,+,+)$	0.36-0.33
			$(+,+,+,-)$	0.22-0.20
			$(+,-,+,+)$	0.13-0.12
			$(+,-,+,-)$	0.13-0.11
			$(+,-,-,-)$	0.11-0.10

than 0.3. Thus, realistic H are far, on a global m-particle space scale, from the simple pairing Hamiltonian. However, it is likely that the generalized pairing or sympletic symmetry may be an effective symmetry for low-lying state and some special high-spin states. Evidence for this will be discussed in Section 4.

3 Multiple Pairing *SU*(1,1) and Complimentary *SO*(*N*) Algebras in Interacting Boson Models

In interacting boson models [19] with identical bosons as in sd, sp, sdg and $sdpf$ IBM's, it is possible to have multiple pairing symmetry algebras as we have several ℓ orbits in these models [11,20,21]. The pairing algebra here is $SU_Q(1,1)$ with the generalized boson pair creation operator $S_+^B = \sum_\ell \beta_\ell S_+^B(\ell)$, $\beta_\ell = \pm 1$. Note that $S_+^B(\ell) = (b_\ell^\dagger \cdot b_\ell^\dagger)/2$. The $SU_Q(1,1)$ is generated by S_+^B, $S_-^B = \left(S_+^B\right)^\dagger$ and $S_0^B = (\hat{n}^B + \Omega^B)/2$ where $\hat{n}^B$ is number operator and $\Omega^B = \sum_\ell \Omega_\ell^B$; $\Omega_\ell^B = (2\ell+1)/2$. Thus, in general for r number of ℓ orbits there will be 2^{r-1} number of $SU_Q^B(1,1)$ algebras. Just as for fermions, in the $(\ell_1, \ell_2, \ldots, \ell_r)^{N^B}$ space, there is a $SO(\mathcal{N}) \leftrightarrow SU_B(1,1)$ in $U(\mathcal{N}) \supset SO(\mathcal{N})$; $\mathcal{N} = 2\Omega^B$. The $U(\mathcal{N})$ generators are $u_q^k(\ell_1,\ell_2) = \left(b_{\ell_1}^\dagger \tilde{b}_{\ell_2}\right)_q^k$ and N^B boson states belong to the irrep $\{N^B\}$ of $U(\mathcal{N})$. The quadratic Casimir invariant of $U(\mathcal{N})$ is easily given by $C_2(U(\mathcal{N})) = \sum_{\ell_1,\ell_2} (-1)^{\ell_1+\ell_2} \sum_k u^k(\ell_1,\ell_2) \cdot u^k(\ell_2,\ell_1)$ with eigenvalues $\langle C_2(U(\mathcal{N}))\rangle^{N^B} = N^B(N^B + \mathcal{N} - 1)$. More importantly, given $SU_B(1,1)$ with given a set of β_ℓ's, the generators of the complimentary $SO(\mathcal{N})$ are [20],

$$
\begin{aligned}
& SO(\mathcal{N}) \; : u_q^k(\ell,\ell) \;\; \text{with} \;\; k \;\; \text{odd} \; , \\
& V_q^k(\ell_1,\ell_2) = \{(-1)^{\ell_1+\ell_2} Y(\ell_1,\ell_2,k)\}^{1/2} [\left(b_{\ell_1}^\dagger \tilde{b}_{\ell_2}\right)_q^k + Y(\ell_1,\ell_2,k) (b_{\ell_2}^\dagger \tilde{b}_{\ell_1})_q^k] \; ; \\
& Y(\ell_1,\ell_2,k) = (-1)^{k+1} \beta_{\ell_1}\beta_{\ell_2} \; .
\end{aligned}
\tag{8}
$$

Using these, it is seen that $4S_+^B S_-^B = C_2(U(\mathcal{N})) - \hat{n}^B - C_2(SO(\mathcal{N}))$ and $\langle C_2(SO(\mathcal{N}))\rangle^{N^B,\omega^B} = \omega^B(\omega^B + \mathcal{N} - 2)$ with the seniority quantum number $\omega^B = N^B$, $N^B - 2, \ldots, 0$ or 1. Now, given a general one-body operator

$$
T_q^k = \sum_\ell \epsilon_{\ell,\ell}^k \left(b_\ell^\dagger \tilde{b}_\ell\right)_q^k + \sum_{\ell_1 > \ell_2} \epsilon_{\ell_1,\ell_2}^k \left[\left(b_{\ell_1}^\dagger \tilde{b}_{\ell_2}\right)_q^k + \frac{\epsilon_{\ell_2,\ell_1}^k}{\epsilon_{\ell_1,\ell_2}^k} \left(b_{\ell_2}^\dagger \tilde{b}_{\ell_1}\right)_q^k\right] \; , \tag{9}
$$

as $SO(N) \leftrightarrow SU_B(1,1)$, it is clear from the generators in Eq. (8) that the diagonal $\left(b_\ell^\dagger \tilde{b}_\ell\right)_q^k$ part will be $SU_Q^B(1,1)$ scalar T_0^0 for k odd and vector T_0^1 for k even (except for $k=0$). Similarly, the off diagonal $\ell_1 \neq \ell_2$ part will be

$$
\frac{\epsilon_{\ell_2,\ell_1}^k}{\epsilon_{\ell_1,\ell_2}^k} = (-1)^{k+1} \beta_{\ell_1}\beta_{\ell_2} \to T_0^0 \; , \quad \frac{\epsilon_{\ell_2,\ell_1}^k}{\epsilon_{\ell_1,\ell_2}^k} = (-1)^k \beta_{\ell_1}\beta_{\ell_2} \to T_0^1 \; . \tag{10}
$$

Thus, the selection rules for the boson systems are similar to those for the fermion systems. Results in Eqs. (8) and (10) together with a condition for the seniority tensorial structure will allow us to write proper forms for the EM operators in boson systems. If we impose the condition that the $T^{E,L=even}$ and $T^{M,L=odd}$ operators are T_0^1 and T_0^0 w.r.t. $SU_Q^B(1,1)$, just as the fermion operators are w.r.t. $SU_Q(2)$ (see Section 2), then

$$T^L = \sum_{\ell} \epsilon^L_{\ell,\ell} \left(b^\dagger_\ell \tilde{b}_\ell\right)^L_q + \sum_{\ell_1>\ell_2} \epsilon^L_{\ell_1,\ell_2} \left[\left(b^\dagger_{\ell_1} \tilde{b}_{\ell_2}\right)^L_q + \beta_{\ell_1}\beta_{\ell_2} \left(b^\dagger_{\ell_2} \tilde{b}_{\ell_1}\right)^L_q\right] . \quad (11)$$

However, if we impose the condition that T^{EL} is T_0^0 w.r.t. $SU_Q^B(1,1)$, then

$$\begin{aligned} T^{E,L=even} &= \sum_{\ell_1>\ell_2} \epsilon^L_{\ell_1,\ell_2} \left[\left(b^\dagger_{\ell_1} \tilde{b}_{\ell_2}\right)^L_q - \beta_{\ell_1}\beta_{\ell_2} \left(b^\dagger_{\ell_2} \tilde{b}_{\ell_1}\right)^L_q\right] \; ; \; (-1)^{\ell_1+\ell_2} = 1 \,, \\ T^{E,L=odd} &= \sum_{\ell_1>\ell_2} \epsilon^L_{\ell_1,\ell_2} \left[\left(b^\dagger_{\ell_1} \tilde{b}_{\ell_2}\right)^L_q + \beta_{\ell_1}\beta_{\ell_2} \left(b^\dagger_{\ell_2} \tilde{b}_{\ell_1}\right)^L_q\right] \; ; \; (-1)^{\ell_1+\ell_2} = -1 \,. \end{aligned} \quad (12)$$

Similarly, T^{ML} can be chosen to be T_0^1 w.r.t. $SU_Q^B(1,1)$.

4 Applications of Multi-Orbit Multiple Pairing Algebras

Firstly, in good seniority states, the matrix elements of T_0^0 type operators do not depend on particle number but only on seniority and they will be only of $v \to v$ type. For T_0^1 type operators we have $v \to v$, $v \pm 2$ transitions and more significantly, the number dependence of these matrix elements can be factored out. The factor for $v \to v$ transitions is $(\Omega - m)/(\Omega - v)$ and for $v-2 \to v$ it is $[(2\Omega - m - v + 2)(m - v + 2)/4(\Omega - v + 1)]^{1/2}$. These results will go to those of boson systems with the well known $\Omega \to -\Omega$ symmetry [20]. Using these, numerical results for the variation of $B(EL)$ with particle number are shown in Figures 1a,b for fermion systems and 1c,d for boson systems.

There is now good data available for $B(E2)$'s and $B(E1)$'s for some high-spin isomer states in even Sn isotopes. These are: $B(E2; 10^+ \to 8^+)$ data for ^{116}Sn to ^{130}Sn and $B(E2; 15^- \to 13^-)$ for ^{120}Sn to ^{128}Sn and $B(E1; 13^- \to 12^+)$ in ^{120}Sn to ^{126}Sn. The states 10^+ and 8^+ are interpreted to be $v = 2$ states while 15^-, 13^- and 12^+ are $v = 4$ states. Therefore, all these transitions are $v \to v$ transitions and their variation with m follows the results in Figure 1a. This is well verified by data [22] by assuming that the active sp orbits are $^0h_{11/2}$, $^1d_{3/2}$ and $^2s_{1/2}$ with $\Omega = 9$ (see also Figure 1a with $\Omega = 9$). The results with $\Omega = 8$ and $\Omega = 7$ obtained by dropping $^2s_{1/2}$ and $^1d_{3/2}$ orbits respectively, are not in good accord with data. In addition, recently $B(E2; 2_1^+ \to 0_1^+)$ data for ^{104}Sn to ^{130}Sn are compiled and the data shows a dip at ^{116}Sn and they are close to adding two displaced parabolas; see Figure 1 in [23]. This is understood by

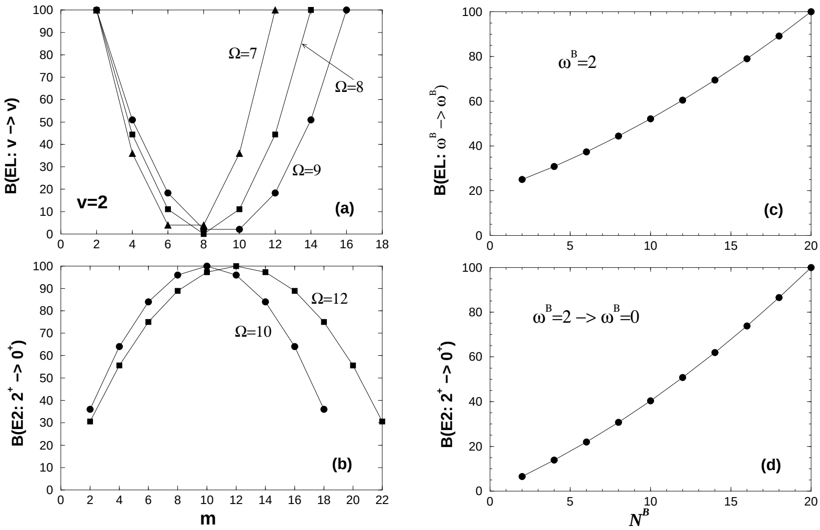

Figure 1. (a) Variation of $B(EL)$ with particle number m for three different values of Ω and seniority $v = 2$ for $v \to v$ transitions. (b) Variation of $B(E2; v = 2, 2^+ \to v = 0, 0^+)$ with particle number for two different values of Ω. (c) Variation of $B(EL)$ with particle number N^B for $\Omega = 16$ and seniority $\omega^B = 2$ for $\omega^B \to \omega^B$ transitions. (d) Variation of $B(E2; \omega^B = 2, 2^+ \to \omega^B = 0, 0^+)$ with particle number N^B. All values are scaled such that the maximum value is 100 and they are not in any units. The T^{EL} operators are assumed to be of T^1_0 type.

employing ${}^0g_{7/2}$, ${}^1d_{5/2}$, ${}^1d_{3/2}$ and ${}^2s_{1/2}$ orbits for neutrons in ^{104}Sn to ^{116}Sn with $\Omega = 10$ and ^{100}Sn core. Similarly, ${}^1d_{5/2}$, ${}^1d_{3/2}$, ${}^2s_{1/2}$ and ${}^0h_{11/2}$ orbits with $\Omega = 12$ and ^{108}Sn core for ^{116}Sn to ^{130}Sn. Then, the $B(E2)$ vs m structure follows from Figure 1b by shifting appropriately the centers of the two parabolas in the figure and defining properly the beginning and end points. These results show that for Sn isotopes generalized seniority, with the choice $\alpha_j = (-1)^{\ell_j}$ and effective Ω values, is possibly an 'emergent symmetry'.

Turning to the interacting boson models, firstly $B(EL)$ values increase with N^B and this trend is seen in $B(E2)$ values of Ru isotopes [24]. More importantly, the results in Section 3 will allow for the determination of the structure of EM operators given the choice for the structure of the pairing operator. This consistency is not maintained in many of the IBM studies presented in literature. Let us consider the example of $sdpf$ model [21] applied recently with good success in describing $E1$ strength distributions in Nd, Sm, Gd and Dy isotopes [25] and also spectroscopy of even-even ${}^{98-110}$Ru isotopes [24]. In $sdpf$IBM, there will be eight generalized pairs S_+ and the algebra complimentary to the $SU_Q(1,1)$ is $SO(16)$ in $U(16) \supset SO(16)$. Keeping intact the $SO(6)$ pair structure of sdIBM as chosen by Arima and Iachello, we will have four S_+ pairs,

$S_+ = s^\dagger s^\dagger - d^\dagger\cdot d^\dagger \pm p^\dagger\cdot p^\dagger \pm f^\dagger\cdot f^\dagger$ giving $SU_Q^{+,-,\pm,\pm}(1,1)$ and correspondingly four $SO^{+,-,\pm,\pm}(16)$ algebras. For each of the four choices, one can write down the T^{E2} and T^{E1} operators that transform as T_0^0 or T_0^1 w.r.t. $SU_Q(1,1)$. In [21], $SU_Q^{+,-,-,-}(1,1)$ is employed. With respect to this, the $E2$ and $E1$ operators employed in [21, 24, 25] will be mixtures of T_0^0 and T_0^1. For example, the $E1$ operator used is $T^{E1} = \alpha_{sp}(s^\dagger\tilde{p}+p^\dagger\tilde{s})^1_\mu + \alpha_{pd}(p^\dagger\tilde{d}+d^\dagger\tilde{p})^1_\mu + \alpha_{df}(d^\dagger\tilde{f}+f^\dagger\tilde{d})^1_\mu$ and this whole operator can be made T_0^0 by using instead $\alpha_{sp}\left(s^\dagger\tilde{p} - p^\dagger\tilde{s}\right)^1_\mu$.

5 Multiple *SU*(3) Algebras in SM and IBM

Given an oscillator shell with shell number η, the orbital angular momentum carried by a particle (fermion or boson) in the shell takes values $\ell = \eta$, $\eta - 2$, …, 0 or 1. For simplicity, let us consider identical bosons in the shell η. Now, the angular momentum operator L^1_q and quadrupole moment operator $Q^2_q = \sqrt{\frac{2\pi}{5}} r^2 Y^2_q(\theta,\phi)$ are given by

$$\begin{aligned}
L^1_q &= \sum_\ell \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{3}} \left(b^\dagger_\ell \tilde{b}_\ell\right)^1_q\,, \\
Q^2_q &= -(2\eta+3)\sum_\ell \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{40(2\ell+3)(2\ell-1)}} (b^\dagger_\ell \tilde{b}_\ell)^2_q \\
&\quad + \sum_{\ell<\eta} \alpha_{\ell,\ell+2} \sqrt{\frac{3(\ell+1)(\ell+2)(\eta-\ell)(\eta+\ell+3)}{20(2\ell+3)}} [(b^\dagger_\ell \tilde{b}_{\ell+2})^2_q + (b^\dagger_{\ell+2}\tilde{b}_\ell)^2_q].
\end{aligned} \tag{13}$$

After some tedious algebra, it can be shown that for $\alpha_{\ell,\ell+2}$ is $+1$ or -1, the eight operators (L^1_q, Q^2_q) form the Lie algebra of $SU(3)$ (the commutators will be independent of the α's). It is important to stress that Eq. (13) and the commutators are also valid for fermion systems (in orbital space) and therefore, there will be multiple $SU(3)$ algebras in both SM and IBM. In general, for a given η there will be $2^{[\eta/2]}$ number of $SU(3)$ algebras; $[\eta/2]$ is the integer part of $\eta/2$. Thus, in sdIBM there will be two $SU(3)$ algebras as discussed in the context of QPT [26]. They generate prolate and oblate shapes respectively. Also, in nuclear $(2s1d)$ shell, with nucleons, there will be two $SU(3)$ algebras [3].

Going beyond sdIBM, in sdgIBM there will be four $SU(3)$ algebras. Employing the coherent state (CS) in terms of $(\beta_2, \beta_4, \gamma)$ for a N^B boson system, as employed before in [27, 28], the CS expectation value of

$$H_{SU_{sdg}(3)} = -Q^2(S)\cdot Q^2(S), \quad \text{with} \quad Q^2_\mu(S) = \sqrt{8/3}Q^2_\mu,$$

is given by

$$
\begin{aligned}
E_{SU_{sdg}(3)}(N^B;\beta_2,\beta_4,\gamma) &= \left\langle N^B;\beta_2;\beta_4,\gamma \mid -Q^2(S)\cdot Q^2(S) \mid N^B;\beta_2;\beta_4,\gamma\right\rangle \\
&= \frac{-(N^B)^2}{(1+\beta_2^2+\beta_4^2)^2}\Big[\frac{448}{15}\,\alpha_{sd}^2\,\beta_2^2 + \frac{384\sqrt{14}}{35}\,\alpha_{sd}\alpha_{dg}\beta_2^2\beta_4 \\
&+ \frac{352\sqrt{35}}{105}\,\alpha_{sd}\beta_2^3\cos 3\gamma + \frac{64\sqrt{35}}{21}\,\alpha_{sd}\beta_2\beta_4^2\cos 3\gamma + \frac{3456}{245}\,\alpha_{dg}^2\beta_2^2\beta_4^2 \\
&+ \frac{1056\sqrt{10}}{245}\,\alpha_{dg}\beta_2^3\beta_4\cos 3\gamma + \frac{484}{147}\,\beta_2^4 + \frac{192\sqrt{10}}{49}\,\alpha_{dg}\beta_2\beta_4^3\cos 3\gamma \\
&+ \frac{880}{441}\,(4-\cos^2 3\gamma)\beta_2^2\beta_4^2 + \frac{400}{1323}\,(16-7\cos^2 3\gamma)\beta_4^4\Big]. \qquad (14)
\end{aligned}
$$

Minimizing $E_{SU_{sdg}(3)}$ with respect to β_2, β_4 and γ will give the equilibrium (ground state) shape parameters (β_2^0, β_4^0, γ^0) and the corresponding equilibrium energy $E^0_{SU_{sdg}(3)}$. As seen from the Table 2, the four values of $(\alpha_{sd}, \alpha_{dg})$ generate the four combinations of (prolate, oblate) deformations in (β_2, β_4) but all with same E^0. This energy value is same as the large N^B eigenvalue of the quadratic Casimir invariant of $SU(3)$ in the $(4N^B, 0)$ irrep.

Table 2. Equilibrium shapes for the four $SU(3)$ algebras in sdgIBM

α_{sd}	α_{dg}	β_2^0	β_4^0	γ^0	$E^0_{SU_{sdg}(3)}$
+1	+1	$\sqrt{20/7}$	$\sqrt{8/7}$	$0°$	$-(64/3)N^2$
-1	+1	$\sqrt{20/7}$	$-\sqrt{8/7}$	$60°$	$(-64/3)N^2$
+1	-1	$\sqrt{20/7}$	$-\sqrt{8/7}$	$0°$	$(-64/3)N^2$
-1	-1	$\sqrt{20/7}$	$\sqrt{8/7}$	$60°$	$(-64/3)N^2$

6 Conclusions

In this article, relationship between quasi-spin tensorial nature of one-body transition operators and the phase choices in the multi-orbit pair creation operator is derived for both identical fermion (described by shell model) and boson (described by interacting boson model) systems. Results for the correlation coefficients between the pairing operator with different choices for phases in S_+ and realistic effective interactions showed that the choice advocated by AM [6] gives maximum correlation though its absolute value is no more than 0.3. Applications of multiple pairing algebras showed that, drawing on Maheswari and Jain work [22, 23], generalized seniority in SM with phase choice advocated by AM along with effective Ω values is seen to be good for Sn isotopes both for low-lying states and high-spin isomeric states. However, direct experimental evidence for pairing algebras with other phase choices is not yet available.

Turning to interacting boson model description of collective states, imposing specific tensorial structure for EM operators with respect to the pairing

$SU_Q(1,1)$ algebra is now possible. It will be interesting to derive results for B(E2)'s (say in sdg and $sdpf$ IBM's) and B(E1)'s (in $sdpf$ IBM) with fixed tensorial structure for the transition operator but with wavefunctions that correspond to different $SU_Q(1,1)$ algebras. Such an exercise was carried out before for sdIBM [29]. A beginning is also made in the study of multiple rotational $SU(3)$ algebras both in SM and IBM's as discussed in Section 5. A more systematic study of multiple $SU(3)$ algebras and also multiple pairing algebras with internal degrees of freedom [20, 30] will be the topics of future papers.

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References

[1] I. Talmi (1993) "*Simple models of complex nuclei: the shell model and interacting boson model*" (Harwood Academic Publishers, Chur, Switzerland).
[2] J. Menendez *et al.* (2009) *Nucl. Phys. A* **818** 139.
[3] J.C. Parikh (1978) "*Group Symmetries in Nuclear Structure*" (Plenum, New York).
[4] P. Van Isacker and S. Heinze (2014) *Ann. Phys. (N.Y.)* **349** 73.
[5] J.B. French *et al.* (1969) *Adv. Nucl. Phys.* **3** 193.
[6] R. Arvieu and S.A. Moszkowski (1966) *Phys. Rev.* **145** 830.
[7] K. Allart *et al.* (1988) *Phys. Rep.* **169** 209.
[8] F. Iachello and I. Talmi (1987) *Rev. Mod. Phys.* **59** 339.
[9] F. Pan, V.G. Gueorguiev, and J.P. Draayer (2004) *Phys. Rev. Lett.* **92** 112503.
[10] Y.M. Zhao and A. Arima (2014) *Phys. Rep.* **545** 1.
[11] V.K.B. Kota and R. Sahu (2017) "*Structure of Medium Mass Nuclei: Deformed Shell Model and Spin-Isospin Interacting Boson Model*" (CRC press, Taylor & Francis, Florida).
[12] J.P. Draayer and K.J. Weeks (1984) *Ann. Phys. (NY)* **156** 41
[13] D. Bonatsos *et al.* (2017) *Phys. Rev. C* **95** 064326.
[14] V.K.B. Kota and R.U. Haq (2010) "*Spectral Distributions in Nuclei and Statistical Spectroscopy*" (World Scientific, Singapore).
[15] K.D. Launey, T. Dytrych, and J.P. Draayer (2012) *Phys. Rev. C* **85** 044003.
[16] M. Honma *et al.* (2004) *Phys. Rev. C* **69** 034335.
[17] M. Honma *et al.* (2009) *Phys. Rev. C* **80** 064323.
[18] C. Qi and Z.X. Xu (2012) *Phys. Rev. C* **86** 044323.
[19] F. Iachello and A. Arima (1987) "*The Interacting Boson Model*" (Cambridge University Press, Cambridge).
[20] V.K.B. Kota (2000) *Ann. Phys. (N.Y.)* **280** 1.
[21] D. Kusnezov (1990) *J. Phys. A* **23** 5673.
[22] B. Maheswari and A.K. Jain (2016) *Phys. Lett. B* **753** 122.

[23] B. Maheswari, A.K. Jain, and B. Singh (2016) *Nucl. Phys. A* **952** 62.
[24] M.A. Jafarizadeh *et al.* (2016) *J. Phys. G: Nucl. Part. Phys.* **43** 095108.
[25] M. Spieker *et al.* (2015) *Phys. Rev. Lett.* **114** 192504.
[26] P. Cejnar, J. Jolie, and R.F. Casten (2010) *Rev. Mod. Phys.* **82** 2155.
[27] Y.D. Devi and V.K.B. Kota (1992) *Pramana-J. Phys.* **39** 413.
[28] P. Van Isacker, A. Bouldjedri, and S. Zerguine (2010) *Nucl. Phys. A* **836** 225.
[29] P. Van Isacker, A. Frank, and J. Dukelsky (1985) Phys. Rev. C **31** 671.
[30] V.K.B. Kota and J.A. Castilho Alcarás (2006) *Nucl. Phys. A* **764** 181.